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Ioannis K. Argyros
Cameron University

Santhosh George
Cameron University

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Unified Ball Convergence of Inexact Methods For Finding Zeros with Multiplicity

[1]Ioannis K. Argyros and [2]Santhosh George

[1]Department of Mathematical Sciences
Cameron University
Lawton, OK 73505, USA
iargyros@cameron.edu

[2]Department of Mathematical and Computational Sciences
National Institute of Technology
Karnataka India-575 025
sgeorge@nitk.ac.in

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Abstract

We present an extended ball convergence of inexact methods for approximating a zero of a nonlinear equation with multiplicity m, where m is a natural number. Many popular methods are special cases of the inexact method.

Keywords: Inexact method; Ball convergence; Radius of convergence; Divided difference; Derivative; Zero with multiplicity

MSC 2010 No.: 49M15, 65D10, 65G99, 47H09

1. Introduction

There is a plethora of problems in applied sciences and also in engineering can be written in a form like

$$F(x) = 0, \tag{1}$$

using mathematical modeling, where function $F : \Omega \subseteq \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ is sufficiently many times differentiable, and $\Omega, \mathcal{B}_1, \mathcal{B}_2$ are convex subsets in $\mathbb{R}$. In the present study, we pay attention to the case of a solution p with multiplicity $m > 1$, namely, $F(p) = 0, F^{(i)}(p) = 0$ for $i = 1, 2, \ldots, m-1$, and

$F^{(m)}(p) \neq 0$. The determination of solutions of multiplicity m is of great interest. As an example, in the study of electron trajectories, when the electron reaches a plate of zero speed, the function distance from the electron to the plate has a solution of multiplicity two. Moreover, the multiplicity of solutions appears in connection to Van Der Waals equation of state and other phenomena. The convergence order of iterative methods decreases, if the equation has solutions of multiplicity m. Modifications in the iterative function are needed to improve the order of convergence.

We present the ball convergence of the inexact method (IM) defined for each $n = 0, 1, 2, \ldots,$ by

$$x_{n+1} = x_n - \epsilon \frac{F(x_n)}{F'(x_n)} - \xi_n, \tag{2}$$

where x_0 is an initial point, $\epsilon \in \mathbb{R}$ a parameter, and $\{\xi_n\} \in \mathbb{R}$ a sequence chosen in such a way as to force convergence for the sequence $\{x_n\}$ to a zero p of multiplicity m for function F. It is important to study the convergence of IM, since many popular methods are special cases of it.

Newton's method ($\epsilon = 1, \xi_n = 0$):

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}. \tag{3}$$

Modified Newton's method ($\epsilon = m, \xi_n = 0$):

$$x_{n+1} = x_n - m\frac{F(x_n)}{F'(x_n)}. \tag{4}$$

Laguerre third order method ($\epsilon = m$) and also choose:

$$\xi_n = m\frac{F(x_n)}{F'(x_n)} - \frac{\lambda F(x_n)}{F'(x_n) \pm \sqrt{\frac{\lambda-m}{m}[(\lambda-1)F'(x_n)^2 - \lambda F(x_n)F''(x_n)]}}.$$

Laguerre method further specializes to Euler-Chebyshev, Halley, Ostrowski and Hansen-Patrick method for $\lambda = 2,\ \lambda = 0,\ \lambda \longrightarrow \infty$ and $\lambda = \frac{1}{\mu} + 1$, respectively for $\mu \neq 0$.

Traub method ($\epsilon = m$) and choose:

$$\xi_n = m\frac{F(x_n)}{F'(x_n)} - \frac{m(3-m)}{2}\frac{F(x_n)}{F'(x_n)} - \frac{m^2}{2}\frac{F(x_n)^2 F''(x_n)}{F'(x_n)^3}.$$

Osada method ($\epsilon = m$) and choose:

$$\xi_n = m\frac{F(x_n)}{F'(x_n)} - \frac{1}{2}m(m+1)\frac{F(x_n)}{F'(x_n)} + \frac{1}{2}(m-1)^2\frac{F'(x_n)}{F''(x_n)}.$$

Neta at al. fourth order method ($\epsilon = m$) and choose:

$$\xi_n = m\frac{F(x_n)}{F'(x_n)} - s(t_n)\frac{F(y_n)}{F'(x_n)},$$

where $t_n = \frac{F(y_n)}{F'(x_n)}$ and s a real function satisfying some initial conditions. Other iterative methods of high convergence order can be found in (Amat and Argyros (2007); Chun and Neta (2009); Hansen and Patrick (1977); Magreñán (2014b); Magreñán (2014a); Neta (2008); Obreshkov (1963); Osada (1994); Petkovic et al. (2013); Schröder (1870); Traub (1982)) and the references therein.

Let $B(p, \lambda) := \{x \in B_1 : |x - p| < \lambda\}$ denote an open ball and let $\bar{B}(p, \lambda)$ denote its closure. It is said that $B(p, \lambda) \subseteq \Omega$ is a convergence ball for an iterative method, if the sequence generated by this iterative method converges to p, provided that the initial point $x_0 \in B(p, \lambda)$. But how close x_0 should be to p so that convergence can take place. Extending the ball of convergence is very important, since it shows the difficulty, we confront to pick initial points. It is desirable to be able to compute the largest convergence ball. This is usually depending on the iterative method and the conditions imposed on the function F and its derivatives. We can unify these conditions by expressing them as:

$$\|(F^{(m)}(p))^{-1}(F^{(m)}(x) - F^{(m)}(y))\| \leq \psi(\|x - y\|), \tag{5}$$

for all $x, y \in \Omega$, where $\psi : \mathbb{R}_+ \cup \{0\} \longrightarrow \mathbb{R}_+ \cup \{0\}$ is a continuous and nondecreasing function satisfying $\psi(0) = 0$. If we specialize function ψ, for $m \geq 1$ and

$$\psi(t) = \mu t^q, \mu > 0, q \in (0, 1), \tag{6}$$

then, we obtain the conditions under which the preceding methods were studied in (Bi et al. (2011); Chun and Neta (2009); Petkovic et al. (2013); Ren and Argyros (2010); Zhou and Song (2011); Zhou et al. (2014)). However, there are cases where even (6) does not hold (see Example 4.1). Moreover, the smaller function ψ is chosen, the larger the radius of convergence becomes. The technique, we present next can be used, for all preceding methods as well as for methods where $m = 1$. However, in the present study, we only use it for IM. This way, we extend the results in (Bi et al. (2011); Chun and Neta (2009); Petkovic et al. (2013); Ren and Argyros (2010); Zhou and Song (2011); Zhou et al. (2014)). In view of (5) there always exists a function $\varphi_0 : \mathbb{R}_+ \cup \{0\} \longrightarrow \mathbb{R}_+ \cup \{0\}$ continuous, nondecreasing, and satisfying

$$\|(F^{(m)}(p))^{-1}(F^{(m)}(x) - F^{(m)}(p))\| \leq \varphi_0(\|x - y\|), \tag{7}$$

for all $x \in \Omega$. We can always choose $\varphi_0(t) = \psi(t)$, for all $t \geq 0$. However, in general

$$\varphi_0(t) \leq \psi(t), \; t \geq 0 \tag{8}$$

holds and $\frac{\psi}{\varphi_0}$ can be arbitrarily large (Argyros (2007)). Denote by r_0 the smallest positive solution of equation $\varphi_0(t) = 1$. Set $\Omega_0 := \Omega \cup B(p, r_0)$. We have again by (5) that there exists function $\varphi : [0, r_0) \longrightarrow \mathbb{R}_+ \cup \{0\}$ continuous, nondecreasing and satisfying $\varphi(0) = 0$ such that for each $x, y \in \Omega_0$

$$\|(F^{(m)}(p))^{-1}(F^{(m)}(x) - F^{(m)}(y))\| \leq \varphi(\|x - y\|). \tag{9}$$

Clearly, we have

$$\varphi(t) \leq \psi(t), \text{ for all } t \in [0, r_0), \tag{10}$$

since $\Omega_0 \subseteq \Omega$. It turns out that more precise (7) (see (8)) can be used than (5) to estimate upper bounds on the inverses of the functions involved (see (32) or (40)). Moreover, for the upper bounds on the numerators (see (33) or (41)) we can use (9) tighter than (5) (see (10)). This way we obtain

(34) or (36) which are tighter than the corresponding ones using only ψ (or its special case (6)). This way we obtain a larger radius of convergence leading to a wider choice of initial guesses and at least as tight error bounds on the distances $|x_n - p|$ resulting to the computation of at least as few iterates to obtain a desired error tolerance (see also the numerical examples). It is worth noticing that these advantages are obtained under the same computational cost as in earlier studies, since in practice the computation of function ψ (or (6)) requires the computation of functions φ_0 and ψ as special cases.

The rest of the paper is structured as follows. Section 2 contains some auxiliary results on divided differences and derivatives. The ball convergence of IM is given in Section 3. The numerical examples appear in the concluding Section 4.

2. Auxiliary results

We need the definition of divided differences, and their properties which can be found in (Bi et al. (2011); Ren and Argyros (2010); Zhou and Song (2011); Zhou et al. (2014)).

Definition 2.1.

The divided differences $F[y_0, y_1, \ldots, y_k]$, on $k+1$ distinct points $y_0, y_1, \ldots, y_k$ of a function $F(x)$ are defined by

$$\begin{aligned} F[y_0] &= F(y_0), \\ F[y_0, y_1] &= \frac{F[y_0] - F[y_1]}{y_0 - y_1}, \\ &\vdots \\ F[y_0, y_1, \ldots, y_k] &= \frac{F[y_0, y_1, \ldots, y_{k-1}] - F[y_0, y_1, \ldots, y_k]}{y_0 - y_k}. \end{aligned} \tag{11}$$

If the function F is sufficiently differentiable, then its divided differences $F[y_0, y_1, \ldots, y_k]$ can be defined even if some of the arguments y_i coincide. For instance, if $F(x)$ has $k-$th derivative at y_0, then it makes sense to define

$$F[\underbrace{y_0, y_1, \ldots, y_k}_{k+1}] = \frac{F^{(k)}(y_0)}{k!}. \tag{12}$$

Lemma 2.2.

The divided differences $F[y_0, y_1, \ldots, y_k]$ are symmetric functions of their arguments, i.e., they are invariant to permutations of the points $y_0, y_1, \ldots, y_k$.

Lemma 2.3.

If the function F has $(k+1)-$th derivative, and p is a zero of multiplicity m, then for every argument

x, the following formula holds

$$F(x) = F[y_0] + \sum_{i=1}^{k} F[y_0, y_1, \ldots, y_k] \prod_{j=0}^{i-1} (x - y_j) + F[y_0, y_1, \ldots, y_k, x] \prod_{i=0}^{k} (x - y_i). \tag{13}$$

Lemma 2.4.

If the function F has $(m+1)-$th derivative, and p is a zero of multiplicity m, then for every argument x, the following formula holds

$$F(x) = F[\underbrace{p, p, \ldots, p}_{m}, x](x-p)^m, \tag{14}$$

$$F'(x) = F[\underbrace{p, p, \ldots, p}_{m}, x, x](x-p)^m + mF[\underbrace{p, p, \ldots, p}_{m}, x](x-p)^{m-1}. \tag{15}$$

We need the following lemma on Genocchi's integral expression formula for divided differences.

Lemma 2.5.

If the function F has continuous $k-$th derivative, then the following formula holds for any points $y_0, y_1, \ldots, y_k$

$$F[y_0, y_1, \ldots, y_k] = \int_0^1 \cdots \int_0^1 F^{)k)}(y_0 + \sum_{i=1}^{k} (y_i - y_{i-1}) \prod_{j=1}^{i} \theta_j) \prod_{i=1}^{k} (\theta_i^{k-i} d\theta_i). \tag{16}$$

We shall also use the following Taylor expansion with integral form reminder.

Lemma 2.6.

Suppose that $F(x)$ is differentiable $n-$times in the ball $B(x_0, r), r > 0$, and $F^{(n)}(x)$ is integrable from a to $x \in B(a, r)$. Then,

$$\begin{aligned} F(x) = {} & F(a) + F'(a)(x-a) + \frac{1}{2} F''(a)(x-a)^2 + \ldots + \frac{1}{n!} F^{(n)}(a)(x-a)^n \\ & + \frac{1}{(n-1)!} \int_0^1 [F^{(n)}(a + t(x-a)) - F^{(n)}(a)](x-a)^n (1-t)^{n-1} dt, \end{aligned} \tag{17}$$

and

$$\begin{aligned} F'(x) = {} & F'(a) + F''(a)(x-a) + \frac{1}{2} F'''(a)(x-a)^2 \\ & + \ldots + \frac{1}{(n-1)!} F^{(n)}(a)(x-a)^{n-1} \\ & + \frac{1}{(n-2)!} \int_0^1 [F^{(n)}(a + t(x-a)) - F^{(n)}(a)](x-a)^{n-1} (1-t)^{n-2} dt. \end{aligned} \tag{18}$$

3. Ball convergence

Let $\varphi_0 : \mathbb{R}_+ \cup \{0\} \longrightarrow \mathbb{R}_+ \cup \{0\}$ be a continuous and nondecreasing function with $\varphi_0(0) = 0$. Moreover, define functions $\beta_0, \beta : \mathbb{R}_+ \cup \{0\} \longrightarrow \mathbb{R}_+ \cup \{0\}$ by

$$\beta_0(t) = (m-1)!(m-1)\int_0^1 \cdots \int_0^1 \varphi_0(t\prod_{i=1}^{m}\theta_i)\prod_{i=1}^{m}\theta_i^{m-i}d\theta_i\ ,$$

and

$$\beta(t) = (m-1)!\int_0^1 \cdots \int_0^1 \varphi_0(t\prod_{i=1}^{m-1}\theta_m)\prod_{i=1}^{m-1}\theta_i^{m-i}d\theta_i + \beta_0(t).$$

We have that functions β_0, β are continuous and nondecreasing with $\beta_0(0) = \beta(0) = 0$. Suppose

$$\beta(t) \longrightarrow 1 \text{ as } t \longrightarrow \text{ a positive number or } +\infty. \tag{19}$$

It follows from the intermediate value theorem that equation $\beta(t) = 1$ has solutions in $(0, +\infty)$. Denote by r_0 the smallest positive solution of equation $\beta(t) = 1$. Set $h_1(t) = 1 - \beta(t)$. Let $\varphi : [0, r_0) \longrightarrow \mathbb{R}_+ \cup \{0\}$ be a continuous and nondecreasing function satisfying $\varphi(0) = 0$. Furthermore, define functions α, h_0 and h on $[0, r_0)$ by

$$\alpha(t) = (m-1)!\int_0^1 \cdots \int_0^1 \varphi(t\prod_{i=1}^{m-1}\theta_i(1-\theta_m))\prod_{i=1}^{m}\theta_i^{m-i}d\theta_i d\theta_m,$$

$$h_0(t) = m^{-1}\alpha(t)t + m^{-1}|m-\epsilon|\beta_0(t) + m^{-1}\alpha(t)at^b + \beta_0(t)at^{b-1},$$

and

$$h(t) = \frac{h_0(t)}{h_1(t)} - 1,$$

where $a \geq 0$ and $b \geq 1$. We get that $h_0(0) = -1 < 0$ and $h(t) \longrightarrow +\infty$ as $t \longrightarrow r_0^-$. Denote by r the smallest solution of equation $h(t) = 0$ in $(0, r_0)$. Then, we have that for each $t \in [0, r)$,

$$0 \leq \beta(t) < 1, \tag{20}$$

and

$$0 \leq h(t) < 1. \tag{21}$$

First, we show the ball convergence of method IM under conditions ($\mathcal{A}$):

($\mathcal{A}_1$) $F : \Omega \subseteq \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ is continuously $m-$times Fréchet-differentiable.
($\mathcal{A}_2$) Function F has a zero p of multiplicity $m, m = 1, 2, \ldots$
($\mathcal{A}_3$) There exists function $\varphi_0 : \mathbb{R}_+ \cup \{0\} \longrightarrow \mathbb{R}_+ \cup \{0\}$ continuous and nondecreasing satisfying $\varphi_0(0) = 0$ such that for each $x \in \Omega$,

$$\|F^{(m)}(p)^{-1}(F^{(m)}(x) - F^{(m)}(p))\| \leq \varphi_0(\|x - p\|).$$

Let $\Omega_0 = \Omega \cup B(p, r_0)$, where r_0 is defined previously.
($\mathcal{A}_4$) There exists $\varphi : [0, r) \longrightarrow \mathbb{R}_+ \cup \{0\}$ continuous and nondecreasing satisfying $\varphi(0) = 0$ such that for each $x, y \in \Omega_0$,

$$\|F^{(m)}(p)^{-1}(F^{(m)}(x) - F^{(m)}(y))\| \leq \varphi(\|x - p\|).$$

($\mathcal{A}_5$) Condition (19) holds.
($\mathcal{A}_6$) $\bar{B}(p,r) \subseteq \Omega$.
($\mathcal{A}_7$) $|\xi_n| \leq a|x_n - p|^b$, for all $n = 0, 1, 2, \ldots$, and some $a \geq 0, b \geq 1$.

Theorem 3.1.

Suppose that the ($\mathcal{A}$) conditions hold. Then, for starting point $x_0 \in B(p,r) - \{p\}$, the sequence $\{x_n\}$ generated by IM is well defined in $B(p,r)$, remains in $B(p,r)$, for all $n = 0, 1, 2, \ldots$, and converges to p.

Proof:

We shall show that sequence

$$\delta_n = x_n - p, \tag{22}$$

is non-increasing and converges to zero. Using $\delta_n = x_n - p$, method (2) for $n = 0$, Lemma 2.4 and the following formulae:

$$g(x) = F[\underbrace{p, p, \ldots, p}_{m}, x],\ g_0(x) = F[\underbrace{p, p, \ldots, p}_{m}, x, x]. \tag{23}$$

$$F(x_0) = g(x_0)\delta_0^m, \tag{24}$$

and

$$F'(x_0) = [g_0(x_0)\delta_0 + mg(x_0)]\delta_0^{m-1}. \tag{25}$$

We can write

$$\delta_1 = \frac{g(p)^{-1}N}{g(p)^{-1}D}, \tag{26}$$

where

$$N = g_0(x_0)\delta_0^2 + [(m-\epsilon)g(x_0) - g_0(x_0)\xi_0]\delta_0 - mg(x_0)\xi_0, \tag{27}$$

and

$$D = g_0(x_0)\delta_0 + mg(x_0). \tag{28}$$

In view of the definition of divided differences, we have

$$g_0(x_0)\delta_0 = F[\underbrace{p, p, \ldots, p}_{m-1}, x_0, x_0] - g(x_0). \tag{29}$$

Then, we obtain from (12) and (29) that

$$\begin{aligned} &|1 - (mg(p))^{-1}[h_0(x_0)\delta_0 + mg(x_0)]| \\ &= |(mg(p))^{-1}[g_0(x_0)\delta_0 + mg(x_0) - mg(p)]| \\ &= (m-1)!F^{(m)}(p)^{-1}(F[\underbrace{p, p, \ldots, p}_{m-1}, x_0, x_0] - g(p) + (m-1)[g(x_0) - g(p)])|. \end{aligned} \tag{30}$$

By Lemma 2.5, we get

$$F[\underbrace{p, p, \ldots, p}_{m-1}, x_0, x_0] \tag{31}$$
$$= \int_0^1 \cdots \int_0^1 F^{(m)}(p + \delta_0 \prod_{i=1}^{m-1} \theta_i) \prod_{i=1}^{m} (\theta_i^{m-1} d\theta_i),$$
$$g(x_0) = \int_0^1 \cdots \int_0^1 F^{(m)}(p + \delta_0 \prod_{i=1}^{m} \theta_i) \prod_{i=1}^{m} (\theta_i^{m-1} d\theta_i), \tag{32}$$
$$g(p) = \int_0^1 \cdots \int_0^1 F^{(m)}(p) \prod_{i=1}^{m} (\theta_i^{m-1} d\theta_i). \tag{33}$$

Substituting (30)–(33) into (29) using condition $(\mathcal{A}_3)$, $x_0 \in B(p, r)$, and the definition of r, we get

$$\begin{aligned}
&|1 - (mg(p))^{-1}]g_0(x_0)\delta_0 + mg(x_0)]| \\
&= (m-1)!|\int_0^1 \cdots \int_0^1 F^{(m)}(p)^{-1}(F^{(m)}(p + \delta_0 \prod_{i=1}^{m-1} \theta_i) - F^{(m)}(p)) \prod_{i=1}^{m} (\theta_i^{m-i} d\theta_i) \\
&\quad +(m-1)F^{(m)}(p)^{-1}(F^{(m)}(p + \delta_0 \prod_{i=1}^{m-1} \theta_i) - F^{(m)}(p)) \prod_{i=1}^{m} (\theta_i^{m-i} d\theta_i)| \\
&\leq (m-1)!(\int_0^1 \cdots \int_0^1 |F^{(m)}(p)^{-1}(F^{(m)}(p + \delta_0 \prod_{i=1}^{m-1} \theta_i) - F^{(m)}(p))| \prod_{i=1}^{m} (\theta_i^{m-i} d\theta_i) \\
&\quad +(m-1) \int_0^1 \cdots \int_0^1 |F^{(m)}(p)^{-1}(F^{(m)}(p + \delta_0 \prod_{i=1}^{m-1} \theta_i) - F^{(m)}(p))| \prod_{i=1}^{m} (\theta_i^{m-i} d\theta_i)| \\
&\leq (m-1)![\int_0^1 \cdots \int_0^1 \varphi_0(|\delta_0| \prod_{i=1}^{m-1} \theta_i) \prod_{i=1}^{m} \theta_i^{m-i} d\theta_i \\
&\quad +(m-1) \int_0^1 \cdots \int_0^1 \varphi_0(|\delta_0| \prod_{i=1}^{m} \theta_i) \prod_{i=1}^{m} \theta_i^{m-i} d\theta_i] \\
&\leq \beta(|\delta_0|) < \beta(r) < 1.
\end{aligned} \tag{34}$$

It follows from the Banach perturbation lemma (Amat and Argyros (2007); Argyros (2003)) and (34) that, $g_0(x_0)\delta_0 + mg(x_0) \neq 0$, and

$$|(mg(p))^{-1}g_0(x_0)\delta_0 + mg(x_0))^{-1}| \leq \frac{1}{1 - \beta(|\delta_0|)} < \frac{1}{1 - \beta(r)}. \tag{35}$$

Moreover, using (29), (31), (32) and ($\mathcal{A}_4$), we have in turn that

$$\begin{aligned}|(mg(p))^{-1}g_0(x_0)\delta_0| &= (m-1)!|\int_0^1\cdots\int_0^1 F^{(m)}(p)^{-1}(F^{(m)}(p+\delta_0\prod_{i=1}^{m-1}\theta_i)\\ &\quad -F^{(m)}(p+\delta_0\prod_{i=1}^{m}\theta_i))\prod_{i=1}^{m}(\theta_i^{m-i}d\theta_i)|\\ &= (m-1)!|\int_0^1\cdots\int_0^1 |F^{(m)}(p)^{-1}(F^{(m)}(p+\delta_0\prod_{i=1}^{m-1}\theta_i)\\ &\quad -F^{(m)}(p+\delta_0\prod_{i=1}^{m}\theta_i))|\prod_{i=1}^{m}(\theta_i^{m-i}d\theta_i)|\\ &\leq (m-1)!\int_0^1\cdots\int_0^1 \varphi_0(|\delta_0|\prod_{i=1}^{m-1}\theta_i(1-\theta_m))\prod_{i=1}^{m}\theta_i^{m-i}d\theta_i d\theta_m\\ &= \alpha(|\delta_0|)<\alpha(r)<1. \end{aligned} \tag{36}$$

Furthermore, we have

$$\begin{aligned}|g(p)^{-1}g(x_0)| &= |g(p)^{-1}(g(x_0)-g(p))|\\ &= |(m-1)!F^{(m)}(p)^{-1}(m-1)(g(x_0)-g(p))|\\ &= (m-1)(m-1)!\int_0^1\cdots\int_0^1 |F^{(m)}(p)^{-1}(F^{(m)}(p+\delta_0\prod_{i=1}^{m}\theta_i)\\ &\quad -F^{(m)}(p)|\prod_{i=1}^{m}\theta_i d\theta_1\\ &\leq (m-1)(m-1)!\int_0^1\cdots\int_0^1 \varphi_0(|\delta_0|)\prod_{i=1}^{m}\theta_i)\prod_{i=1}^{m}\theta_i^{m-i}d\theta_i. \end{aligned} \tag{37}$$

Using (34) – (37), we obtain that

$$|\delta_1| \leq c|\delta_0| < |\delta_0| < r,$$

where $c = h(|\delta_0|) \in [0,1)$, so $x_1 \in B(p,r)$. By simply replacing x_0, x_1, by x_k, x_{k+1}, we arrive at

$$|x_{k+1}-p| \leq c|x_k-p| < r, \tag{38}$$

which shows $\lim_{k\longrightarrow+\infty} x_k = p$, and $x_{k+1} \in B(p,r)$. ■

Concerning the uniqueness of the solution p, we have the following.

Proposition 3.2.

Suppose that conditions ($\mathcal{A}$) and

$$\frac{m}{(s_2-s_1)^m}\int_{s_1}^{s_2}\varphi_0(|t-s_1|)|s_2-t|^{m-1}dt < 1, \tag{39}$$

for all s_1, t, s_2 with $0 \leq s_1 \leq t \leq s_2 \leq \bar{r}$ for some $\bar{r} \geq r$ hold. Then, the solution p of equation $F(x) = 0$ is unique in $\Omega_0 = \Omega \cup \bar{B}(p,\bar{r})$.

Proof:

Suppose that $p^* \in \Omega_0$ is a solution of equation $F(x) = 0$ with $p \neq p^*$. Without loss of generality suppose $p < p^*$. We can write

$$F(p^*) - F(p) = \frac{1}{(m-1)!} \int_p^{p^*} F^{(m)}(t)(p^* - t)^{m-1} dt. \tag{40}$$

Using $(\mathcal{A}_3)$ and (38), we obtain in turn that

$$\begin{aligned} &\left| 1 - \left(\frac{(p^* - p)^m}{m} F^{(m)}(p) \right)^{-1} \int_p^{p^*} F^{(m)}(t)(p^* - t)^{m-1} dt \right| \\ &= \left| \left(\frac{(p^* - p)^m}{m} F^{(m)}(p) \right)^{-1} \int_p^{p^*} [F^{(m)}(t) - F^{(m)}(p)](p^* - t)^{m-1} dt \right| \\ &\leq \frac{m}{(p^* - p)^m} \int_p^{p^*} \varphi_0(|t - p|)|p^* - t|^{m-1} dt < 1, \end{aligned} \tag{41}$$

so $\left(\frac{(p^*-p)^m}{m} F^{(m)}(p) \right)^{-1} \int_p^{p^*} F^{(m)}(t)(p^* - t)^{m-1} dt$ is invertible, i.e., $\int_p^{p^*} F^{(m)}(t)(p^* - t)^{m-1} dt$ is invertible. ■

4. Numerical Examples

We apply Theorem 3.1 to the case of the modified Newton method, so we choose $\xi_n = 0$. In the first example condition (5) is not satisfied. Hence, earlier results based on it (Bi et al. (2011); Chun and Neta (2009); Petkovic et al. (2013); Ren and Argyros (2010); Zhou and Song (2011); Zhou et al. (2014)) cannot guarantee the convergence of method (2), but our conditions hold, so convergence is assured. Hence, we extended the applicability of method (2). The second example is used to show how we choose functions φ_0 and φ appearing in Theorem 3.1.

Example 4.1.

Let $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}, \Omega = [0, 1], m = 2, p = 0$. Define function F on Ω by

$$F(x) = \frac{4}{35} x^{\frac{7}{2}} + \frac{1}{6} x^3 + \frac{1}{2} x^2. \tag{42}$$

We have by (42) $F'(x) = \frac{2}{5} x^{\frac{5}{2}} + \frac{x^2}{2} + x$, $F''(x) = x^{\frac{3}{2}} + x + 1$, $F'''(x) = x^{\frac{1}{2}} + 1$ and $F''(0) = 1$. Function F'' cannot satisfy (5) with ψ given by (6). Hence, the results in (Bi et al. (2011); Chun and Neta (2009); Petkovic et al. (2013); Ren and Argyros (2010); Zhou and Song (2011); Zhou et al. (2014)) cannot apply. However, the new results apply, since (7) and (9) are satisfied for $\varphi_0(t) = \varphi(t) = t^{\frac{3}{2}} + t$, respectively. The convergence radius is $r = 0.6511$, obtained by solving the equation $h(t) = 0$.

Example 4.2.

Let $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}, \Omega = [-1, 1], m = 2, p = 0$. Define function F on Ω by

$$F(x) = e^x - x - 1.$$

We get by (7) and (9), respectively that $\varphi_0(t) = (e-1)t$ and $\varphi(t) = et$. Then, the convergence radius is $r = 0.7163$, obtained by solving the equation $h(t) = 0$.

5. Conclusion

There are many iterative methods used to generate a sequence converging to zeros with multiplicity of nonlinear equations defined on a subset of the real line. We have mentioned several such methods in the introduction. These methods converge under some conditions.

In this article we introduce inexact method (2) that contains all these and other methods as special cases. This allows us to study these methods in a uniform way. In particular, we have provided a ball convergence for IM using generalized Lipschitz type functions. Moreover, using the center-Lipschitz condition first, we have located a subset of the original domain containing the iterates. This way, the majorizing functions are tighter leading to a finer convergence analysis than in the earlier studies, and under the same computational cost (see also the numerical examples). Hence, we extended the applicability of these methods.

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