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Analysis of a recent heat conduction model with a delay for thermoelastic interactions in an unbounded medium with a spherical cavity

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Abstract

In this work, we study the thermoelastic interactions in an unbounded medium with a spherical cavity in the context of a very recently proposed heat conduction model established by Quintanilla (2011). This model is a reformulation of three-phase-lag conduction model and is an alternative heat conduction theory with a single delay term. We make an attempt to study the thermoelastic interactions in an isotropic elastic medium with a spherical cavity subjected to three types of thermal and mechanical loads in the contexts of two versions of this new model. Analytical solutions for the distributions of the field variables are found out with the help of the integral transform technique. A detailed analysis of analytical results is provided by short-time approximation concept. Further, the numerical solutions of the problems are obtained by applying numerical inversion of Laplace transform. We observe significant variations in the analytical results predicted by different heat conduction models. Numerical values of field variables are also observed to show significantly different results for a particular material. Several important points related to the prediction of the new model are highlighted.

Keywords: Generalized thermoelasticity; Non-Fourier heat conduction model; Quintanilla model; Spherical Cavity; Ramp type heating; Thermal shock

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1. Introduction

It has been realized in recent years that Fourier law is applicable only to the problems that involve large spatial dimension and when the focus is on long time behaviour. However, it yields unacceptable results in situations involving temperature near absolute zero, extreme thermal gradients, high heat flux conduction and short time behaviour, such as laser-material interactions. Hence, intense efforts are put forth since 1950s to better understand the limitations of Fourier law and for more accurate predictions of temperature. Accordingly, some non-Fourier heat conduction models are proposed and they have become the centre of active research for last few decades. Classical coupled dynamic thermoelasticity theory is based on Fourier law of heat conduction and suffers from the drawback of infinite speed of thermal signal. Generalized coupled dynamical thermoelasticity theory is free from this so called classical paradox. Furthermore, besides the paradox of infinite propagation speed, the classical coupled dynamic thermoelasticity theory predicts poor description of a solid's response to a fast transient loading (say, due to short laser pulses) and at low temperatures. During last few decades, such drawbacks have been addressed by researchers and the generalized theories are advocated accordingly. A systematic development in the heat conduction theory and thermoelasticity theory can be found out in the review articles/books (Ignaczak, 1989; Joseph and Preziosi, 1989; Chandrasekhariah, 1998; Hetnarski and Ignaczak, 1999; Ignaczak and Starzewski, 2010; Straughan, 2011). Here, we also find studies concerning applicability of these generalized theories.

The first pioneering contribution to this field is provided by Lord and Shulman (1967) who derived a generalized thermoelasticity theory with the modification of governing equations of classical coupled theory with the introduction of one relaxation time. The heat conduction law is therefore replaced with the Cattaneo-Vernotte heat conduction model. This theory overcomes the paradox of infinite propagation of thermal signal. Subsequently, another modification of classical thermoelasticity theory was reported by Green and Lindsay (1972) in an alternative way. This theory is based on a generalized dissipation inequality and introduces two relaxation times in the description of thermoelastic process. Later on, Green and Naghdi (1993; 1992; 1993) proposed a completely new thermoelasticity theory by developing an alternative formulation of heat propagation. They incorporated the approach based on Fourier law (refereed as type-I), the theory without energy dissipation (type-II), and theory with energy dissipation (type-III). The type-III model is more general one. In the theories by Green and Naghdi, the gradient of thermal displacement, ν is considered as a new constitutive variable. The thermal displacement, ν satisfies $\dot{\nu} = \theta$, where, θ is the temperature. The heat conduction law for GN-III model is given by

$$q = -[k\nabla\theta + k^{*}\nabla\nu],$$

where, q is the heat flux vector, k is thermal conductivity and k^{*}, a material parameter called as conductivity rate of the material, is also newly introduced in this theory and is considered as the characteristic of the theory. In an attempt to model ultra fast processes of heat transport, Tzou (1995) proposed a dual-phase-lag heat conduction model (DPLM) in which heat transport equation relating q at a point r to $\nabla\theta$ at r is introduced in the form

$$q(r, t + \tau_q) = -\left[k\nabla\theta(r, t + \tau_\theta)\right],$$

where, τ_q and τ_θ stands for the phase-lag of the heat flux and temperature gradient, respectively. The phase-lags are assumed to be positive and they are intrinsic properties of the medium (Tzou, 1995). Chandrasekharaiah (1998) extended this DPL model of heat conduction to a generalized thermoelasticity theory. Subsequently, Roychoudhari (2007) established the three phase-lag (TPL) thermoelasticity theory in which a modified constitutive realtion for heat conduction is taken by introducing the phase-lag of the heat flux, temperature gradient and thermal displacement gradient in heat conduction equation proposed in type-III model by Green and Naghdi. The constitutive equation for the heat flux vector in the three-phase-lag theory (2007) is proposed to be

$$q(r, t + \tau_q) = -\left[k\nabla\theta(r, t + \tau_\theta) + k^*\nabla\nu(r, t + \tau_\nu)\right].$$

Here, τ_ν is the additional phase-lag known as phase-lag of thermal displacement gradient. This phase-lag is understood as a delay in terms of the micro-structure of the material.

In recent years, the above referred thermoelasticity theories have drawn the significant attention of researchers and they find distinct attributes of various models. A big interest has been evolved to analyze the different Taylor approximations to these heat transport equations (Horgon, 2005; Kumar and Mukhopadhyay, 2010; Mukhopadhayay et al., 2010; Quintanilla, 2002; Quintanilla, 2003; Quintanilla and Racke, 2006; Quintanilla and Racke, 2007; Quintanilla and Racke, 2008). Some critical analysis on these models are also discussed. For example, critical analysis on dual phase-lag and three phase-lag heat conduction models is reported by Dreher et al. (2009). They have addressed the point that when we combine this constitutive equation with energy equation

$$-\nabla q(x, t) = c\dot{\theta}(x, t),$$

we find that a sequence of eigenvalues in the point spectrum always exists and the real parts of eigen values tend to infinity (Dreher et al., 2009). This signify the ill-posedness of the problem in the Hadamard sense. We can not obtain continuous dependence of the solution with respect to initial parameters. The mathematical consequences of such theories do not agree with what one would expect a priori (Dreher et al., 2009, see also Quintanilla, 2011). Recently, Quintanilla (2011) has reformulated the three-phase-lag heat conduction model in an alternative way and described the stability and spatial behavior of the new proposed model. By defining $\tau_\nu < \tau_q = \tau_\theta$ and $\tau = \tau_q - \tau_\nu$, and adjoining three-phase-lag equation with energy equation they obtain the field equation with a single delay term in the form

$$c\ddot{\nu}(t) = k\triangle\theta(t) + k^*\triangle\nu(t - \tau).$$

This newly developed model has been studied in a detailed way by Leseduarte and Quintanilla (2013). The spatial behavior of the solutions for this theory are explored by them. They found that a Phragmen- Lindelof type alternative can be obtained and it has been shown that the solutions either decay in an exponential way or blow-up at infinity in an exponential way. The obtained results are extended to a thermoelasticity theory by considering the Taylor series approximation of the equation of heat conduction to the delay term and Phragmen-Lindelof type alternative is obtained for the forward and backward in time equations. Stability of the model is also discussed. Recently, Kumari and Mukhopadhyay (2016) established uniqueness of solution of this theory (Quintanilla, 2011; Leseduarte and Quintanilla, 2013). They also discussed variational principle and reciprocity theorem of this model. Kumar and Mukhopadhyay (2016) investigated a problem of thermoelastic interactions on this theory in which state-space approach is used to formulate the problem and the formulation is then applied to a problem of an isotropic elastic half-space with its plane boundary subjected to sudden increase in temperature and zero stress.

In the present problem, we consider the newly developed thermoelasticity theory based on the heat conduction model with a single delay term (Quintanilla, 2011; Leseduarte and Quintanilla, 2013). We make an attempt to study the thermoelastic interactions in an isotropic elastic medium with a spherical cavity subjected to three different types of thermal and mechanical boundary conditions. The Laplace transform technique is applied to obtain the solution of the problem. In view of the fact that delay time parameter is of small value, we attempted to obtain the short-time approximated solution for the field variables. We provide a detailed analysis of analytical results. An attempt has also been made to illustrate the problem and numerical values of field variables are obtained for a particular material. We analyze the results with different graphs and compare these results with GN-III model. It is believed that the analyses of the present study would be helpful to understand the basic features of this new heat conduction model.

2. Governing Equations

By following Quintanilla (2011) and Leseduarte and Qunitanilla (2013), we consider the basic governing equations in the absence of body forces and heat sources for an isotropic elastic medium in indicial notation as follows:

Stress-strain-temperature relation:

$$\sigma_{ij} = \lambda e \delta_{ij} + 2\mu e_{ij} - \gamma\theta\delta_{ij}, \tag{1}$$

Stress equation of motion:

$$\sigma_{ij,j} = \rho \frac{\partial^2 u_i}{\partial t^2}, \tag{2}$$

Heat conduction equation:

$$\{k\frac{\partial}{\partial t} + k^*(1 + \tau\frac{\partial}{\partial t} + \frac{1}{2}\tau^2\xi\frac{\partial^2}{\partial t^2})\}\theta_{,ii} = \gamma\theta_0\frac{\partial^2 e}{\partial t^2} + \rho c_v\frac{\partial^2\theta}{\partial t^2}, \tag{3}$$

where, $e = e_{ii}$.

Here, λ and μ are Lame elastic constants, u_i is the component of displacement vector, θ is the temperature above reference temperature θ_0. σ_{ij} and e_{ij} are the components of stress tensor and strain tensor, respectively. ρ is the mass density, c_v is the specific heat at constant strain, $\gamma = (3\lambda + 2\mu)\alpha_t$, where, α_t is the coefficient of linear thermal expansion, k is the thermal conductivity and k^* is the thermal conductivity rate of the medium, and τ is the delay time due to new model (Leseduarte and Qunitanilla, 2013). The subscripts followed by comma denote the derivatives w.r.t space co-ordinate x_i. The parameter ξ has been used here to formulate the problem under two different models of heat conduction in a unified way. We obtain two different versions of heat conduction equation by Leseduarte and Quintanilla (2013) by assuming $\xi = 1$ and $\xi = 0$, whereas in the second case ($i.e.$,when $\xi = 0$), we neglect the second order effect in the Taylor series approximation of the equation of heat conduction to the delay term. The case when we assume $\tau = 0$ represents the case in the context of GN-III model.

Hence, we study our problem by considering equations (1), (2) and (3) to analyze the results under two different cases as follows:

(1) **New model 1** :$\tau \neq 0, \xi = 1$,
(2) **New model 2** :$\tau \neq 0, \xi = 0$.

3. Problem Formulation

We consider an infinitely extended homogeneous isotropic elastic medium with a spherical cavity of radius a. The center of the cavity is taken to be the origin of the spherical polar co-ordinate system (r,θ,ϕ). By assuming spherical symmetry in our problem, the displacement and temperature components are taken to be function of r and t only. The non-zero strain component are therefore given by

$$e_{rr} = \frac{\partial u}{\partial r},\ e_{\theta\theta} = e_{\phi\phi} = \frac{u}{r}.$$

Then, non-zero stress components are obtained as

$$\sigma_{rr} = \lambda(\frac{\partial u}{\partial r} + \frac{2u}{r}) + 2\mu\frac{\partial u}{\partial r} - \gamma\theta, \tag{4}$$

$$\sigma_{\theta\theta} = \sigma_{\phi\phi} = \lambda(\frac{\partial u}{\partial r} + \frac{2u}{r}) + 2\mu\frac{\mu}{r} - \gamma\theta. \tag{5}$$

Hence, the equation of motion (1) reduces to the form

$$\rho\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial r}\sigma_{rr} + \frac{2}{r}(\sigma_{rr} - \sigma_{\theta\theta}). \tag{6}$$

Now, for simplicity we introduce the following dimensionless variables and quantities:

$$r' = \frac{r}{a},\ t' = \frac{c_1}{a}t,\ \theta' = \frac{\theta}{\theta_0},\ u' = \frac{u}{a},\ \tau' = \frac{c_1}{a}\tau,\ a_0 = \frac{ak^*}{kc_1},\ a_1 = \frac{\gamma\theta_0}{\lambda+\mu},$$
$$a_2 = \frac{a\rho c_v c_1}{k},\ a_3 = \frac{a\gamma c_1}{k},\ \sigma'_{rr} = \frac{\sigma_{rr}}{\lambda+2\mu},\ \sigma'_{\phi\phi} = \frac{\sigma_{\phi\phi}}{\lambda+2\mu},\ \text{and}\ \lambda_1 = \frac{\lambda}{\lambda+2\mu}.$$

Then, (4), (5) and (6) reduce to

$$\sigma_{rr} = \frac{\partial u}{\partial r} + 2\lambda_1\frac{u}{r} - a_1\theta, \tag{7}$$

$$\sigma_{\phi\phi} = \lambda_1\frac{\partial u}{\partial r} + (\lambda_1 + 1)\frac{u}{r}, \tag{8}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial e}{\partial r} - a_1\frac{\partial \theta}{\partial r}. \tag{9}$$

In above equations, we omit the primes for simplicity.

Further, the equation of heat conduction is simplified to the dimensionless form as

$$\{\frac{\partial}{\partial t}+a_0(1+\tau\frac{\partial}{\partial t}+\frac{1}{2}\tau^2\xi\frac{\partial^2}{\partial t^2})\}\nabla^2\theta=a_3\frac{\partial^2 e}{\partial t^2}+a_2\frac{\partial^2\theta}{\partial t^2}. \tag{10}$$

We consider that the medium is at rest and undisturbed at the beginning and the initial conditions are given by

$$\begin{aligned}u(r,t)\mid_{t=0}&=\frac{\partial}{\partial t}u(r,t)\mid_{t=0}=0,\\ \theta(r,t)\mid_{t=0}&=\frac{\partial}{\partial t}\theta(r,t)\mid_{t=0}=0,\\ \sigma_{rr}(r,t)\mid_{t=0}&=0.\end{aligned}$$

4. Solution of the problem

We introduce the Laplace transform defined by

$$\bar{f}(r,p)=\int_0^\infty \mathrm{e}^{-pt}f(r,t)dt,$$

where, p is a Laplace transform parameter.

Therefore, by applying the Laplace transform on time to equations (7)-(10) with respect to time t, we get following equations:

$$\bar{\sigma}_{rr}=\bar{e}-2(1-\lambda_1)\frac{\bar{u}}{r}-a_1\bar{\theta}, \tag{11}$$

$$\bar{\sigma}_{\phi\phi}=\lambda_1\bar{e}+(1-\lambda_1)\frac{\bar{u}}{r}-a_1\bar{\theta}, \tag{12}$$

$$p^2\bar{u}=\frac{\partial\bar{e}}{\partial r}-a_1\frac{\partial\bar{\theta}}{\partial r}, \tag{13}$$

$$\{p+a_0(1+\tau p+\frac{1}{2}\xi\tau^2p^2)\}\nabla^2\bar{\theta}=a_3p^2\bar{e}+a_2p^2\bar{\theta}. \tag{14}$$

Decoupling equations (13) and (14) and solving them, we get the general solutions for $\bar{e}$ and $\bar{\theta}$ bounded at infinity as

$$\bar{e}=\frac{1}{\sqrt{r}}[X_1K_{\frac{1}{2}}(m_1r)+X_2K_{\frac{1}{2}}(m_2r)], \tag{15}$$

$$\bar{\theta}=\frac{1}{\sqrt{r}}[Y_1K_{\frac{1}{2}}(m_1r)+Y_2K_{\frac{1}{2}}(m_2r)]. \tag{16}$$

Here, X_i, Y_i, $i=1,2$ are the arbitrary constants independent of r, $K_{\frac{1}{2}}(m_ir)$ are modified Bessel functions of order half and m_1, m_2 satisfy the following equation:

$$\begin{aligned}&\{p+a_0(1+\tau p+\frac{1}{2}\xi\tau^2p^2)\}m^4\\ &-\{p^3+\epsilon p^2+a_2p^2+a_0p^2(1+\tau p+\frac{1}{2}\xi\tau^2p^2)\}m^2+a_2p^4=0.\end{aligned} \tag{17}$$

Here, $\epsilon = \frac{\gamma^2\theta_0}{\rho^2 c_v c_1^2}$ is the thermoelastic coupling constant.

Now, by using (13), (15), and (16), we find the relation between X_i and Y_i as

$$Y_i = F_i X_{i,} \quad i = 1, 2, \tag{18}$$

where,

$$F_i = \frac{m_i^2 - p^2}{a_1 m_i^2}, \quad i = 1, 2.$$

We apply the following relation:

$$\frac{d}{dr} K_{\frac{1}{2}}(m_i r) = \frac{1}{2r} K_{\frac{1}{2}}(m_i r) - m_i K_{\frac{3}{2}}(m_i r) \quad \text{for } i = 1, 2.$$

in order to find the solutions for displacement and stresses from equations (7)-(10), (15) and (16) as

$$\bar{u} = -\frac{1}{\sqrt{r}}[\frac{1}{m_1} X_1 K_{\frac{3}{2}}(m_1 r) + \frac{1}{m_2} X_2 K_{\frac{3}{2}}(m_2 r)], \tag{19}$$

$$\bar{\sigma}_{rr} = \frac{1}{\sqrt{r}}[X_1 \sigma_{r1} + X_2 \sigma_{r2}], \tag{20}$$

$$\bar{\sigma}_{\phi\phi} = \frac{1}{\sqrt{r}}[X_1 \sigma_{\phi 1} + X_2 \sigma_{\phi 2}], \tag{21}$$

where,

$$\sigma_{ri} = [1 - a_1 F_i] K_{\frac{1}{2}}(m_i r) + \frac{2(1-\lambda_1)}{m_i r} K_{\frac{3}{2}}(m_i r), \quad i = 1, 2,$$

$$\sigma_{\phi i} = [\lambda_1 - a_1 F_i] K_{\frac{1}{2}}(m_i r) - \frac{(1-\lambda_1)}{m_i r} K_{\frac{3}{2}}(m_i r), \quad i = 1, 2.$$

We can obtain the values of X_1 and X_2 with the help of boundary conditions.

4.1. Applications:

Now, we apply above formulation to investigate the nature of solutions in different cases of thermoelastic interactions. Hence, we consider three problems that correspond to three types of thermal and mechanical boundary conditions as follows:

Problem-1: Ramp-type varying temperature and zero stress on the boundary of the spherical cavity of the medium

We consider a homogeneous isotropic thermoelastic solid with homogeneous initial conditions.The surface of the cavity, $r = 1$ is considered to be stress free and is affected by ramp-type heating which depends on the time t. Hence, we assume boundary conditions as:

$$\sigma_{rr}\mid_{r=1}=0, \tag{22}$$

$$\theta\mid_{r=1}=\begin{cases} 0 & t<0,\\ T_0\frac{t}{t_0} & 0<t<t_0,\\ T_0 & t>t_0.\end{cases} \tag{23}$$

where, t_0 indicates the length of time to rise the heat and T_0 is a constant. This means that the boundary of the surface of cavity, which is initially at rest is suddenly raised to an increased temperature equal to the function $F(t)=T_0\frac{t}{t_0}$ and after the instance $t=t_0$ coming, we let the temperature at constant value T_0 maintain from then on.

The Laplace transform of equations (22) and (23) yield

$$\bar{\sigma}_{rr}\mid_{r=1}=0, \tag{24}$$

$$\bar{\theta}\mid_{r=1}=\frac{T_0(1-e^{-pt_0})}{t_0p^2}. \tag{25}$$

Therefore, with the help of boundary conditions (24) and (25), we find the constants X_1 and X_2 involved in solutions of different field variables in Laplace transform domain as:

$$X_1=-\frac{\sigma^0_{r2}T_0(1-e^{-pt_0})}{t_0p^2[\sigma^0_{r1}F_2K_{\frac{1}{2}}(m_2)-\sigma^0_{r2}F_1K_{\frac{1}{2}}(m_1)]},$$

$$X_2=\frac{\sigma^0_{r1}T_0(1-e^{-pt_0})}{t_0p^2[\sigma^0_{r1}F_2K_{\frac{1}{2}}(m_2)-\sigma^0_{r2}F_1K_{\frac{1}{2}}(m_1)]},$$

where,

$$\sigma^0_{ri}=[1-a_1F_i]K_{\frac{1}{2}}(m_i)+\frac{2(1-\lambda_1)}{m_i}K_{\frac{3}{2}}(m_i),\quad i=1,2.$$

Problem-2: Unit step increase in temperature and zero stress on the boundary of the cavity

Here, we assume that the surface of cavity, $r=1$ is stress free and is subjected to a unit step increase in temperature, i.e., we consider

$$\sigma_{rr}\mid_{r=1}=0, \tag{26}$$

$$\theta\mid_{r=1}=T^*H(t), \tag{27}$$

where, T^* is a constant temperature and $H(t)$ is the Heaviside unit step function.

The Laplace transform of equations (26) and (27) are given by

$$\bar{\sigma}_{rr}\mid_{r=1}=0, \tag{28}$$

$$\bar{\theta}\mid_{r=1}=\frac{T^*}{p}. \tag{29}$$

Similarly, with the help of (28) and (29), the constants X_1, X_2 are obtained as:

$$X_1 = -\frac{\sigma^0_{r2} T^*}{p[\sigma^0_{r1} F_2 K_{\frac{1}{2}}(m_2) - \sigma^0_{r2} F_1 K_{\frac{1}{2}}(m_1)]},$$
$$X_2 = \frac{\sigma^0_{r1} T^*}{p[\sigma^0_{r1} F_2 K_{\frac{1}{2}}(m_2) - \sigma^0_{r2} F_1 K_{\frac{1}{2}}(m_1)]}.$$

Problem-3: Normal load on the boundary of the spherical cavity

In this case, we assume that the surface of the cavity is kept at the constant reference temperature, and it is subjected to a normal load. Therefore, the boundary conditions for this problem are of the forms

$$\sigma_{rr} \mid_{r=1} = -\sigma^* H(t), \tag{30}$$
$$\theta \mid_{r=1} = 0. \tag{31}$$

The Laplace transform of (30) and (31) yields

$$\bar{\sigma}_{rr} \mid_{r=1} = -\frac{\sigma^*}{p}, \tag{32}$$
$$\bar{\theta} \mid_{r=1} = 0. \tag{33}$$

In this case, the constants X_1, X_2 are found to be of the forms

$$X_1 = \frac{F_2 K_{\frac{1}{2}}(m_2)\sigma^*}{p[\sigma^0_{r2} F_1 K_{\frac{1}{2}}(m_1) - \sigma^0_{r1} F_2 K_{\frac{1}{2}}(m_2)]},$$
$$X_2 = -\frac{F_1 K_{\frac{1}{2}}(m_1)\sigma^*}{p[\sigma^0_{r2} F_1 K_{\frac{1}{2}}(m_1) - \sigma^0_{r1} F_2 K_{\frac{1}{2}}(m_2)]}.$$

5. Short-time approximation

We will find the solutions of distributions of displacement, temperature and stresses in the physical domain (r, t) by inverting the expressions for $\bar{u}$, $\bar{\theta}$, $\bar{\sigma}_{rr}$, $\bar{\sigma}_{\phi\phi}$ obtained in previous section by using the Laplace inverse transforms. However, these expressions involve complicated functions of Laplace transform parameter, p. Hence, the closed form inversion of Laplace transforms for any values of p is formidable task. Therefore, in this section, we attempt to get the short-time approximated solutions of the field variables in the time domain for small values of time, i.e., for large values of p.

Firstly, with the help of Maclaurin's series expansion and neglecting the higher powers of small terms, we get the roots m_1, m_2 of equation (17) under different models as follows:

For New Model -I:

$$m_1 = b_1^1 p + b_2^1 \frac{1}{p}, \tag{34}$$

$$m_2 = b_3^1 - b_4^1 \frac{1}{p}, \tag{35}$$

For New Model -II:

$$m_1 = b_1^2 p + b_2^2, \tag{36}$$

$$m_2 = b_3^2 \sqrt{p} - b_4^2 \frac{1}{\sqrt{p}}, \tag{37}$$

where,

$$b_1^1 = 1,\; b_2^1 = \frac{A_2 - A_0 - a_2}{2A_3},\; b_3^1 = \sqrt{\frac{a_2}{A_3}},\; b_4^1 = \frac{A_1\sqrt{a_2}}{2(A_3)^{\frac{3}{2}}},$$

$$b_1^2 = 1,\; b_2^2 = \frac{A_2 - A_0 - a_2}{2A_1},\; b_3^2 = \sqrt{\frac{a_2}{A_1}},\; b_4^2 = \frac{\sqrt{a_2}}{2(A_1)^{\frac{3}{2}}}(A_0 + \epsilon),$$

$$A_0 = a_0,\; A_1 = 1 + a_0\tau,\; A_2 = \epsilon + a_0 + a_2,\; A_3 = \frac{1}{2}a_0\tau^2.$$

Now, by substituting m_1 and m_2 from equations (34)-(37) into solutions given by equations (16) and (19)-(21) and using the expression

$$K_\nu(z) \approx \sqrt{\frac{\pi}{2z}} e^{-z} \left[1 + \frac{4\nu^2 - 1}{8z} + \frac{(4\nu^2 - 1)(4\nu^2 - 9)}{2(8z)^2} + ...\right],$$

we get the short-time approximated solutions for the distributions of displacement, temperature and stresses in the Laplace transform domain (r, p) for different problems 1-3 under New model-I and New model-II as follows:

Case of prolem-1:

In the context of New model-I:

$$\overline{u}\,(r,p) \simeq \frac{T_0 a_1}{r t_0} \sum_{i=1}^{2} e^{-m_i(r-1)}(1 - e^{-pt_0}) \left[\frac{B_{i1}^1}{p^{i+2}} + \frac{B_{i2}^1}{p^{i+3}}\right], \tag{38}$$

$$\overline{\theta}\,(r,p) \simeq \frac{T_0}{r t_0} \sum_{i=1}^{2} e^{-m_i(r-1)}(1 - e^{-pt_0}) \left[\frac{C_{i1}^1}{p^{6-2i}} + \frac{C_{i2}^1}{p^{6-i}}\right], \tag{39}$$

$$\overline{\sigma}_{rr}\,(r,p) \simeq \frac{T_0 a_1}{r t_0} \sum_{i=1}^{2} e^{-m_i(r-1)}(1 - e^{-pt_0}) \left[\frac{D_{i1}^1}{p^{2}} + \frac{D_{i2}^1}{p^{i+2}}\right], \tag{40}$$

$$\overline{\sigma}_{\phi\phi}\,(r,p) \simeq \frac{T_0 a_1}{r t_0} \sum_{i=1}^{2} e^{-m_i(r-1)}(1 - e^{-pt_0}) \left[\frac{E_{i1}^1}{p^{2}} + \frac{E_{i2}^1}{p^{i+2}}\right]. \tag{41}$$

In the context of New model-II:

$$\overline{u}\,(r,p) \simeq \frac{T_0 a_1}{r t_0} \sum_{i=1}^{2} e^{-m_i(r-1)}(1-e^{-pt_0}) \left[\frac{B_{i1}^2}{p^{\frac{i+5}{2}}} + \frac{B_{i2}^2}{p^4}\right], \tag{42}$$

$$\overline{\theta}\,(r,p) \simeq \frac{T_0}{r t_0} \sum_{i=1}^{2} e^{-m_i(r-1)}(1-e^{-pt_0}) \left[\frac{C_{i1}^2}{p^{4-i}} - \frac{C_{i2}^2}{p^{5-i}}\right], \tag{43}$$

$$\overline{\sigma}_{rr}\,(r,p) \simeq \frac{T_0 a_1}{r t_0} \sum_{i=1}^{2} e^{-m_i(r-1)}(1-e^{-pt_0}) \left[\frac{D_{i1}^2}{p^2} + \frac{D_{i2}^2}{p^3}\right], \tag{44}$$

$$\overline{\sigma}_{\phi\phi}\,(r,p) \simeq \frac{T_0 a_1}{r t_0} \sum_{i=1}^{2} e^{-m_i(r-1)}(1-e^{-pt_0}) \left[\frac{E_{i1}^2}{p^2} + \frac{E_{i2}^2}{p^3}\right]. \tag{45}$$

Case of problem-2:

In the context of New model-I:

$$\overline{u}\,(r,p) \simeq \frac{T^* a_1}{r} \sum_{i=1}^{2} e^{-m_i(r-1)} \left[\frac{B_{i1}^{1'}}{p^{i+1}} + \frac{B_{i2}^{1'}}{p^{i+2}}\right], \tag{46}$$

$$\overline{\theta}\,(r,p) \simeq \frac{T^*}{r} \sum_{i=1}^{2} e^{-m_i(r-1)} \left[\frac{C_{i1}^{1'}}{p^{5-2i}} + \frac{C_{i2}^{1'}}{p^{5-i}}\right], \tag{47}$$

$$\overline{\sigma}_{rr}\,(r,p) \simeq \frac{T^* a_1}{r} \sum_{i=1}^{2} e^{-m_i(r-1)} \left[\frac{D_{i1}^{1'}}{p} + \frac{D_{i2}^{'}}{p^{i+1}}\right], \tag{48}$$

$$\overline{\sigma}_{\phi\phi}\,(r,p) \simeq \frac{T^* a_1}{r} \sum_{i=1}^{2} e^{-m_i(r-1)} \left[\frac{E_{i1}^{1'}}{p} + \frac{E_{i2}^{1'}}{p^{i+1}}\right]. \tag{49}$$

In the context of New model-II:

$$\overline{u}\,(r,p) \simeq \frac{T^* a_1}{r} \sum_{i=1}^{2} e^{-m_i(r-1)} \left[\frac{B_{i1}^{2'}}{p^{\frac{i+3}{2}}} + \frac{B_{i2}^{2'}}{p^3}\right], \tag{50}$$

$$\overline{\theta}\,(r,p) \simeq \frac{T^*}{r} \sum_{i=1}^{2} e^{-m_i(r-1)} \left[\frac{C_{i1}^{2'}}{p^{3-i}} + \frac{C_{i2}^{2'}}{p^{4-i}}\right], \tag{51}$$

$$\overline{\sigma}_{rr}\,(r,p) \simeq \frac{T^* a_1}{r} \sum_{i=1}^{2} e^{-m_i(r-1)} \left[\frac{D_{i1}^{2'}}{p} + \frac{D_{i2}^{2'}}{p^2}\right], \tag{52}$$

$$\overline{\sigma}_{\phi\phi}\,(r,p) \simeq \frac{T^* a_1}{r} \sum_{i=1}^{2} e^{-m_i(r-1)} \left[\frac{E_{i1}^{2'}}{p} + \frac{E_{i2}^{2'}}{p^2}\right]. \tag{53}$$

Case of problem-3:

In the context of New model-I:

$$\overline{u}\left(r,p\right) \simeq \frac{\sigma^*}{r}\sum_{i=1}^{2} e^{-m_i(r-1)}\left[\frac{B_{i1}^{1''}}{p^{3i-1}} + \frac{B_{i2}^{1''}}{p^{3i}}\right], \tag{54}$$

$$\overline{\theta}\left(r,p\right) \simeq \frac{\sigma^*}{ra_1}\sum_{i=1}^{2} e^{-m_i(r-1)}\left[\frac{C_{i1}^{1''}}{p^{3}} + \frac{C_{i2}^{1''}}{p^{4}}\right], \tag{55}$$

$$\overline{\sigma}_{rr}\left(r,p\right) \simeq \frac{\sigma^*}{r}\sum_{i=1}^{2} e^{-m_i(r-1)}\left[\frac{D_{i1}^{1''}}{p^{2i-1}} + \frac{D_{i2}^{1''}}{p^{2i}}\right], \tag{56}$$

$$\overline{\sigma}_{\phi\phi}\left(r,p\right) \simeq \frac{\sigma^*}{r}\sum_{i=1}^{2} e^{-m_i(r-1)}\left[\frac{E_{i1}^{1''}}{p^{2i-1}} + \frac{E_{i2}^{1''}}{p^{2i}}\right]. \tag{57}$$

In the context of New model-II:

$$\overline{u}\left(r,p\right) \simeq \frac{\sigma^*}{r}\sum_{i=1}^{2} e^{-m_i(r-1)}\left[\frac{B_{i1}^{2''}}{p^{\frac{3i+1}{2}}} + \frac{B_{i2}^{2''}}{p^{i+2}}\right], \tag{58}$$

$$\overline{\theta}\left(r,p\right) \simeq \frac{\sigma^*}{ra_1}\sum_{i=1}^{2} e^{-m_i(r-1)}\left[\frac{C_{i1}^{2''}}{p^{2}} + \frac{C_{i2}^{2''}}{p^{3}}\right], \tag{59}$$

$$\overline{\sigma}_{rr}\left(r,p\right) \simeq \frac{\sigma^*}{r}\sum_{i=1}^{2} e^{-m_i(r-1)}\left[\frac{D_{i1}^{2''}}{p^{i}} + \frac{D_{i2}^{2''}}{p^{i+1}}\right], \tag{60}$$

$$\overline{\sigma}_{\phi\phi}\left(r,p\right) \simeq \frac{\sigma^*}{r}\sum_{i=1}^{2} e^{-m_i(r-1)}\left[\frac{E_{i1}^{2''}}{p^{i}} + \frac{E_{i2}^{2''}}{p^{i+1}}\right]. \tag{61}$$

where, different notations used in solutions given by (38)-(61) are given by following expressions:

$B_{11}^1 = B_{11}^{1'} = D_{21}^1 = D_{21}^{1'} = D_{11}^{1''} = E_{21}^1 = E_{21}^{1'} = -1,$

$C_{21}^1 = C_{21}^{1'} = D_{11}^1 = D_{11}^{1'} = B_{11}^{1''} = 1,$

$B_{12}^1 = B_{12}^{1'} = -\frac{1-2r+2r\lambda_1}{r}$, $B_{21}^1 = B_{21}^{1'} = \frac{1+b_3^1 r}{r}$, $B_{12}^{1''} = \frac{1-2r+2r\lambda_1}{r},$

$B_{22}^1 = B_{22}^{1'} = -b_4^1, C_{11}^1 = C_{11}^{1'} = C_{21}^{1''} = C_{22}^{1'} = D_{21}^{1''} = 2b_2^1,$

$C_{12}^1 = C_{12}^{1'} = C_{22}^{1''} = D_{22}^{1''} = -4b_2^1(1-\lambda_1),$

$E_{22}^{1''} = C_{12}^{1''} = 4b_2^1(1-\lambda_1)$, $E_{21}^{1''} = C_{11}^{1''} = C_{22}^1 = -2b_2^1,$

$D_{12}^1 = D_{12}^{1'} = D_{12}^{1''} = \frac{2(1-\lambda_1)(r-1)}{r}$, $E_{12}^{1''} = -\frac{2(1+2r\lambda_1)(\lambda_1-1)}{r}$

$D_{22}^1 = D_{22}^{1'} = -\frac{2+2b_3^1 r-2b_2^1 r^2+(b_3^1)^2 r^2-2\lambda_1-2b_3^1 r\lambda_1}{r^2}$, $E_{11}^1 = \lambda_1,$

$E_{12}^{1'} = \frac{\lambda_1-2r\lambda_1+2r\lambda_1^2-1}{r}$, $E_{22}^1 = E_{22}^{1'} = -\frac{b_3^1 r^2\lambda_1+b_3^1 r\lambda_1+\lambda_1-2b_2^1 r-b_3^1 r-1}{r^2},$

$B_{21}^{1''} = \frac{2b_2^1(1+b_3^1 r)}{r}$, $B_{22}^{1''} = \frac{2b_2^1(-2-2b_3^1 r - b_4^1 r + 2\lambda_1 + 2b_3^1 r\lambda_1)}{r}$, $E_{11}^{''1} = -\lambda_1$,

$B_{11}^2 = B_{11}^{2'} = B_{11}^{2''} = D_{21}^2 = D_{11}^{2''} = E_{21}^2 = E_{21}^{2'} = D_{21}^{2'} = -1$,

$D_{11}^2 = D_{11}^{2'} = C_{21}^2 = C_{21}^{2'} = 1$, $B_{12}^2 = B_{12}^{2'} = \frac{-(1-2r-rb_2^2+rb_3^2+2\lambda_1)}{r}$,

$B_{21}^2 = B_{21}^{2'} = b_3^2$, $B_{22}^2 = B_{22}^{2'} = \frac{1}{r}$,

$C_{12}^2 = -b_2^2(-4-3b_2^2+2(b_3^2)^2+4\lambda_1)$, $E_{11}^2 = E_{11}^{2'} = \lambda_1$, $E_{11}^{2''} = -\lambda_1$,

$D_{12}^2 = D_{12}^{2'} = \frac{1}{r}(2-2r-2rb_2^2+(rb_3^2)^2-2\lambda_1+2r\lambda_1)$,

$D_{22}^2 = D_{22}^{2'} = -(-2b_2^2+(b_3^2)^2)$, $C_{22}^2 = C_{22}^{2'} = C_{11}^{2''} = D_{21}^{2''} = E_{21}^{2''} = -2b_2^2$,

$E_{12}^2 = E_{12}^{2'} = \frac{1}{r}(-1-2rb_2^2+\lambda_1-2r\lambda_1+r\lambda_1(b_3^2)^2+2r\lambda_1^2)$, $E_{22}^2 = -\frac{1}{2b_3^2}$,

$C_{12}^{2'} = b_2^2(-4-3b_2^2+(b_3^2)^2+4\lambda_1)$, $E_{22}^{2'} = -4b_2^2b_3^2+(b_3^2)^2\lambda_1$,

$B_{12}^{2''} = -\frac{1}{r}(-1+2r+rb_2^2-2r\lambda_1)$, $B_{21}^{2''} = -2b_2^2b_3^2$, $B_{22}^{2''} = \frac{1}{r}(-2b_2^2)$,

$C_{12}^{2''} = C_{22}^{2''} = b_2^2(4+3b_2^2-4\lambda_1)$, $D_{12}^{2''} = \frac{2}{r}(-1+r+rb_2^2+\lambda_1-r\lambda_1)$,

$D_{22}^{2''} = (4+3b_2^2-2(b_3^2)^2-4\lambda_1)$, $E_{12}^2 = \frac{1}{r}(-1-2rb_2^2+\lambda_1-2r\lambda_1+2r\lambda_1^2)$,

$E_{22}^{2''} = 4+3b_2^2-4\lambda_1-2(b_3^2)^2\lambda_1$.

5.1. Solutions in the physical domain:

Now, we use inversion of the Laplace transform for the above obtained Laplace transforms. For this we use the convolution theorem and the formulae listed below (Oberhettinger,1973).

$$L^{-1}\left[\frac{e^{-\frac{a}{p}}}{p^{\nu+1}}\right] = \left(\frac{t}{a}\right)^{\frac{\nu}{2}} J_\nu(2\sqrt{at}), \quad Re(\nu) > -1,\ a > 0,$$

$$L^{-1}\left[\frac{e^{\frac{a}{p}}}{p^{\nu+1}}\right] = \left(\frac{t}{a}\right)^{\frac{\nu}{2}} I_\nu(2\sqrt{at}), \quad Re(\nu) > -1,\ a > 0,$$

$$L^{-1}\left[\frac{e^{-a\sqrt{p}}}{p^{\frac{\nu}{2}+1}}\right] = (4t)^{\frac{\nu}{2}}\, \boldsymbol{i}^\nu \mathrm{erfc}(\frac{a}{2\sqrt{t}}), \quad \nu = 0,1,2... \quad .$$

Here, J_ν, I_ν are the Bessel and modified Bessel functions of order ν and of the first kind, and $\boldsymbol{i}^n\mathrm{erfc}(x)$ is complementary error function of n^{th} order defined by

$$\boldsymbol{i}^n\mathrm{erfc}(x) = \int_x^\infty \boldsymbol{i}^{n-1}\mathrm{erfc}(u)du, \qquad n = 1,2,3...,$$

with

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty \mathrm{e}^{-\mathrm{t}^2} dt.$$

Now, taking inverse Laplace transforms of equations (38)-(61), we get the approximated analytical solutions for the distributions of displacement, temperature and stresses in space-time domain for all three problems in the following forms:

Problem-1:

Solutions in the context of New model- I:

$$\begin{aligned} u\left(r,t\right) &= \frac{T_0 a_1}{r t_0} \sum_{i=1}^{2} \left[B_{1i}^1 \left(\frac{t_1}{b_2^1 r_1} \right)^{i+1/2} J_{i+1}\left(2\sqrt{b_2^1 r_1 t_1}\right) H(t_1) \right. \\ &\quad \left. + B_{2i}^1 e^{-b_3^1 r_1} \left(\frac{t}{b_4^1 r_1} \right)^{\frac{2+i}{2}} I_{2+i}\left(2\sqrt{b_4^1 r_1 t}\right) \right] \\ &\quad - \frac{T_0 a_1}{r t_0} \sum_{i=1}^{2} \left[B_{1i}^1 \left(\frac{t_2}{b_2^1 r_1} \right)^{i+1/2} J_{i+1}\left(2\sqrt{b_2^1 r_1 t_2}\right) H(t_2) \right. \\ &\quad \left. + B_{2i}^1 e^{-b_3^1 r_1} \left(\frac{t'}{b_4^1 r_1} \right)^{\frac{2+i}{2}} I_{2+i}\left(2\sqrt{b_4^1 r_1 t'}\right) H(t') \right], \end{aligned} \tag{62}$$

$$\begin{aligned} \theta\left(r,t\right) &= \frac{T_0}{r t_0} \sum_{i=1}^{2} \left[C_{1i}^1 \left(\frac{t_1}{b_2^1 r_1} \right)^{(i+2)/2} J_{i+2}\left(2\sqrt{b_2^1 r_1 t_1}\right) H(t_1) \right. \\ &\quad \left. + C_{2i}^1 e^{-b_3^1 r_1} \left(\frac{t}{b_4^1 r_1} \right)^{(2i-1)/2} I_{2i-1}\left(2\sqrt{b_4^1 r_1 t}\right) \right] \\ &\quad - \frac{T_0}{r t_0} \sum_{i=1}^{2} \left[C_{1i}^1 \left(\frac{t_2}{b_2^1 r_1} \right)^{(i+2)/2} J_{i+2}\left(2\sqrt{b_2^1 r_1 t_2}\right) H(t_2) \right. \\ &\quad \left. + C_{2i}^1 e^{-b_3^1 r_1} \left(\frac{t'}{b_4^1 r_1} \right)^{(2i-1)/2} I_{2i-1}\left(2\sqrt{b_4^1 r_1 t'}\right) H(t') \right], \end{aligned} \tag{63}$$

$$\begin{aligned} \sigma_{rr}\left(r,t\right) &= \frac{T_0 a_1}{r t_0} \sum_{i=1}^{2} \left[D_{1i}^1 \left(\frac{t_1}{b_2^1 r_1} \right)^{i/2} J_{i}\left(2\sqrt{b_2^1 r_1 t_2}\right) H(t_1) \right. \\ &\quad \left. + D_{2i}^1 e^{-b_3^1 r_1} \left(\frac{t}{b_4^1 r_1} \right)^{(2i-1)/2} I_{2i-1}\left(2\sqrt{b_4^1 r_1 t}\right) \right] \\ &\quad - \frac{T_0 a_1}{r t_0} \sum_{i=1}^{2} \left[D_{1i}^1 \left(\frac{t_2}{b_2^1 r_1} \right)^{i/2} J_{i}\left(2\sqrt{b_2^1 r_1 t_2}\right) H(t_2) \right. \\ &\quad \left. + D_{2i}^1 e^{-b_3^1 r_1} \left(\frac{t'}{b_4^1 r_1} \right)^{(2i-1)/2} I_{2i-1}\left(2\sqrt{b_4^1 r_1 t'}\right) H(t') \right], \end{aligned} \tag{64}$$

$$\begin{aligned}\sigma_{\phi\phi}(r,t) = \frac{T_0 a_1}{r t_0}\sum_{i=1}^{2}\Bigg[&E_{1i}^{1}\left(\frac{t_1}{b_2^1 r_1}\right)^{i/2} J_i\left(2\sqrt{b_2^1 r_1 t_1}\right)H(t_1)\\ &+E_{2i}^{1} e^{-b_3^1 r_1}\left(\frac{t}{b_4^1 r_1}\right)^{(2i-1)/2} I_{2i-1}\left(2\sqrt{b_4^1 r_1 t}\right)\Bigg]\\ -\frac{T_0 a_1}{r t_0}\sum_{i=1}^{2}\Bigg[&E_{1i}^{1}\left(\frac{t_2}{b_2^1 r_1}\right)^{i/2} J_i\left(2\sqrt{b_2^1 r_1 t_2}\right)H(t_2)\\ &+E_{2i}^{1} e^{-b_3^1 r_1}\left(\frac{t'}{b_4^1 r_1}\right)^{(2i-1)/2} I_{2i-1}\left(2\sqrt{b_4^1 r_1 t'}\right)H(t')\Bigg].\end{aligned} \tag{65}$$

Solutions in the context of New model-II:

$$\begin{aligned}u(r,t) = &\frac{T_0 a_1}{r t_0}\sum_{i=1}^{2} e^{-b_2^2 r_1}\left[B_{1i}^{1}\frac{(t_3)^{i+1}}{4i-2}H(t_3) - B_{1i}^{2}H(t_4)\frac{(t_4)^{i+1}}{4i-2}\right]\\ &+\frac{T_0 a_1}{r t_0}\sum_{i=1}^{2}\left[B_{2i}^{2}(4t)^{\frac{i+2}{2}}\boldsymbol{i}^{i+4}\mathrm{erfc}\Big(\frac{b_3^2 r_1}{2\sqrt{t}}\Big) - B_{2i}^{2}(4t)^{\frac{i+2}{2}}\boldsymbol{i}^{i+4}\mathrm{erfc}\Big(\frac{b_3^2 r_1}{2\sqrt{t'}}\Big)H(t')\right],\end{aligned} \tag{66}$$

$$\begin{aligned}\theta(r,t) = &\frac{T_0}{r t_0}\sum_{i=1}^{2} e^{-b_2^2 r_1}\left[C_{1i}^{2}\frac{(t_3)^{i+1}}{4i-2}H(t_3) - C_{1i}^{2}H(t_4)\frac{(t_4)^{i+1}}{4i-2}\right]\\ &+\frac{T_0}{r t_0}\sum_{i=1}^{2}\left[C_{2i}^{2}(4t)^{i}\boldsymbol{i}^{2i}\mathrm{erfc}\Big(\frac{b_3^2 r_1}{2\sqrt{t}}\Big) - C_{2i}^{2}(4(t-t_0))^{i}\boldsymbol{i}^{2i}\mathrm{erfc}\Big(\frac{b_3^2 r_1}{2\sqrt{t'}}\Big)H(t')\right],\end{aligned} \tag{67}$$

$$\begin{aligned}\sigma_{rr}(r,t) = &\frac{T_0 a_1}{r t_0}\sum_{i=1}^{2} e^{-b_2^2 r_1}\left[D_{1i}^{2}\frac{(t_3)}{i}H(t_3) - D_{1i}^{2}H(t_4)\frac{(t_4)^{i}}{i}\right]\\ &+\frac{T_0 a_1}{r t_0}\sum_{i=1}^{2}\left[D_{2i}^{2}(4t)^{i}\boldsymbol{i}^{2i}\mathrm{erfc}\Big(\frac{b_3^2 r_1}{2\sqrt{t}}\Big) - D_{2i}^{2}(4(t'))^{i}\boldsymbol{i}^{2i}\mathrm{erfc}\Big(\frac{b_3^2 r_1}{2\sqrt{t'}}\Big)H(t')\right],\end{aligned} \tag{68}$$

$$\begin{aligned}\sigma_{\phi\phi}(r,t) = &\frac{T_0 a_1}{r t_0}\sum_{i=1}^{2} e^{-b_2^2 r_1}\left[E_{1i}^{2}\frac{(t_3)^{i}}{i}H(t_3) - E_{1i}^{2}H(t_4)\frac{(t_4)^{i}}{i}\right]\\ &+\frac{T_0 a_1}{r t_0}\sum_{i=1}^{2}\left[E_{2i}^{2}(4t)^{i}\boldsymbol{i}^{2i}\mathrm{erfc}\Big(\frac{b_3^2 r_1}{2\sqrt{t}}\Big) - E_{2i}^{2}(4t')^{i}\boldsymbol{i}^{2i}\mathrm{erfc}\Big(\frac{b_3^2 r_1}{2\sqrt{t'}}\Big)H(t')\right].\end{aligned} \tag{69}$$

Problem-2:

Solutions in the context of New model-I:

$$u\,(r,t) = \frac{T^* a_1}{r}\Bigg[\sum_{i=1}^{2} - B_{1i}^{1'}\left(\frac{t_1}{b_2^1 r_1}\right)^{\frac{i}{2}} J_i\left(2\sqrt{b_2^1 r_1 t_1}\right) H(t_1) + \sum_{i=1}^{2} B_{2i}^{1'} e^{-b_3^1 r_1}\left(\frac{t}{b_4^1 r_1}\right)^{\frac{i+1}{2}} I_{i+1}\left(2\sqrt{b_4^1 r_1 t}\right)\Bigg], \tag{70}$$

$$\theta\,(r,t) = \frac{T^*}{r}\Bigg[\sum_{i=1}^{2} C_{1i}^{1'}\left(\frac{t_1}{b_2^1 r_1}\right)^{(i+1)/2} J_{i+1}\left(2\sqrt{b_2^1 r_1 t_1}\right) H(t_1) + \sum_{i=1}^{2} (-1)^{i+1} C_{2i}^{1'} e^{-b_3^1 r_1}\left(\frac{t}{b_4^1 r_1}\right)^{2i-2/2} I_{2i-2}\left(2\sqrt{b_4^1 r_1 t}\right)\Bigg], \tag{71}$$

$$\sigma_{rr}\,(r,t) = \frac{T^* a_1}{r}\Bigg[\sum_{i=1}^{2} D_{1i}^{1'}\left(\frac{t_1}{b_2^1 r_1}\right)^{(i-1)/2} J_{i-1}\left(2\sqrt{b_2^1 r_1 t_1}\right) H(t_1) + \sum_{i=1}^{2} D_{2i}^{1'} e^{-b_3^1 r_1}\left(\frac{t}{b_4^1 r_1}\right)^{(2i-2)/2} I_{2i-2}\left(2\sqrt{b_4^1 r_1 t}\right)\Bigg], \tag{72}$$

$$\sigma_{\phi\phi}\,(r,t) = \frac{T^* a_1}{r}\Bigg[\sum_{i=1}^{2} E_{1i}^{1'}\left(\frac{t_1}{b_2^1 r_1}\right)^{(i-1)/2} J_{i-1}\left(2\sqrt{b_2^1 r_1 t_1}\right) H(t_1) + \sum_{i=1}^{2} E_{2i}^{1'} e^{-b_3^1 r_1}\left(\frac{t}{b_4^1 r_1}\right)^{(2i-2)/2} I_{2i-2}\left(2\sqrt{b_4^1 r_1 t}\right)\Bigg]. \tag{73}$$

Solutions in the context of New model-II:

$$u\,(r,t) = \frac{T^* a_1}{r}\sum_{i=1}^{2}\left[e^{-b_2^2 r_1}\left\{B_{1i}^{2'}\frac{(t_3)^i}{i} H(t_3)\right\} + \left\{B_{2i}^{2'}(4t)^{\frac{i+2}{2}}\boldsymbol{i}^{i+2}\mathrm{erfc}\left(\frac{b_3^2 r_1}{2\sqrt{t}}\right)\right\}\right], \tag{74}$$

$$\theta\,(r,t) = \frac{T^*}{r}\sum_{i=1}^{2}\left[e^{-b_2^2 r_1}\left\{C_{1i}^{2'}\frac{(t_3)^i}{i} H(t_3)\right\} + \left\{C_{2i}^{2'}(4t)^{i-1}\boldsymbol{i}^{2i-2}\mathrm{erfc}\left(\frac{b_3^2 r_1}{2\sqrt{t}}\right)\right\}\right], \tag{75}$$

$$\sigma_{rr}\,(r,t) = \frac{T^* a_1}{r}\sum_{i=1}^{2}\left[e^{-b_2^2 r_1}\left\{D_{1i}^{2'}(t_3)^{i-1} H(t_3)\right\} + \left\{D_{2i}^{2'}(4t)^{i-1}\boldsymbol{i}^{2i-2}\mathrm{erfc}\left(\frac{b_3^2 r_1}{2\sqrt{t}}\right)\right\}\right], \tag{76}$$

$$\sigma_{\phi\phi}\,(r,t) = \frac{T^* a_1}{r}\sum_{i=1}^{2}\left[e^{-b_2^2 r_1}\left\{E_{1i}^{2'}(t_3)^{i-1} H(t_3)\right\} + \left\{E_{2i}^{2'}(4t)^{i-1}\boldsymbol{i}^{2i-2}\mathrm{erfc}\left(\frac{b_3^2 r_1}{2\sqrt{t}}\right)\right\}\right]. \tag{77}$$

Problem-3:

Solutions in the context of New model-I:

$$u\left(r,t\right)=\frac{\sigma^{*}}{r}\Bigg[\sum_{i=1}^{2}B_{1i}^{1''}\left(\frac{t_1}{b_2^1r_1}\right)^{\frac{i}{2}}J_i\left(2\sqrt{b_2^1r_1t_1}\right)H(t_1)+\sum_{i=1}^{2}B_{2i}^{1''}e^{-b_3^1r_1}\left(\frac{t}{b_4^1r_1}\right)^{\frac{i+3}{2}}I_{i+3}\left(2\sqrt{b_4^1r_1(t)}\right)\Bigg], \tag{78}$$

$$\theta\left(r,t\right)=\frac{\sigma^{*}}{ra_1}\Bigg[\sum_{i=1}^{2}C_{1i}^{1''}\left(\frac{t_1}{b_2^1r_1}\right)^{(i+1)/2}J_{i+1}\left(2\sqrt{b_2^1r_1t_1}\right)H(t_1)+\sum_{i=1}^{2}C_{2i}^{1''}e^{-b_3^1r_1}\left(\frac{t}{b_4^1r_1}\right)^{(i+1)/2}I_{i+1}\left(2\sqrt{b_4^1r_1t}\right)\Bigg], \tag{79}$$

$$\sigma_{rr}\left(r,t\right)=\frac{\sigma^{*}}{r}\Bigg[\sum_{i=1}^{2}D_{1i}^{1''}\left(\frac{t_1}{b_2^1r_1}\right)^{(i-1)/2}J_{i-1}\left(2\sqrt{b_2^1r_1t_1}\right)H(t_1)+\sum_{i=1}^{2}D_{2i}^{1''}e^{-b_3^1r_1}\left(\frac{t}{b_4^1r_1}\right)^{(i+1)/2}I_{i+1}\left(2\sqrt{b_4^1r_1t}\right)\Bigg], \tag{80}$$

$$\sigma_{\phi\phi}\left(r,t\right)=\frac{\sigma^{*}}{r}\Bigg[\sum_{i=1}^{2}E_{1i}^{1''}\left(\frac{t_1}{b_2^1r_1}\right)^{(i-1)/2}J_{i-1}\left(2\sqrt{b_2^1r_1t_1}\right)H(t_1)+\sum_{i=1}^{2}E_{2i}^{1''}e^{-b_3^1r_1}\left(\frac{t}{b_4^1r_1}\right)^{(i+1)/2}I_{i+1}\left(2\sqrt{b_4^1r_1t}\right)\Bigg]. \tag{81}$$

Solutions in the context of New model-II:

$$u\left(r,t\right)=\frac{\sigma^{*}}{r}\sum_{i=1}^{2}\left[e^{-b_2^2r_1}\left\{B_{1i}^{2''}\frac{(t_3)^i}{i}H(t_3)\right\}+\left\{B_{2i}^{2''}(4t)^{\frac{3i}{2}}\boldsymbol{i}^{3i}\mathrm{erfc}(\frac{b_3^2r_1}{2\sqrt{t}})\right\}\right], \tag{82}$$

$$\theta\left(r,t\right)=\frac{\sigma^{*}}{ra_1}\sum_{i=1}^{2}\left[e^{-b_2^2r_1}\left\{C_{1i}^{2''}\frac{(t_3)^i}{i}H(t_3)\right\}+\left\{C_{2i}^{2''}(4t)^{i}\boldsymbol{i}^{2i}\mathrm{erfc}(\frac{b_3^2r_1}{2\sqrt{t}})\right\}\right], \tag{83}$$

$$\sigma_{rr}\left(r,t\right)=\frac{\sigma^{*}}{r}\sum_{i=1}^{2}\left[e^{-b_2^2r_1}\left\{D_{1i}^{2''}(t_3)^{i-1}H(t_3)\right\}+\left\{D_{2i}^{2''}(4t)^{i}\boldsymbol{i}^{2i}\mathrm{erfc}(\frac{b_3^2r_1}{2\sqrt{t}})\right\}\right], \tag{84}$$

$$\sigma_{\phi\phi}\left(r,t\right)=\frac{\sigma^{*}}{r}\sum_{i=1}^{2}\left[e^{-b_2^2r_1}\left\{E_{1i}^{2''}(t_3)^{1-i}H(t_3)\right\}+\left\{E_{2i}^{2''}(4t)^{i}\boldsymbol{i}^{2i}\mathrm{erfc}(\frac{b_3^2r_1}{2\sqrt{t}}\right\}\right], \tag{85}$$

where,

$$r_1=(r-1),\ t^{'}=t-t_0,\ \ t_1=t-b_1^1r_1,$$
$$t_2=t-b_1^1r_1-t_0,\ t_3=t-b_1^2r_1,\ t_4=t-b_1^2r_1-t_0.$$

6. Analysis of analytical results

From the short-time approximated solutions obtained for the **case of problem-1**, i.e., when the boundary of the spherical cavity is subjected to a ramp-type heating, we observe that the distributions of displacement, temperature and stresses for the case of New model-I consist of four different terms. The terms involving $H(t - b_1^1 r_1)$ and $H(t - b_1^1 r_1 - t_0)$ correspond to a wave propagating with non dimensional finite speed $\frac{1}{b_1^1}$. These waves can be identified as predominantly elastic waves propagating with different phases. The expressions given by equations (62)–(65) reveal that under New model-I, the modified elastic waves propagate without any attenuation. This is a distinct feature predicted by the theory and it has a similarity with case of GN-II model. Furthermore, we note that the speed of elastic waves do not depend on the delay parameter or any other material parameter. The second and fourth part of the solutions of the field variables do not indicate to be a contribution of a wave. On the contrary, it is diffusive in nature indicating an exponential decay with distance with an attenuating coefficient b_3^1. This represents that under this present model, the thermal wave do not propagate with finite wave speed like other generalized thermoelasticity theories, namely Lord-Shulman model, Green-Lindsay model, GN-II model, dual-phase-lag model, or three-phase-lag model. However, a similar nature of the second part of the solution of field variables is observed under GN-III model. Furthermore, we note that the solutions for all fields given by equations (62) - (65) are continuous in nature.

In the context of New model-II, we observe that solutions for the distributions of displacement, temperature and stresses given by equations (66) - (69) also consist of four different contributions. As in the previous case, here the terms involving $H(t - b_1^2 r_1)$ and $H(t - b_1^2 r_1 - t_0)$ correspond to two waves propagating with non dimensional finite speed $\frac{1}{b_1^2}$, which correspond to modified elastic waves. Unlike New model-I, this New model-II predicts that elastic waves decay exponentially with an attenuating coefficient b_2^2. The wave speeds in case of New model-I and New model-II is exactly same and do not depend on any material parameter. However, the attenuation coefficient of modified elastic wave in case of New model-II depend on the delay time as well as on the material parameters. The other parts of solution are diffusive in nature due to the damping term. The solutions of all fields are continuous in nature in this case too.

From the solutions given by equations (70)–(77) **in case of problem-2**, we observe that as in problem-1, each of the distributions of temperature, displacement, and stress consist of two main parts. In the first part, the terms involving $H(t-b_1^1 r_1)$ and $H(t-b_1^2 r_1)$ correspond to pre-dominantly elastic wave propagating with non dimensional finite speeds of $\frac{1}{b_1^1}$ and $\frac{1}{b_1^2}$ for New model-I and New model- II, respectively. The wave speeds are same as in case of problem-1. In view of the expressions represented by equations (70)–(73) under New model- I, we note that the elastic waves propagate without any attenuation, whereas in the solutions represented by equations (74)-(77) under New model- II, elastic waves propagate with an attenuation coefficient b_2^2 that depends on the delay time and material parameter. The second part of the solutions given by equations (70)-(77) show that the field variables do not indicate to be a contribution of a wave. Instead of that it is diffusive in nature. It indicates an exponential decay with distance with an attenuating coefficient b_3^1 for New model- I. Furthermore, we note that the solutions for temperature and displacement are continuous in nature for both the models. However, the analytical results given by equations

(72)-(73) for New model- I show that stress distributions suffer from finite discontinuity at elastic wave front. The finite jumps are obtained as follows:

$$[\sigma_{rr}]_{r_1=\frac{t}{b_1^1}} = \sigma_{rr}\big|_{r_1=\frac{t+}{b_1^1}} - \sigma_{rr}\big|_{r_1=\frac{t-}{b_1^1}} = D_{11}^{1'}\frac{T^*a_1}{r},$$
$$[\sigma_{\phi\phi}]_{r_1=\frac{t}{b_1^1}} = \sigma_{\phi\phi}\big|_{r_1=\frac{t+}{b_1^1}} - \sigma_{\phi\phi}\big|_{r_1=\frac{t-}{b_1^1}} = E_{11}^{1'}\frac{T^*a_1}{r}.$$

From the solutions of radial stress and circumferential stress distributions given by equations (76)-(77) for New model- II, we observe similar characteristic like the solutions for New model-I. In this case the finite jumps at the elastic wave front are found out as follows:

$$[\sigma_{rr}]_{r_1=\frac{t}{b_1^2}} = \sigma_{rr}\big|_{r_1=\frac{t+}{b_1^2}} - \sigma_{rr}\big|_{r_1=\frac{t-}{b_1^2}} = D_{11}^{2'}e^{-\frac{b_2^2 t}{b_1^1}}\frac{T^*a_1}{r},$$
$$[\sigma_{\phi\phi}]_{r_1=\frac{t}{b_1^2}} = \sigma_{\phi\phi}\big|_{r_1=\frac{t+}{b_1^2}} - \sigma_{\phi\phi}\big|_{r_1=\frac{t-}{b_1^2}} = E_{11}^{2'}e^{-\frac{b_2^2 t}{b_1^1}}\frac{T^*a_1}{r}.$$

In case of problem- 3, we note that the solutions for the field variables have two main parts like the case of problem-II. The first part in each solution represents elastic wave propagating with the speed $\frac{1}{b_1^1}$ and it propagates without any attenuation in the case of New model-I, whereas New model-II predicts that the predominating elastic wave have speed $\frac{1}{b_1^2}$ and an attenuation coefficient b_2^2. The second parts of the solutions are of diffusive nature which is because of the parabolic nature of heat transport equation for this case. As in case of problem-2, in the present case also the radial stress and circumferential stress distributions show discontinuities with finite jumps at the elastic wave front under both the models, i.e., under New model-I and New model-II.

7. Numerical Results and discussion

In the previous section, we derived short-time approximated analytical solutions for the physical fields and we noted some significant features in the nature of solutions predicted by New model-I and New model-II. In this section, we make an attempt to find the numerical solutions of the three problems in the present work by employing a numerical procedure for Laplace inversions of temperature, displacement, radial stress and shear stress with the help of Matlab software. We employ here the method given by Honig and Hirdes (1984), which is based on the method proposed by Durbin(1973) for the Laplace inversion. We also compare our results with the corresponding results under GN-III model.

We assume that the spherical cavity is made of copper material and the physical data for which are taken as below (Sherief and Salah, 2005).

$\lambda = 7.76\times10^{10}\ Nm^{-2}$, $\mu = 3.86\times10^{10}\ Nm^{-2}$, $\alpha_t = 1.78\times10^{-5}\ K^{-1}$, $\eta = 8886.73\ sm^{-2}$,

$c_E = 383.1\ JKg^{-1}K^{-1}$, $\rho = 8954\ Kg\ m^{-3}$, $T_0 = 293K$.

In order to analyze the solutions for non-dimensional temperature, displacement, radial stress and tangential stress in space-time domain inside the spherical cavity, the results under three different problems are displayed in Figures $1(a,b,c,d)$-$3(a,b,c,d)$. In each figure, we plotted the graphs

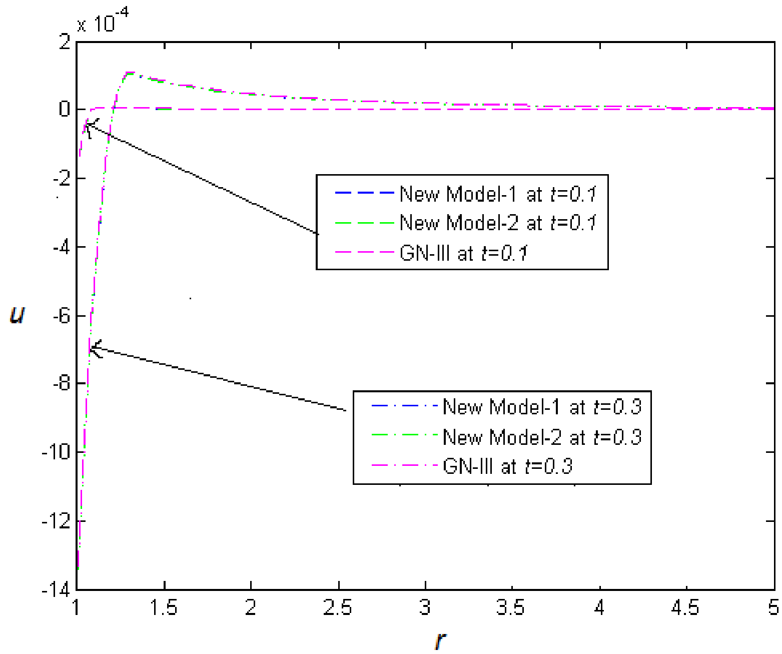

Figure 1(a): Variation of u vs. r for different value of t under problem 1

for the fields at two different times, $t = 0.1$, $t = 0.3$ and for three different thermoelastic models namely, New model-I, New model-II and GN-III model. Specially, we aim to understand the effects of thermal relaxation parameter τ on the solutions at various times and highlight the specific features of the recent heat conduction model. In the domain of influence, we observe that the nature of variations for some field variables at any time in the region near the boundary (from r = 1 to r = 1.5) that is the region near the source of thermoelastic disturbance is different as compared to the variation at a distance far away from the boundary. The location of finite jumps at the elastic wave front is indentified in the region for from r = 1 to$r$ = 1.5. We observe different nature in the solutons for three different problems. Some important facts under various cases of prescribed boundary conditions arising out from our investigation are highlighted as follows:

Discussion on results of Problem-1:

Figures $1(a, b, c, d)$ show the variations in field variables u, θ, σ_{rr} and $\sigma_{\phi\phi}$ with respect to radial distance r for problem-1. From the figures, it is clear that at any particular time there is no prominent difference in the solutions of the field variables under three models in the problem when boundary is subjected to a ramp-type heating. This implies that there is not much effect of delay parameter, τ for the distributions of the field variables and the recent model predicts similar results like the Green-Naghdi model of thermoelasticity of type-III. However it is noted that under each model, the region of influence for temperature and stress fields is much larger and it increases significantly with the increase of time as compared to the region of influence of displacement. Displacement distribution shows a local maximum and both the stress fields show existence of local minima.

Discussion on results of Problem-2:

Figures $2(a, b, c, d)$ display the distributions of the field variables under three models for the case of

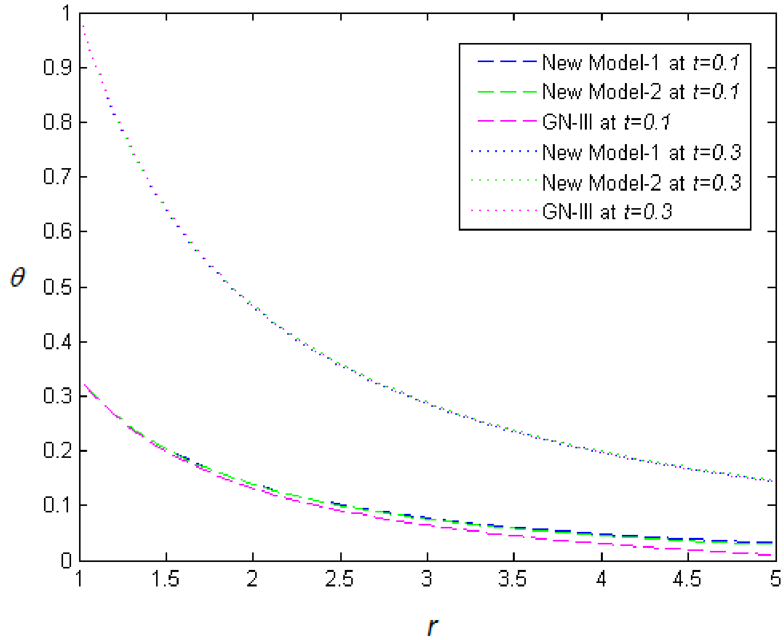

Figure 1(b): Variation of θ vs. r for different value of t under problem 1

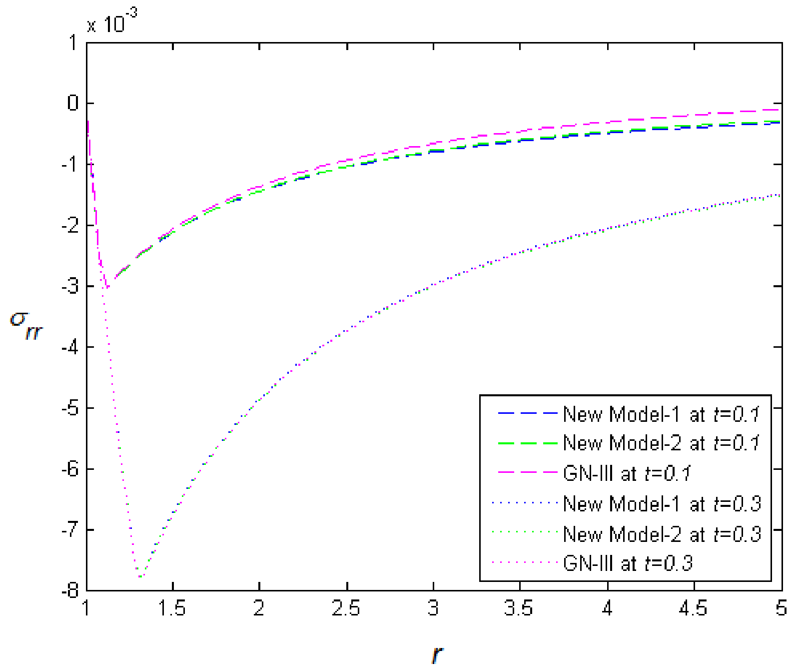

Figure 1(c): Variation of σ_{rr} vs. r for different value of t under problem 1

problem-2, i.e., when the boundary of the cavity is subjected to a unit step increase of temperature (thermal shock). Figure 2(a) displays that for this case, delay time parameter has not much effect on displacement at any instant of time and all models predict almost similar results for displacement. The region of influence is also very less. However, we find from figures 2(b, c, d) that temperature

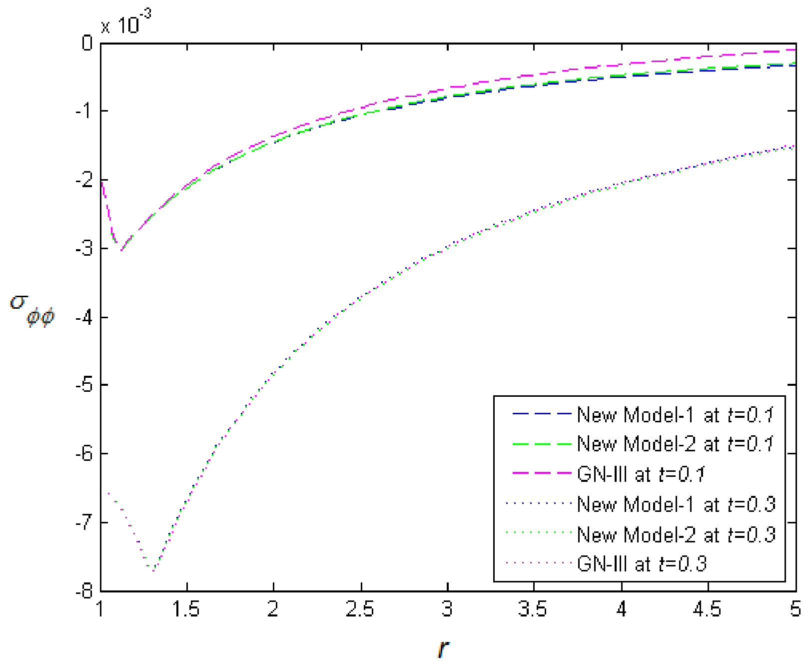

Figure 1(d): Variation of $\sigma_{\phi\phi}$ vs. r for different value of t under problem 1

and stress fields show prominent differences in the results under different models and there is a significant role of the delay time parameter, τ. The disagreement in predictions of different models is more prominent beyond $r > 1.5$. Figure 2(b) shows that temperature, θ increases with increase of time t. Further, New model-I shows the highest value and GN-III predicts the lowest value for temperature at a fixed time, i.e., τ has an increasing effect on temperature, θ. Figures 2(c, d) indicate that as time increases, the stresses σ_{rr} and $\sigma_{\phi\phi}$ show oscillatory nature near the boundary of the cavity. There exist a local maximum and a local minimum for redial stress as well as for circumferential stress in the region from $r = 1$ to $r = 1.5$. σ_{rr} and $\sigma_{\phi\phi}$ show highest values in the context of GN-III model and the lowest value under New model-I at a fixed time implying that there is a decreasing effect of τ on σ_{rr} and $\sigma_{\phi\phi}$. From figures 2(a, b, c, d), it is observed that the effects of delay time parameter, τ on the field variables for problem-2 is significant for temperature and stress fields. However, the effect decreases with the increase of time, although the region of influence for each variable increases with time.

Discussion on results of Problem-3:

Figures 3(a, b, c, d) reveal that for problem-3 when the boundary of the cavity is subjected to a normal load, there is no prominent effect of delay parameter on u, σ_{rr} nd $\sigma_{\phi\phi}$. However, it affects the temperature, θ very prominently and the region of influence is much larger for temperature field as compared to displacement and stress fields. Like the cases of problem-I and problem 2, the region of influence for each field increases with respect to time t for problem 3 too. Further more, figure 3(b) shows that at lower time, the GN-III model predicts the highest value and New model-I shows the lowest value for temperature θ. However, at higher time, the result is reverse i.e., GN-III model predicts the lowest value and New model-I predicts the highest value for θ. Locations of finite jumps at elastic wave front are identified for temperature and stress fields.

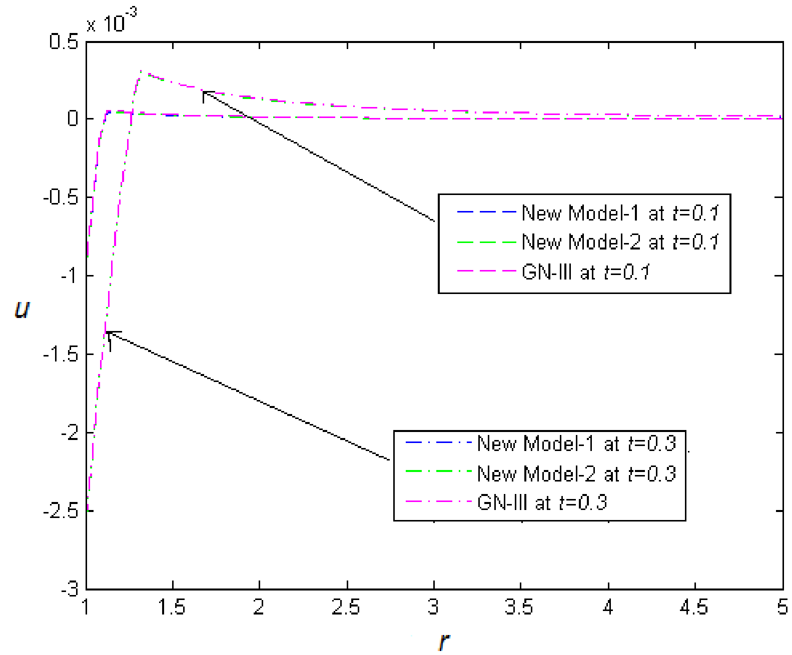

Figure 1. Figure 2(a): Variation of u vs. r for different value of t under problem 2

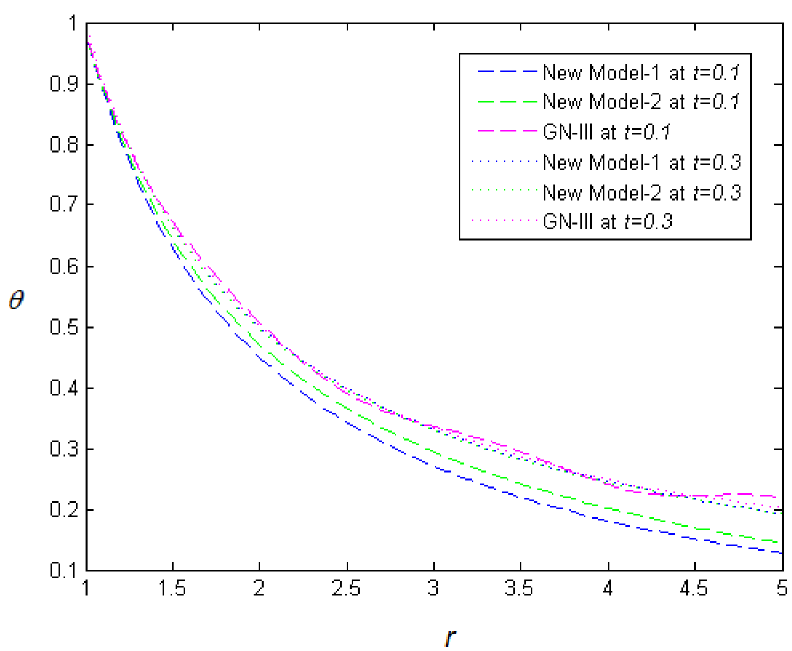

Figure 2(b): Variation of θ vs. r for different value of t under problem 2

8. Conclusion

In the present paper, we have investigated the thermoelastic interactions in an unbounded elastic medium with a spherical cavity under the very recent heat conduction model with single delay term introduced by Quintanilla (2011). This model is an alternative formulation of the three-phase-

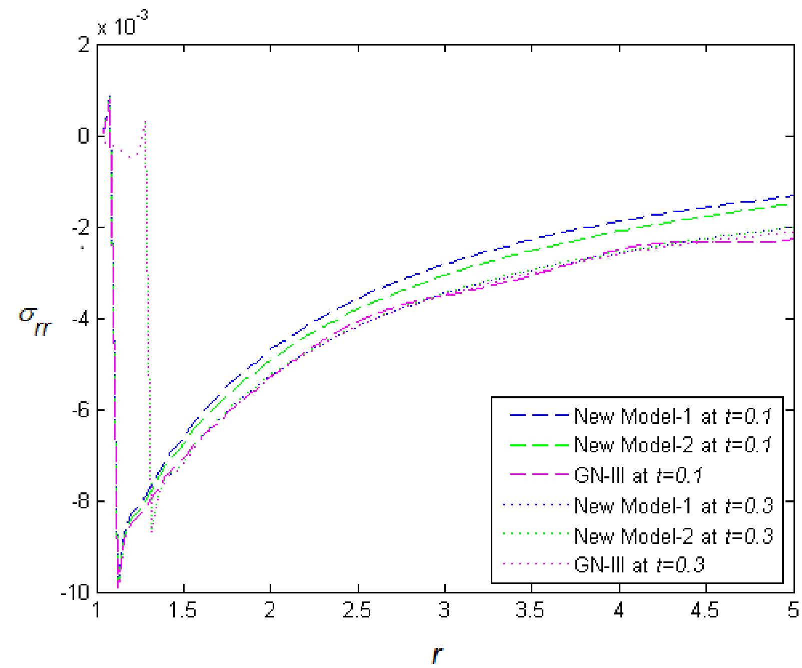

Figure 2(c): Variation of σ_{rr} vs. r for different value of t under problem 2

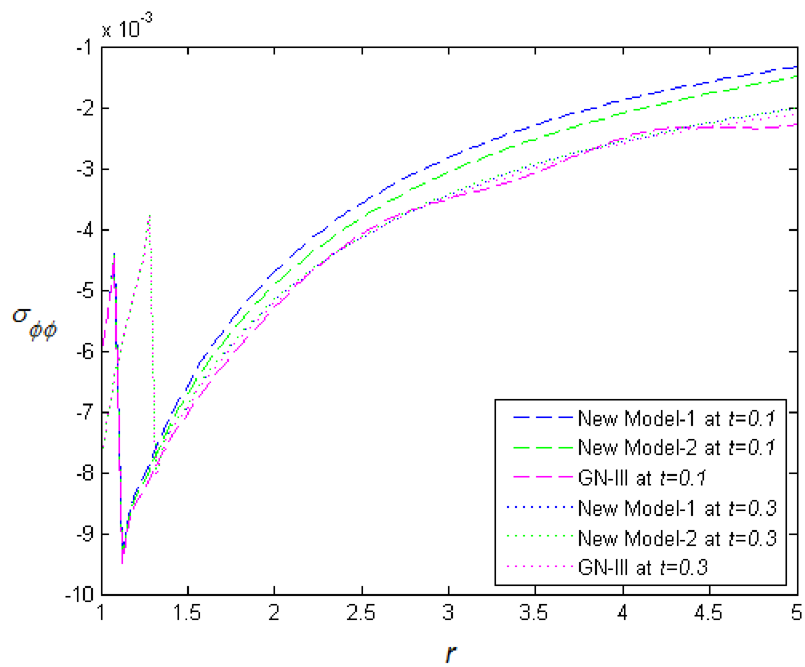

Figure 2(d): Variation of $\sigma_{\phi\phi}$ vs. r for different value of t under problem 2

lag model. We have studied the thermoelastic interactions in an isotropic elastic medium with a spherical cavity subjected to three types of thermal and mechanical loads named as problem-1, problem-2, and problem-3 in the contexts of two versions of this New model. We find analytical as well as numerical solutions for the distributions of displacement, temperature and stresses with the help of the integral transform technique. We also compare numerical results with the corresponding

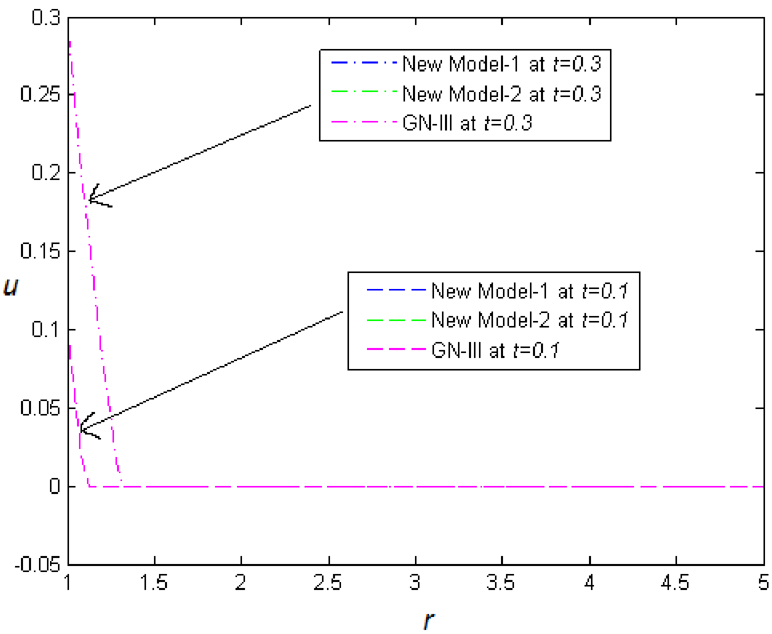

Figure 3(a): Variation of u vs. r for different value of t under problem 3

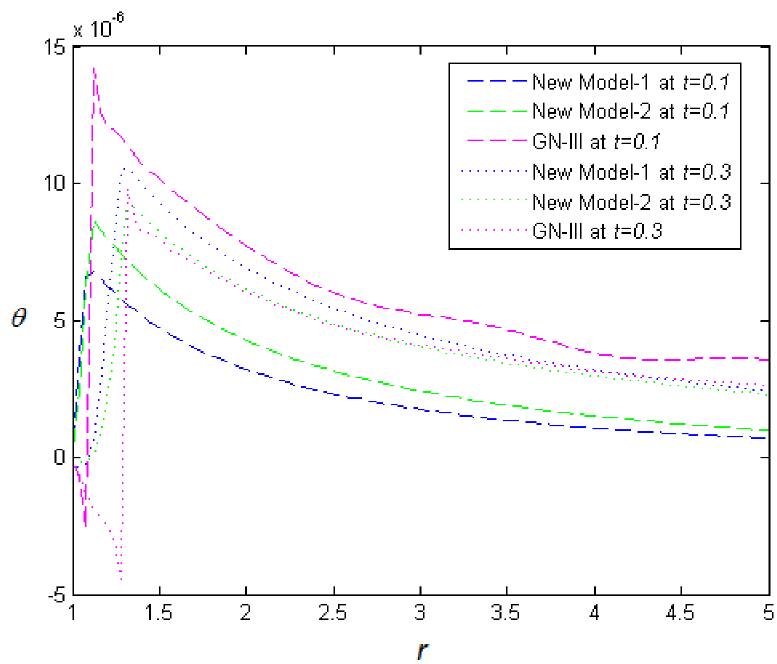

Figure 3(b): Variation of θ vs. r for different value of t under problem 3

results of Green-Naghdi thermoelasticity theory of type-III. We observed following significant facts while analyzing the numerical and analytical results:

(1) In the case of problem-1, when the boundary of the cavity is subjected to a ramp type heating, we observe that elastic waves are propagating with finite speed. However, in the context of New model-I, the elastic wave propagates without any attenuation and under New model-II, the elastic

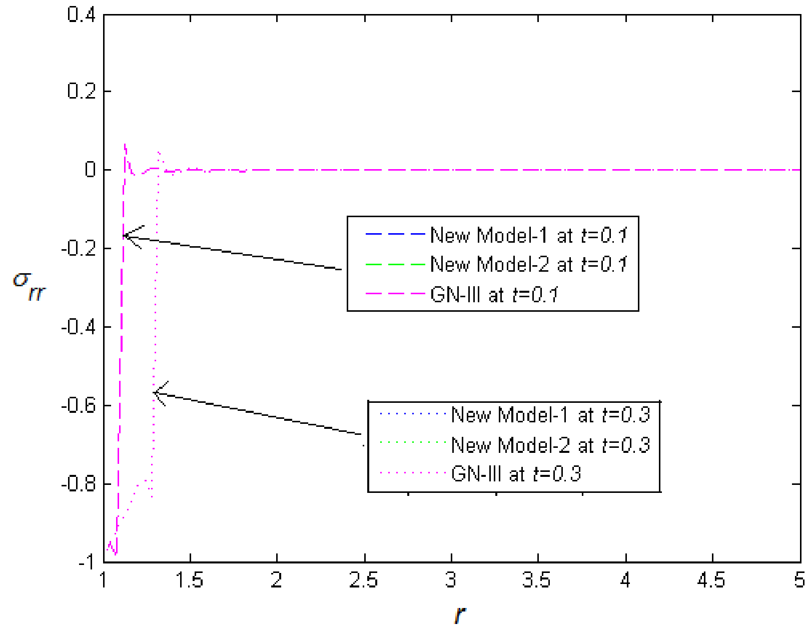

Figure 3(c): Variation of σ_{rr} vs. r for different value of t under problem 3

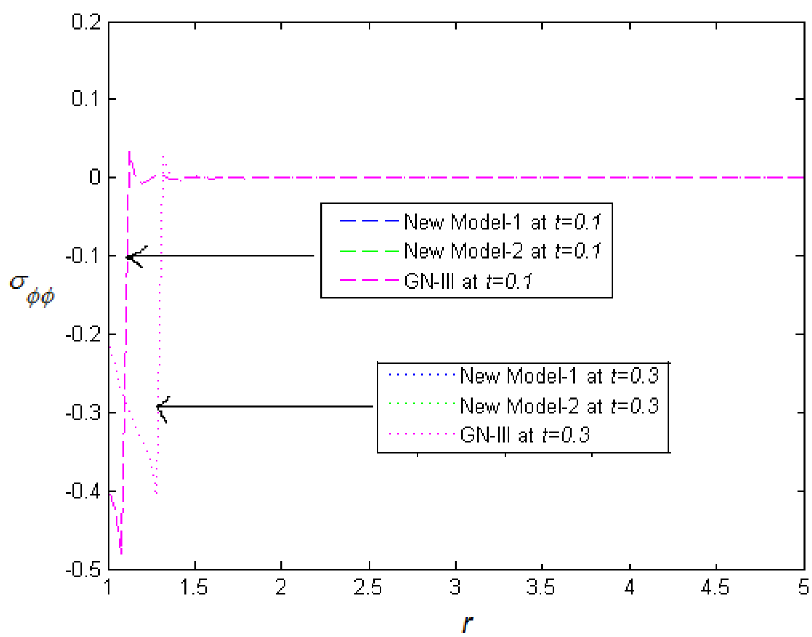

Figure 3(d): Variation of $\sigma_{\phi\phi}$ vs. r for different value of t under problem 3

wave decays exponentially. In this problem, the speed of elastic waves do not depend on the delay parameter as well as on any material parameter. On the other hand, the thermal waves propagate with infinite speed like classical thermoelasticity theory. In this case, all the fields are continuous in nature and the results are of similar nature like Green-Naghdi model of type -III.

(2) In problem-2, when the boundary of the cavity is subjected to thermal shock, we observe

that elastic wave propagates with finite speed but attenuation coefficient of elastic wave for New model-II is totally dependent on the delay time and other material parameters. We also find that temperature and stress fields show prominent differences under different models and there is a significant role of the delay time parameter.

(3) In problem-3, when the boundary of the cavity is subjected to a normal load, there is no prominent effect of delay parameter on u, σ_{rr} and $\sigma_{\phi\phi}$. However, it affects the temperature, θ very prominently.

(4) In problem-2 and problem-3, we also observe that there is a finite discontinuity of radial and circumferential stress at elastic wave front under for both the models.

(5) A similar nature of variation in the field variables is predicted by different models near the boundary. This implies that the difference in predictions by different models is not very significant near the source of disturbance. However, the difference of models for each plot is prominent beyond the region $r > 1.5$. It is more prominent for temperature and stress fields as compared to displacement. The locations of wave fronts are identified to exist in the region from $r = 1$ to $r = 1.5$.

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