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R. Jana

Jadavpur University

A. K. Das

Indian Statistical Institute

S. Sinha

Jadavpur University

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On Processability of Lemke's Algorithm

¹ *R. Jana, ²A. K. Das and ³S. Sinha

^{1, 3}Department of Mathematics
Jadavpur University
188, Raja S.C. Mallick Rd
Kolkata, India

¹rwitamjanaju@gmail.com; ³sagnik62@yahoo.co.in

²SQC & OR Unit
Indian Statistical Institute
203, BT Road
Kolkata, India
akdas@isical.ac.in

*Corresponding Author

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Abstract

Lemke's algorithm is a pivotal kind of algorithm which is developed based on principal pivot transform. We consider several matrix classes to study the relationship among them in the context of linear complementarity problem. These classes are important from Lemke's algorithmic point of view. In this article we discuss about the processability of Lemke's algorithm with respect to some selective matrix classes.

Keywords: Linear complementarity problem; Lemke's algorithm; principal pivot transform; column sufficient matrix; almost P_0 -matrix; Q_0 -matrix; Q -matrix

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1. Introduction

We start with the definition of linear complementarity problem. Given $A \in R^{n \times n}$ and a vector $q \in R^n$, the *linear complementarity problem* $LCP(q, A)$ is the problem of finding a solution $x \in R^n$

and $y \in R^n$ to the following system of linear equations and inequalities;

$$y - Ax = q, \quad y \geq 0, \quad x \geq 0, \quad (1.1)$$

and

$$y^T x = 0. \quad (1.2)$$

Linear complementarity problem is well studied in the literature of optimization theory. For details see Dubey et al. (2018), Husain et al. (2013) and Gupta et al. (2013). Several mathematical programming problems which include linear programming problem, quadratic programming problem, integral equation [Mishra (2017)] can be posed as linear complementarity problem. Linear complementarity problem is studied in Banach spaces [Mishra (2007)]. Lemke's algorithm is a path-following algorithm to solve linear complementarity problem when the domain space is finite dimensional. Lemke's algorithm does not solve every instance of the linear complementarity problem and in some instances of the problem may terminate inconclusively without either computing a solution to it or showing that no solution to it exists. Lemke's algorithm is a pivotal kind of technique to compute $LCP(q, A)$. We provide a brief description of this algorithm.

Step 1: Decrease x_0 so that one of the variables y_i , $1 \leq i \leq n$, say y_r is reduced to zero. We now have a basic feasible solution with x_0 in place of y_r and with exactly one pair of complementary variables (y_r, x_r) being non-basic.

Step 2: At each iteration, the complement of the variable which has been removed in the previous iteration is to be increased. In the second iteration, for instance, x_r will be increased.

Step 3: If the variable selected at step 2 to enter the basis can be arbitrarily increased, then the procedure terminates in a *secondary ray*. If a new basic feasible solution is obtained with $x_0 = 0$, we have solved (1.1) and (1.2). If in the new basic feasible solution $x_0 > 0$, we have obtained a new basic pair of complementary variables (y_s, x_s) . We repeat step 2.

Lemke's algorithm consists of the repeated applications of steps 2 and 3. If non-degeneracy is assumed, the procedure terminates either in a secondary ray or in a solution to (1.1) and (1.2). Ramamurthy (1986) showed that Lemke's algorithm for the linear complementarity problem can be used to check whether a given Z -matrix is a P_0 -matrix and it can also be used to analyze the structure of finite Markov chains. Lemke's algorithm is used in the area of game theory [Aumann (2017)], market equilibrium problems. For details see Duan et al. (1989), Garg et al. (2015).

Extending the applicability of Lemke's algorithm to more matrix classes have been considered by many researchers like Eaves (1971), Garcia (1973). The concept of principal pivot transform and the matrix classes play important role in this context. Some of the matrix classes are invariant under the principal pivot transform. The *principal pivot transform* (PPT) of A , a real $n \times n$ matrix, with respect to $\alpha \subseteq \{1, 2, \dots, n\}$ is defined as the matrix M given by

$$M = \begin{bmatrix} M_{\alpha\alpha} & M_{\alpha\bar{\alpha}} \\ M_{\bar{\alpha}\alpha} & M_{\bar{\alpha}\bar{\alpha}} \end{bmatrix},$$

where $M_{\alpha\alpha} = (A_{\alpha\alpha})^{-1}$, $M_{\alpha\bar{\alpha}} = -(A_{\alpha\alpha})^{-1}A_{\alpha\bar{\alpha}}$, $M_{\bar{\alpha}\alpha} = A_{\bar{\alpha}\alpha}(A_{\alpha\alpha})^{-1}$, $M_{\bar{\alpha}\bar{\alpha}} = A_{\bar{\alpha}\bar{\alpha}} - A_{\bar{\alpha}\alpha}(A_{\alpha\alpha})^{-1}A_{\alpha\bar{\alpha}}$. Note that PPT is only defined with respect to those α for which $\det A_{\alpha\alpha} \neq 0$. When $\alpha = \emptyset$, by convention $\det A_{\alpha\alpha} = 1$ and $M = A$. For further details see Cottle et al. (1992), Das et al. (2017), Das et al. (2018) and Karamardian (2014).

Let us consider $\text{FEA}(q, A) = \{x : q + Ax \geq 0\}$ and $\text{SOL}(q, A) = \{x \in \text{FEA}(q, A) : x^T(q + Ax) = 0\}$ are said to be feasible and solution set of $\text{LCP}(q, A)$ respectively. A matrix is said to be Q -matrix if for every q , $\text{LCP}(q, A)$ has at least one solution. A matrix is said to be Q_0 -matrix if for $\text{FEA}(q, A) \neq \emptyset \implies \text{SOL}(q, A) \neq \emptyset$. In this article we discuss about the processability of Lemke's algorithm.

2. Preliminaries

We denote the n dimensional real space by R^n . We consider vectors and matrices with real entries. Any vector $x \in R^n$ is a column vector unless otherwise specified and x^T denotes the row transpose of x . The value of a matrix $v(A) > 0$ if \exists a $0 \neq x \geq 0$ such that $Ax > 0$. Similarly, $v(A) < 0$ if \exists a $0 \neq y \geq 0$ such that $y^T A < 0$. Now we give the definitions of some matrix classes which will be required in the next section.

Definition 2.1.

A matrix $A \in R^{n \times n}$ is said to be;

- (i) *column sufficient* if $x_i(Ax)_i \leq 0 \forall i \implies x_i(Ax)_i = 0 \forall i$.
- (ii) *row sufficient* if A^T is column sufficient.
- (iii) *sufficient* if A is both column and row sufficient.

Definition 2.2.

A matrix $A \in R^{n \times n}$ is said to be;

- (i) $P(P_0)$ if all its principal minors are positive (nonnegative).
- (ii) *almost P_0* if all its principal minors upto order $(n - 1)$ are nonnegative and $\det A < 0$.

Definition 2.3.

A matrix $A \in R^{n \times n}$ is said to be;

- (i) *copositive* (C_0) if $x^T Ax \geq 0, \forall x \geq 0$.
- (ii) *strictly C_0* if $x^T Ax > 0, \forall 0 \neq x \geq 0$.
- (iii) *copositive star* (C_0^*) matrix if A is copositive and $x^T Ax = 0, Ax \geq 0, x \geq 0 \implies A^T x \leq 0$.

Definition 2.4.

A matrix A is said to be E_0 if for every $0 \neq x \geq 0 \exists$ an index k such that $x_k > 0$ and $(Ax)_k \geq 0$.

Definition 2.5.

A matrix $A \in R^{n \times n}$ is said to be L_2 if for every $0 \neq x \geq 0$ such that $Ax \geq 0$, $x^T Ax = 0$, \exists two diagonal matrices $D_1 \geq 0$ and $D_2 \geq 0$ such that $D_2 x \neq 0$ and $(D_1 A + A^T D_2)x = 0$.

Definition 2.6.

A matrix $A \in L$ if $A \in E_0 \cap L_2$.

Definition 2.7.

A matrix $A \in R^{n \times n}$ is said to be Z if $a_{ij} \leq 0$.

Now we give some theorems which will be required for discussion in the next section.

Theorem 2.8.

[Eaves (1971)] L -matrices are Q_0 -matrix.

Theorem 2.9.

[Cottle et al. (1992)] Z -matrices are Q_0 -matrix.

Theorem 2.10.

[Gowda (1989)] C_0^* -matrices are L -matrix.

3. Main results

In this article we discuss the processability of Lemke's algorithm by addressing the following three cases.

Case I: *Is it true that a subclass of column sufficient matrix which is not row sufficient is processable by Lemke's algorithm?* To address the Case I, first we establish the following result.

Theorem 3.1.

Suppose A is column sufficient matrix. Then, $A \in P_0$.

Proof:

Let us consider $0 \neq x \in R^n$ be arbitrary. Then, \exists at least one index k such that $x_k \neq 0$. Suppose $x_k(Ax)_k < 0$. Then, it will contradict the fact that A is column sufficient matrix. Again A is said to be P_0 matrix [Cottle et al. (1992)] if for every $0 \neq x \in R^n \exists$ an index k such that $x_k(Ax)_k \geq 0$. Therefore $A \in P_0$. ■

Now we consider the following two examples which are column sufficient but not row sufficient.

Example 3.2.

Consider

$$E = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now we show that $E \notin Q_0$. We consider $q = \begin{bmatrix} -8 \\ -5 \\ 1 \end{bmatrix}$ and so $LCP(q, E)$ is feasible however $LCP(q, E)$ has no solution.

Example 3.3.

Consider

$$F = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now we show $F \notin L$. For any nonnegative vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $x^T F x = (x_1 - x_2)^2 + x_3^2 + 2x_1 x_3$. Now

consider $x_1 = x_2 = k (> 0)$ and $x_3 = 0$. Therefore $x = \begin{bmatrix} k \\ k \\ 0 \end{bmatrix}$ is the only non-zero vector for which

$x^T F x = 0$. Let us consider $D_1 = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$ and $D_2 = \begin{bmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{bmatrix}$ be two nonnegative diagonal matrices.

Then, $D_1 F + F^T D_2 = \begin{bmatrix} e_1 + d_1 & -d_2 & -2d_1 \\ -d_2 & e_2 + d_2 & 0 \\ -2e_1 & 0 & e_3 + d_3 \end{bmatrix}$.

Hence it is clear that for $x = \begin{bmatrix} k \\ k \\ 0 \end{bmatrix}$, $(D_1 F + F^T D_2)x = 0$ and $D_2 x \neq 0$ does not hold simultaneously.

However it is easy to show $F \in Q_0$.

It is well known that $LCP(q, A)$ is processable by Lemke's algorithm [Aganagic et al. (1987)] if $A \in P_0 \cap Q_0$. This implies any row sufficient matrix is processable by Lemke's algorithm. However same phenomenon is not applicable in case of column sufficient matrix. For Example 3.2 we say $E \in P_0$ and $E \notin Q_0$. Hence we can not conclude about the processability of $LCP(q, E)$ by Lemke's algorithm. For Example 3.3, note that F is neither row sufficient nor L . However $F \in P_0 \cap Q_0$.

We can conclude about the processability of $LCP(q, F)$ by Lemke's algorithm. Hence we conclude that a subclass of column sufficient matrix which is neither row sufficient nor L is processable by Lemke's algorithm.

Case II: Lemke's algorithm with Z -matrices. It is known that a Z -matrix is processable by Lemke's algorithm [Cottle et al. (1992)]. We prove the following results related to Z -matrices in the context of linear complementarity problem. To start with we establish the following result to show the conditions that Q -matrix is processable by Lemke's algorithm.

Theorem 3.4.

Let A be Z -matrix and \exists a positive diagonal matrix D such that $(DA + A^T D)$ is strictly C_0 -matrix. Then, $A \in Q$.

Proof:

Let A is Z -matrix. Therefore $A \in Q_0$ by the Theorem 2.9. To show $A \in Q$ it is enough to show that $v(A) > 0$. Suppose not, then $v(A) \leq 0$. Therefore \exists a $y \geq 0$ such that $y^T A \leq 0$. As D is positive diagonal matrix $y^T A D \leq 0$. Now $y^T (DA + A^T D)y = y^T A D y + y^T A^T D y$ which is clearly ≤ 0 . It contradicts the fact that $(DA + A^T D)$ is strictly C_0 -matrix. Hence $v(A) > 0$ and $A \in Q$. ■

Now we discuss about the relationship between the Z and almost P_0 -matrices. It is very easy to show that not every almost P_0 -matrices are Z -matrices. For example $\begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}$ is almost P_0 -matrix but it is not a Z -matrix. We prove the following theorem.

Theorem 3.5.

Let $A \in R^{n \times n} \cap$ almost P_0 -matrix with at least one PPT of A is a Z -matrix. Assume that for some $i_0, j_0 \in \{1, 2, \dots, n\}$, $a_{i_0 i_0} = 0$ and $a_{i_0 j_0} > 0$. Then, \exists a $k \in \{1, 2, \dots, n\}$ such that $a_{k i_0} < 0$.

Proof:

As $a_{i_0 j_0} > 0$, then without loss of generality we choose $q \in R^n$ such that $q_{i_0} < 0$, $q_j > 0$ for all $j \neq i_0$ and $FEA(q, A) \neq \emptyset$. Again A has at least one PPT, say B , which is a Z -matrix. Then, B is a Q_0 -matrix by Theorem 2.9. Hence $A \in Q_0$. Therefore $SOL(q, A) \neq \emptyset$. Let $z \in SOL(q, A)$ and $\alpha = \{i : z_i \neq 0\}$. Take $\beta = \alpha \setminus \{i_0\}$. Clearly $\beta \neq \emptyset$ as $a_{i_0 i_0} = 0$ and $q_{i_0} < 0$. Since $z_\beta > 0$, we can write $A_{\beta i_0} z_{i_0} + A_{\beta \beta} z_\beta + q_\beta = 0$. Note that $q_\beta > 0$. Now if $A_{i_0} \geq 0$, then $A_{\beta \beta} z_\beta = -q_\beta - A_{\beta i_0} z_{i_0} < 0$. This implies $v(A_{\beta \beta}) < 0$. Again $A \in$ almost P_0 . So $A_{\beta \beta} \in P_0$ which implies $A_{\beta \beta} \in E_0$. Hence $v(A_{\beta \beta}) \geq 0$ [Cottle et al. (1992)]. So $v(A_{\beta \beta}) < 0$ is not possible. Therefore A_{i_0} must contain a negative entry and subsequently \exists a $k \in \{1, 2, \dots, n\}$ such that $a_{k i_0} < 0$. ■

Consider the following example to illustrate our result.

Example 3.6.

Consider

$$M = \begin{bmatrix} 1 & -3 & 1 \\ 1 & 0 & -1 \\ -2 & 1 & 1 \end{bmatrix}.$$

Note that $M \in \text{almost } P_0\text{-matrix}$ and

$$M^{-1} = \begin{bmatrix} -1 & -4 & -3 \\ -1 & -3 & -2 \\ -1 & -5 & -3 \end{bmatrix}.$$

Corollary 3.7.

Suppose $A \in R^{n \times n} \cap \text{almost } P_0\text{-matrix}$ with at least one PPT of A is a Z -matrix. Assume that every row of A contains a positive entry. Then, every nontrivial solution of $\text{LCP}(0, A)$ contains at least two positive coordinates.

We show that $A \in L$ does not imply $A \in Z$ and vice versa.

Example 3.8.

Consider

$$G = \begin{bmatrix} 0 & 7 & 8 \\ 5 & 0 & 6 \\ -2 & -1 & 0 \end{bmatrix}.$$

It is easy to show that $G \in C_0^*$. Hence by the Theorem 2.10, $G \in L$ but $G \notin Z$. For the reverse part we consider the matrix F in Example 3.3 given above. Note that $F \in Z$ but $F \notin L$.

Note that $\text{LCP}(q, A)$ is processable by Lemke's algorithm where $A \in Z$ or $A \in P_0 \cap L$ by the Theorem 2.8.

Case III: Are all Q -matrices processable by Lemke's algorithm? Our answer is negative.

Example 3.9.

Consider

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}.$$

Now by taking a PPT with respect to $\alpha = \{1, 3\}$, we get

$$M = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix}.$$

Now it is easy to show $M \in Q$. Hence $A \in Q$. Let B be a PPT of A with respect to $\alpha = \{1, 2\}$. Then, $B \in Q$ but $LCP(q, B)$ is not processable by Lemke's algorithm.

4. Conclusion

In this article we discuss about the processability of Lemke's algorithm. We show that a subclass of column sufficient matrix is processable by Lemke's algorithm. We also prove some results related to Z -matrices in the context of linear complementarity problem. We give an example of an almost P_0 -matrix and show that at least one PPT of this matrix is Z -matrix. Finally we show that not all Q -matrices are processable by Lemke's algorithm by giving an example of Q -matrix. However the complete characterization of the class of Q -matrices which are processable by Lemke's algorithm remains an interesting open problem.

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