Applications and Applied Mathematics: An International Journal (AAM)

Volume 13 | Issue 2 | Article 5

12-2018

An M^x/G(a,b)/1 queue with breakdown and delay time to two phase repair under multiple vacation

G. Ayyappan
Pondicherry Engineering College

M. Nirmala
Pondicherry Engineering College

Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Vol. 13, Issue 2 (December 2018), pp. 639 - 663

Applications and Applied Mathematics: An International Journal (AAM)

An $M^X/G(a,b)/1$ queue with breakdown and delay time to two phase repair under multiple vacation

[1]G. Ayyappan and [2]M. Nirmala

Department of Mathematics
Pondicherry Engineering College
Puducherry, India
[1]ayyappanpec@hotmail.com; [2]jayaudhya@gmail.com;

Received: May 3, 2018; Accepted: September 5, 2018

Abstract:

In this paper, we consider an $M^X/G(a,b)/1$ queue with active breakdown and delay time to two phase repair under multiple vacation policy. A batch of customers arrive according to a compound Poisson process. The server serves the customers according to the "General Bulk Service Rule" (GBSR) and the service time follows a general (arbitrary) distribution. The server is unreliable and it may breakdown at any instance. As the result of breakdown, the service is suspended, the server waits for the repair to start and this waiting time is called as 'delay time' and is assumed to follow general distribution. Further, the repair process involves two phases of repair with different general (arbitrary) repair time distributions. Immediately after the repair, the server is ready to start its remaining service to the customers. After each service completion, if the queue length is less than 'a', the server will avail a multiple vacation of random length. In the proposed model, the probability generating function of the queue size at an arbitrary and departure epoch in steady state are obtained using the supplementary variable technique. Various performance indices, namely mean queue length, mean waiting time of the customers in the queue etc. are obtained. In order to validate the analytical approach, we compute numerical results.

Keywords: General bulk service; Non-Markovian queue; Breakdown; Multiple vacation; Delay time to two phase repair; Supplementary variable

MSC 2010 No.: 60K25, 68M30, 90B22

1. Introduction

In Queueing theory, many authors have made an extensive study on general bulk service and vacations. Server vacation makes the queueing model more realistic to face real life queueing situations. Doshi (1986) and Takagi (1991) gave an excellent survey of queuing systems with server vacations. Lee et al. (1994) analyzed a batch arrival queue with multiple vacations and N-policy. A non-Markovian batch arrival queue with multiple vacations and restricted admissibility policy with two types of service and random breakdown have been studied by Ayyappan and Sathiya (2013). Aliakbar Montazer Haghighi et al. (2013) analysed a bulk arrival stochastic three-stage hiring model as a Tandem queueing process with Erlang phase-type selection. Jeyakumar and Senthilnathan (2012) have studied a non-markovian bulk service queue with multiple vacations, server breakdown without interruption and closedown time. Recently, Rajadurai et al. (2017) have investigated a single server non-Markovian feedback retrial queue with breakdown, repair under multiple working vacation. Manoharan and Shakir Majid (2017) analysed a multiserver queue with multiple working vacation and impatient customers.

In real life, we often come across a queueing situation, where the service is interrupted due to a breakdown in the service facility and the server will not be able to continue his service unless the system is repaired. In many papers, the server is immediately repaired upon failure. However, it may not be feasible to start the repair immediately due to different reasons. So, the server has to wait for repair, which is referred as waiting period or delay time. Ben-Israel and Naor (1960) dicussed a problem of delayed-service. Ayyappan and Shymala (2014) have discussed an $M^X/G_1,G_2/1$ queue with setup time, Bernoulli vacations, breakdown and delayed repair. Gautham Choudhary (2009), studied a non Markovian queue with two phases of service with breakdown and delayed repair. In (2012), he also analyzed a batch arrival retrial queue with general retrial times for unreliable server under Bernoulli vacation and delaying repair. Aliakbar Montazer Haghighi et al. (2016) studied delayed and network queues. In 2016, he discussed a single-server Poisson queueing system with splitting and batch delayed-feedback. Rajadurai et al. (2015) analyzed a batch arrival non-Markovian retrial queue with two phase service under Bernoulli vacation schedule with random breakdown.

Montazer-Haghighi et al. (2011) analyzed a single-server poisson queueing system with delayed-feedback. Recently, Madan et al. (2015) have studied a bulk arrival queue with single server, random breakdown and two phases of repair with delay. In queueing literature, there are very few papers discussed individually batch arrival, delayed repair and multiple vacation. Thus, in this work, we intend to study batch arrival bulk service queueing system with active breakdown, two phases of repair with delay. After the first phase of repair the server will be send for second phase of repair after some delay. This delay may occur due to the unavailability of the repairman or the necessary equipments needed for repair. Madan (2003) used this type of delay time in analyzing an non-Markovian queueing model with random breakdown, with general delay time and exponential repair time. We also, assume that after the two stage of repair, the server is ready to start its remaining service to the batch of customers whose service was interrupted due to breakdown. Also, we used the elapsed service time, elapsed vacation time, elapsed delay time and elapsed repair time as supplementary variables.

The model discussed above has application in several real world systems. We consider a manufacturing system where production orders arrive in batches of random size that can be modeled as a Poisson process. The production starts only when the orders are accumulated to some predetermined quantity. The production time is a random variable which follows a general distribution. It is desirable that the production begins whenever the number of orders reaches a critical value. If the number of orders is less than the required value, the production waits for minimum number of orders to arrive. Whenever the production is completed or no sufficient orders are present, the system will perform optional jobs (vacation) for a random length of time. On completion of each optional job the manufacturing system verify the orders whether or not to restart the production. If at that moment, the orders did not reach the critical value, a decision is made for other optional job to be performed next. The optional job can be referred as machine maintenances or utilized for other secondary works. During production, the system gets breakdown and there may be a delay to repair due to non-availability of repairmen. After repair completion, the system will start the production with the remaining orders in the service station.

We organized the paper as follows. Mathematical analysis of the model is briefly described in Section 2. Schematic representation of the model is given in Figure 1.1. Section 3 focuses on the queue size distribution at a random epoch. PGF of queue size is obtained in Section 4. Stability condition and some particular cases are examined in Section 5 and Section 6. In Section 7, we derive the mean queue length and expected waiting time of the customers. Sections 8 and 9 concern with distribution of queue size at departure epoch and special cases for the proposed model. Numerical illustration, conclusions are presented in Section 10 and Section 11 respectively.

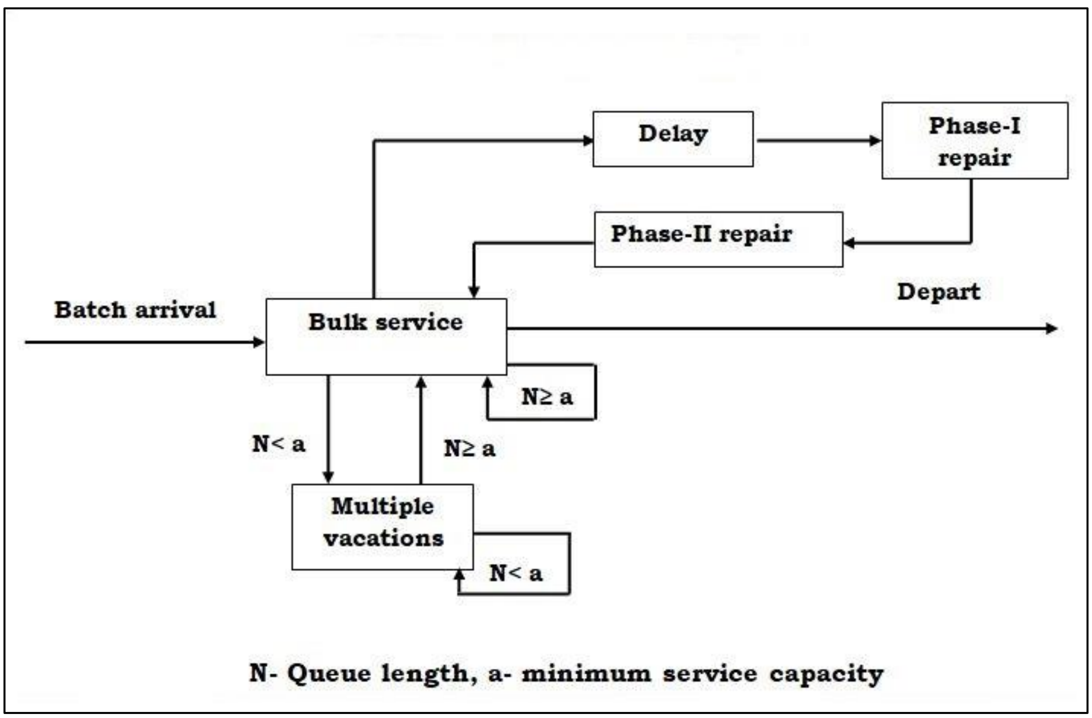

Figure 1.1: Schematic representation of the proposed model

2. The Mathematical model

We consider an $M^X/G(a,b)/1$ queueing system, where the customers arrive according to a compound Poisson process with rate λ. Let $X_1, X_2, \ldots$ be the size of successive arriving batches which are independent and identically distributed (i.i.d) random variables with probability mass function (p.m.f) $c_j = Pr\{X_k = j\}; j \geq 1,\ k$ =1,2,..., and PGF *C(z)*. Service follows the "General Bulk Service Rule". The service discipline is FCFS. The service times $\{Q_j, j \geq 1\}$ of the customers are i.i.d. random variable with distribution function $Q(u)$, Laplace-Stieltjes Transform (LST) $\bar{Q}(s)$ and finite moments $E(Q^k)(k \geq 1)$. While the server is serving a batch of customers, it may breakdown at any instant with a rate 'η' and the server will be down for a short interval of time. At the server breakdown, it is sent for repair during which the server stops providing service to the batch of customers and waits for repair to start, we define this waiting time as the delay time and the delay times $\{L_j, j \geq 1\}$ of the server are i.i.d random variables with distribution function $L(v)$, LST $\bar{L}(s)$ and finite moments $E(L^k)(k \geq 1)$.

Next, we assume that the repair process comprises two phases of repairs, the first phase repair (FPR) followed by the second phase repair(SPR). The repair times $\{R_j^f, j \geq 1\}$, $\{R_j^s, j \geq 1\}$ of the server are i.i.d random variable with distribution function $R^f(v)$, $R^s(v)$, LST $\bar{R}^f(s)$, $\bar{R}^s(s)$ and finite moments $E((R^f)^k)(k \geq 1)$, $E((R^s)^k)(k \geq 1)$. Immediately after the SPR is completed, the server is ready to start its remaining service to the batch of customers whose service was interrupted due to breakdown. After each service completion, if the queue length is less than 'a', the server leaves for a vacation of random length. When he returns, if he finds less than 'a' customers, he leaves for another vacation and so on, until he finds atleast 'a' customers in the queue. The vacation times $\{F_j, j \geq 1\}$ of the customers are i.i.d random variables with distribution function $F(u)$, LST $\bar{F}(s)$ and finite moments $E(F^k)(k \geq 1)$. In addition, we assume that the inter arrival times, the service times, the delay times, FPR times and SPR times are mutually independent of each other.

3. Queue size distribution at a random epoch

In this section, we obtain the PGF of the server's state and number in the queue (the number of customers in the queue excluding the batch being served, if any) at a random epoch by treating the elapsed service times, elapsed vacation times, elapsed delay times, elapsed first phase repair times and elapsed second phase repair times of the server as supplementary variables.

Let $H(t)$ be the queue size, $Q^0(t)$ be the elapsed service time, $F^0(t)$ be the elapsed vacation time, $L^0(t)$ be the elapsed delay time and $(R^f)^0(t)$, $(R^s)^0(t)$ be the elapsed repair time for the first and second phase repair during breakdown in the system at time 't'.

For further development, we introduce the random variable,

$$Y(t)=\begin{cases}1, \text{ if the server is busy with regular service at time `}t\text{'},\\ 2, \text{ if the server is on vacation at time `}t\text{'},\\ 3, \text{ if the server is in delay period at time `}t\text{'},\\ 4, \text{ if the server is under FPR at time `}t\text{'},\\ 5, \text{ if the server is under SPR at time `}t\text{'}.\end{cases}$$

We introduce the supplementary variables $Q^0(t)$, $F^0(t)$, $L^0(t)$, $(R^f)^0(t)$ and $(R^s)^0(t)$ in order to obtain a bivariate Markov process $\{H(t), U(t)\}$, where $U(t) = Q^0(t)$ if $Y(t) = 1$, $U(t) = F^0(t)$ if $Y(t) = 2$, $U(t) = L^0(t)$ if $Y(t) = 3$, $U(t) = (R^f)^0(t)$ if $Y(t) = 4$ and $U(t) = (R^s)^0(t)$ if $Y(t) = 5$.

Now, we define the following probabilities:

$Q_j(u,t)$ = Probability that there are exactly 'j' customers in the queue at time 't', excluding the batch under service and the server is busy with regular service with elapsed service time 'u'.

$F_j(u,t)$ = Probability that there are exactly 'j' customers in the queue at time 't' and the server is on vacation with the elapsed vacation time 'u'.

$L_j(u,v,t)$ = Probability that there are exactly 'j' customers in the queue at time 't', the server is waiting for repair with the elapsed service time of the batch of customers undergoing service is 'u' and the elapsed delay time to repair for the server is 'v'.

$R_j^f(u,v,t)$ = Probability that there are exactly 'j' customers in the queue at time 't', with elapsed service time of the batch of customers undergoing service is 'u' and the elapsed repair time of server for FPR is 'v'.

$R_j^s(u,v,t)$ = Probability that there are exactly 'j' customers in the queue at time 't', with elapsed service time of the batch of customers undergoing service is 'u' and the elapsed repair time of server for SPR is 'v'.

For the process,

$$Q_j(u)du = \lim_{t\to\infty} Pr[H(t) = j, U(t) = Q^0(t); u < Q^0(t) \le u + du]; u > 0, \qquad j \ge 0,$$
$$F_j(u)du = \lim_{t\to\infty} Pr[H(t) = j, U(t) = F^0(t); u < F^0(t) \le u + du]; u > 0, \qquad j \ge 0,$$

and for fixed value of u and $j \ge 0$,

$$L_j(u,v)dv = \lim_{t\to\infty} Pr\left[H(t) = j, U(t) = L^0(t);\ v < L^0(t) \le v + \frac{dv}{Q^0(t)} = u\right], (u,v) > 0,$$
$$R_j^f(u,v)dv = \lim_{t\to\infty} Pr[H(t) = j, U(t) = (R^f)^0(t);\ v < \left(R^f\right)^0(t) \le v + \frac{dv}{Q^0(t)} = u\Big],$$

$$(u,v) > 0,$$

$$R_j^s(u,v)dv = \lim_{t\to\infty} Pr[H(t) = j, U(t) = (R^s)^0(t);\ v < \left(R^s)^0(t) \le v + \frac{dv}{Q^0(t)} = u\right], \quad (u,v) > 0.$$

Further, it is assumed that

$$Q(0) = F(0) = L(0) = R^f(0) = R^s(0) = 0, \qquad \text{and}$$
$$Q(\infty) = F(\infty) = L(\infty) = R^f(\infty) = R^s(\infty) = 1.$$

$Q(u)$, $F(u)$ are continuous at $u = 0$ and $L(v)$, $R^f(v)$, $R^s(v)$ are continuous at $v = 0$.

So that,

$$\mu(u)du = \frac{dQ(u)}{1 - Q(u)}, \qquad \nu(u)du = \frac{dF(u)}{1 - F(u)}, \quad \beta(v)dv = \frac{dL(v)}{1 - L(v)},$$

and

$$\zeta_1(v)dv = \frac{dR^f(v)}{1 - R^f(v)} \ and\ \zeta_2(v)dv = \frac{dR^s(v)}{1 - R^s(v)}$$

are the first order differential functions (hazard rates) of $Q(u), F(u), L(v), R^f(v)$ and $R^s(v)$ respectively.

The Kolmogorov forward equations governing the system in transient state is given as follows:

$$\frac{\partial}{\partial u} Q_j(u,t) + \frac{\partial}{\partial t} Q_j(u,t) + (\lambda + \mu(u) + \eta) Q_j(u,t) = \lambda(1 - \delta_{j,0}) \sum_{k=1}^{j} c_k Q_{j-k}(u,t)$$
$$+ \int_0^\infty R_j^s(u,v,t)\zeta_2(v)dv,\ j \ge 0, \qquad (1)$$

$$\frac{\partial}{\partial u} F_j(u,t) + \frac{\partial}{\partial t} F_j(u,t) + (\lambda + \nu(u)) F_j(u,t) = \lambda(1 - \delta_{j,0}) \sum_{k=1}^{j} c_k F_{j-k}(u,t),$$
$$j \ge 0, \qquad (2)$$

$$\frac{\partial}{\partial v} L_j(u,v,t) + \frac{\partial}{\partial t} L_j(u,v,t) + (\lambda + \beta(v)) L_j(u,v,t) = \lambda(1 - \delta_{j,0}) \sum_{k=1}^{j} c_k L_{j-k}(u,v,t),$$
$$j \ge 0, \qquad (3)$$

$$\frac{\partial}{\partial v} R_j^f(u,v,t) + \frac{\partial}{\partial t} R_j^f(u,v,t) + (\lambda + \zeta_1(v)) R_j^f(u,v,t)$$
$$= \lambda(1 - \delta_{j,0}) \sum_{k=1}^{j} c_k R_{j-k}^f(u,v,t), j \ge 0, \qquad (4)$$

$$\frac{\partial}{\partial v}R_j^s(u,v,t)+\frac{\partial}{\partial t}R_j^s(u,v,t)+\big(\lambda+\zeta_2(v)\big)R_j^s(u,v,t)$$
$$=\lambda\big(1-\delta_{j,0}\big)\sum_{k=1}^{j} c_kR_{j-k}^s(u,v,t), j\geq 0, \tag{5}$$

where $\delta_{\mathrm{m,j}}$ denotes Kroneckers's function.

Boundary conditions at 'u = 0'

$$Q_0(0,t)=\sum_{r=a}^{b}\int_0^\infty Q_r(u,t)\mu(u)du+\sum_{r=a}^{b}\int_0^\infty F_r(u,t)\nu(u)du, \tag{6}$$

$$Q_j(0,t)=\int_0^\infty Q_{j+b}(u,t)\mu(u)du+\int_0^\infty F_{j+b}(u,t)\nu(u)du, j\geq 1, \tag{7}$$

$$F_j(0,t)=\int_0^\infty Q_j(u,t)\mu(u)du+F_j(u,t)\nu(u)du, j=0,1,2,\dots,a-1, \tag{8}$$

$$F_j(0,t)=0,\quad j\geq a, \tag{9}$$

and at ν=0 and fixed value of 'u'

$$L_j(u,0,t)=\eta Q_j(u,t), j\geq 0, \tag{10}$$

$$R_j^f(u,0,t)=\int_0^\infty L_j(u,v,t)\beta(v)dv, j\geq 0, \tag{11}$$

$$R_j^s(u,0,t)=\int_0^\infty R_j^f(u,v,t)\zeta_1(v)dv, j\geq 0. \tag{12}$$

The initial conditions are:

$$Q_j(0)=F_j(0)=L_j(0)=R_j^f(0)=R_j^s(0)=0, \mathrm{j}=0,1,2,\dots \tag{13}$$

The normalizing consition is:

$$\sum_{j=0}^{\infty}\int_0^\infty Q_j(u)du+\sum_{j=0}^{\infty}\int_0^\infty F_j(u)du+\sum_{j=0}^{\infty}\int_0^\infty\int_0^\infty L_j(u,v)dudv$$
$$+\sum_{j=0}^{\infty}\int_0^\infty\int_0^\infty R_j^f(u,v)dudv+\sum_{j=0}^{\infty}\int_0^\infty\int_0^\infty R_j^s(u,v)dudv=1 \tag{14}$$

Probability generating functions for $|z| \leq 1$:

$$Q(u,z,t) = \sum_{j=0}^{\infty} z^j Q_j(u,t), Q(0,z,t) = \sum_{j=0}^{\infty} z^j Q_j(0,t),$$

$$F(u,z,t) = \sum_{j=0}^{\infty} z^j F_j(u,t), F(0,z,t) = \sum_{j=0}^{\infty} z^j F_j(0,t),$$

$$L(u,v,z,t) = \sum_{j=0}^{\infty} z^j L_j(u,v,t), L(u,0,z,t) = \sum_{j=0}^{\infty} z^j L_j(u,0,t), \tag{15}$$

$$R^f(u,v,z,t) = \sum_{j=0}^{\infty} z^j R_j^f(u,v,t), R^f(u,0,z,t) = \sum_{j=0}^{\infty} z^j R_j^f(u,0,t),$$

$$R^s(u,v,z,t) = \sum_{j=0}^{\infty} z^j R_j^s(u,v,t), R^s(u,0,z,t) = \sum_{j=0}^{\infty} z^j R_j^s(u,0,t).$$

Laplace transform of $h(t)$ is defined as:

$$\bar{h}(s) = \int_0^{\infty} e^{-st} h(t) dt, \Re(s) > 0. \tag{16}$$

Applying Laplace transform techniques on both sides of the equations (1) to (12) and using equation (13), we get

$$\frac{\partial}{\partial u}\bar{Q}_j(u,s) + (s+\lambda+\mu(u)+\eta)\bar{Q}_j(u,s) = \lambda(1-\delta_{j,0})\sum_{k=1}^{j} c_k \bar{Q}_{j-k}(u,s) + \int_0^{\infty} \bar{R}_j^s(u,v,s)\zeta_2(v)dv, j \geq 0 \tag{17}$$

$$\frac{\partial}{\partial u}\bar{F}_j(u,s) + \big(s+\lambda+\nu(u)\big)\bar{F}_j(u,s) = \lambda\big(1-\delta_{j,0}\big)\sum_{k=1}^{j} c_k \bar{F}_{j-k}(u,s), \mathrm{j} \geq 0 \tag{18}$$

$$\frac{\partial}{\partial v}\bar{L}_j(u,v,s) + \big(s+\lambda+\beta(v)\big)\bar{L}_j(u,v,s) = \lambda\big(1-\delta_{j,0}\big)\sum_{k=1}^{j} c_k \bar{L}_{j-k}(u,v,s), j \geq 0 \tag{19}$$

$$\frac{\partial}{\partial v}\bar{R}_j^f(u,v,s)+\big(s+\lambda+\zeta_1(v)\big)\bar{R}_j^f(u,v,s)=\lambda\big(1-\delta_{j,0}\big)\sum_{k=1}^{j}c_k\bar{R}_{j-k}^f(u,v,s), j\geq 0, \quad (20)$$

$$\frac{\partial}{\partial v}\bar{R}_j^s(u,v,s)+\big(s+\lambda+\zeta_2(v)\big)\bar{R}_j^s(u,v,s)=\lambda\big(1-\delta_{j,0}\big)\sum_{k=1}^{j}c_k\bar{R}_{j-k}^s(u,v,s), j\geq 0, \quad (21)$$

$$\bar{Q}_0(0,s)=\sum_{r=a}^{b}\int_0^\infty\bar{Q}_r(u,s)\mu(u)du+\sum_{r=a}^{b}\int_0^\infty\bar{F}_r(u,s)\nu(u)du\,, \quad (22)$$

$$\bar{Q}_j(0,s)=\int_0^\infty\bar{Q}_{j+b}(u,s)\mu(u)du+\int_0^\infty\bar{F}_{j+b}(u,s)\nu(u)du, j\ \geq 1, \quad (23)$$

$$\bar{F}_j(0,s)=\int_0^\infty\bar{Q}_j(u,s)\mu(u)du+\int_0^\infty\bar{F}_j(u,s)\nu(u)du, \mathrm{j}=0,1,2\ldots,a-1, \quad (24)$$

$$\bar{F}_j(0,s)=0, j\geq a, \quad (25)$$

$$\bar{L}_j(u,0,s)=\eta\bar{Q}_j(u,s), j\geq 0, \quad (26)$$

$$\bar{R}_j^f(u,0,s)=\int_0^\infty\bar{L}_j(u,v,s)\beta(v)dv, j\geq 0, \quad (27)$$

$$\bar{R}_j^s(u,0,s)=\int_0^\infty\bar{R}_j^f(u,v,s)\zeta_1(v)dv, j\geq\ 0. \quad (28)$$

Multiplying equations (17) to (21) by z^j, and taking summation over all possible values of $j,(j=0,1,2\ldots)$ and using (15) we get

$$\frac{\partial}{\partial u}\bar{Q}(u,z,s)+\big[s+\lambda\big(1-C(z)\big)+\mu(u)+\eta\big]\bar{Q}(u,z,s)$$
$$=\int_0^\infty\bar{R}^s(u,v,z,s)\zeta_2(v)dv, \quad (29)$$

$$\frac{\partial}{\partial u}\bar{F}(u,z,s)+\big[s+\lambda\big(1-C(z)\big)+\nu(u)\big]\bar{F}(u,z,s)=0, \quad (30)$$

$$\frac{\partial}{\partial v}\bar{L}(u,v,z,s)+\big[s+\lambda\big(1-C(z)\big)+\beta(v)\big]\bar{L}(u,v,z,s)=0, \quad (31)$$

$$\frac{\partial}{\partial v}\bar{R}^f(u,v,z,s)+\big[s+\lambda\big(1-C(z)\big)+\zeta_1(v)\big]\bar{R}^f(u,v,z,s)=0, \quad (32)$$

$$\frac{\partial}{\partial v}\bar{R}^s(u,v,z,s) + \left[s + \lambda\big(1 - C(z)\big) + \zeta_2(v)\right]\bar{R}^s(u,v,z,s) = 0. \tag{33}$$

Multiplying equations (22) to (28) by suitable powers of 'z' and taking summation over all possible values of 'j' and using the definition of PGF, we get

$$\begin{aligned} z^b\bar{Q}(0,z,s) = &\int_0^\infty \bar{Q}(u,z,s)\mu(u)du + \int_0^\infty \bar{F}(u,z,s)\nu(u)du - \sum_{r=0}^{a-1}\int_0^\infty \bar{Q}_r(u,s)\mu(u)z^r du \\ &- \sum_{r=0}^{a-1}\int_0^\infty \bar{F}_r(u,s)\nu(u)z^r du + \sum_{r=a}^{b-1}\int_0^\infty \bar{Q}_r(u,s)\mu(u)(z^b - z^r)du \\ &+ \sum_{r=a}^{b-1}\int_0^\infty \bar{F}_r(u,s)\nu(u)(z^b - z^r)du, \end{aligned} \tag{34}$$

$$\bar{F}(0,z,s) = \sum_{r=0}^{a-1}\int_0^\infty \bar{Q}_r(u,s)\mu(u)z^r du + \sum_{r=0}^{a-1}\int_0^\infty \bar{F}_r(u,s)\nu(u)z^r du, \tag{35}$$

$$\bar{L}(u,0,z,s) = \eta\bar{Q}(u,z,s), \tag{36}$$

$$\bar{R}^f(u,0,z,s) = \int_0^\infty \bar{L}(u,v,z,s)\beta(v)dv, \tag{37}$$

$$\bar{R}^s(u,0,z,s) = \int_0^\infty \bar{R}^f(u,v,z,s)\zeta_1(v)dv. \tag{38}$$

Integrating equations (30) to (33), we get

$$\bar{F}(u,z,s) = \bar{F}(0,z,s)\big(1 - \bar{F}(u)\big)e^{-\{T(z)\}u}, \tag{39}$$

$$\bar{L}(u,v,z,s) = \bar{L}(u,0,z,s)\big(1 - \bar{L}(v)\big)e^{-\{T(z)\}v}, \tag{40}$$

$$\bar{R}^f(u,v,z,s) = \bar{R}^f(u,0,z,s)\left(1 - \bar{R}^f(v)\right)e^{-\{T(z)\}v}, \tag{41}$$

$$\bar{R}^s(u,v,z,s) = \bar{R}^s(u,0,z,s)\big(1 - \bar{R}^s(v)\big)e^{-\{T(z)\}v}, \tag{42}$$

where $T(z)$ is given in Appendix.

Multiplying both sides of the equation (39) by $\nu(u)$ and integrating over 'u', we obtain

$$\int_0^\infty \bar{F}(u,z,s)\nu(u)du = \bar{F}(0,z,s)\bar{F}[T(z)]. \tag{43}$$

Multiplying both sides of the equations (40) to (42) by $\beta(v)$, $\zeta_1(v)$ and $\zeta_2(v)$ respectively and integrating over 'v', we obtain

$$\int_0^\infty \bar{L}(u,v,z,s)\beta(v)dv = \bar{L}(u,0,z,s)\bar{L}[T(z)], \tag{44}$$

$$\int_0^\infty \bar{R}^f(u,v,z,s)\zeta_1(v)dv = \bar{R}^f(u,0,z,s)\bar{R}^f[T(z)], \tag{45}$$

$$\int_0^\infty \bar{R}^s(u,v,z,s)\zeta_2(v)dv = \bar{R}^s(u,0,z,s)\bar{R}^s[T(z)]. \tag{46}$$

Again integrating equations (39) to (42), we have

$$\int_0^\infty \bar{F}(u,z,s)du = \bar{F}(z,s) = \bar{F}(0,z,s)\left[\frac{1-\bar{F}\big(T(z)\big)}{T(z)}\right], \tag{47}$$

$$\int_0^\infty \bar{L}(u,v,z,s)dv = \bar{L}(u,z,s) = \bar{L}(u,0,z,s)\left[\frac{1-\bar{L}\big(T(z)\big)}{T(z)}\right], \tag{48}$$

$$\int_0^\infty \bar{R}^f(u,v,z,s)dv = \bar{R}^f(u,z,s) = \bar{R}^f(u,0,z,s)\left[\frac{1-\bar{R}^f\big(T(z)\big)}{T(z)}\right], \tag{49}$$

$$\int_0^\infty \bar{R}^s(u,v,z,s)dv = \bar{R}^s(u,z,s) = \bar{R}^s(u,0,z,s)\left[\frac{1-\bar{R}^s\big(T(z)\big)}{T(z)}\right]. \tag{50}$$

Utilizing equations (36) to (38) and (44) to (46) in equation (29) and simplifying, we obtain

$$\begin{aligned}\frac{\partial}{\partial u}\bar{Q}(u,z,s) + (s+\lambda(1-C(z))+\mu(u)+\eta)\bar{Q}(u,z,s)\\ = \eta\bar{L}(T(z))\bar{R}^f(T(z))\bar{R}^s(T(z))\bar{Q}(u,z,s).\end{aligned} \tag{51}$$

On integrating the above equation, we get,

$$\bar{Q}(u,z,s) = \bar{Q}(0,z,s)\big(1-\bar{Q}(u)\big)e^{-\{\psi(z)\}u}, \tag{52}$$

where $\psi(z)$ is given in Appendix.

Multiplying both sides of (52) by $\mu(u)$ and integrating over 'u', we get

$$\int_0^\infty \bar{Q}(u,z,s)\mu(u)du = \bar{Q}(0,z,s)\bar{Q}[\psi(z)]. \tag{53}$$

Again integrating equation (52), we get

$$\int_0^\infty \bar{Q}(u,z,s)du = \bar{Q}(z,s) = \bar{Q}(0,z,s)\left[\frac{1-\bar{Q}\big(\psi(z)\big)}{\psi(z)}\right]. \tag{54}$$

Substituting equations (43) and (53) in (34) and on simplifying, we get

$$\bar{Q}(0,z,s) = \frac{\left[\begin{array}{c}\sum_{r=0}^{a-1}\int_0^\infty [\bar{Q}_r(u,s)\mu(u)+\bar{F}_r(u,s)\nu(u)]z^r[\bar{F}(T(z))-1]du \\ +\sum_{r=a}^{b-1}\int_0^\infty [\bar{Q}_r(u,s)\mu(u)+\bar{F}_r(u,s)\nu(u)](z^b-z^r)du\end{array}\right]}{z^b-\bar{Q}(\psi(z))}. \tag{55}$$

From equation (48),

$$\bar{L}(z,s) = \int_0^\infty \bar{L}(u,z,s)du = \int_0^\infty \bar{L}(u,0,z,s)\left[\frac{1-\bar{L}(T(z))}{T(z)}\right]du.$$

Utilizing equations (36) and (54) in the above equation, we obtain

$$\bar{L}(z,s) = \eta\bar{Q}(0,z,s)\left[\frac{1-\bar{Q}\big(\psi(z)\big)}{\psi(z)}\right]\left[\frac{1-\bar{L}\big(T(z)\big)}{T(z)}\right]. \tag{56}$$

From equation (49),

$$\bar{R}^f(z,s) = \int_0^\infty \bar{R}^f(u,z,s)du = \int_0^\infty \bar{R}^f(u,0,z,s)\left[\frac{1-\bar{R}^f(T(z))}{T(z)}\right]du.$$

Using equations (36), (37), (44) and (54) in the above equation, we get

$$\bar{R}^f(z,s) = \eta\bar{Q}(0,z,s)\bar{L}\big(T(z)\big)\left[\frac{1-\bar{Q}\big(\psi(z)\big)}{\psi(z)}\right]\left[\frac{1-\bar{R}^f\big(T(z)\big)}{T(z)}\right]. \tag{57}$$

From equation (50),

$$\bar{R}^s(z,s) = \int_0^\infty \bar{R}^s(u,z,s)du = \int_0^\infty \bar{R}^s(u,0,z,s)\left[\frac{1-\bar{R}^s(T(z))}{T(z)}\right]du.$$

Similarly, by using equations (36) to (38), (44), (45) and (54) in the above equation, we get

$$\bar{R}^s(z,s) = \eta\bar{Q}(0,z,s)\bar{L}\big(T(z)\big)\bar{R}^f\big(T(z)\big)\left[\frac{1-\bar{Q}\big(\psi(z)\big)}{\psi(z)}\right]\left[\frac{1-\bar{R}^s\big(T(z)\big)}{T(z)}\right]. \tag{58}$$

Thus, $\overline{Q}(z,s)$, $\overline{F}(z,s)$, $\overline{L}(z,s)$, $\bar{R}^f(z,s)$ and $\bar{R}^s(z,s)$ are completely determined from equations (54), (47), (56), (57) and (58).

4. Probability generating unction of queue length

The PGF of the queue size in steady state is obtained by the Tauberian property,

$$\lim_{s\to 0} s\bar{h}(s) = \lim_{t\to\infty} h(t). \tag{59}$$

Therefore, in steady state, the PGF of queue size when the server is busy, on vacation, delay to repair, first and second phases of repair are given below:

$$Q(z) = \frac{\left[\sum_{r=0}^{a-1} q_r z^r\big(\bar{F}\big(T_1(z)\big)-1\big) + \sum_{r=a}^{b-1}\omega_r(z^b - z^r)\right]\left[1-\bar{Q}\big(\psi_1(z)\big)\right]}{\psi_1(z)\left[z^b - \bar{Q}\big(\psi_1(z)\big)\right]}, \tag{60}$$

$$F(z) = \frac{\sum_{r=0}^{a-1} q_r z^r\left[1-\bar{F}\big(T_1(z)\big)\right]}{T_1(z)}, \tag{61}$$

$$L(z) = \frac{\left[\begin{array}{c}\eta\left[1-\bar{L}\big(T_1(z)\big)\right]\left[1-\bar{Q}\big(\psi_1(z)\big)\right]\times \\ \left[\sum_{r=0}^{a-1} q_r z^r\left[\bar{F}\big(T_1(z)\big)-1\right] + \sum_{r=a}^{b-1}\omega_r(z^b - z^r)\right]\end{array}\right]}{T_1(z)\psi_1(z)\left[z^b - \bar{Q}\big(\psi_1(z)\big)\right]}, \tag{62}$$

$$R^f(z) = \frac{\left[\begin{array}{c}\eta\bar{L}\big(T_1(z)\big)\left[1-\bar{R}^f\big(T_1(z)\big)\right]\left[1-\bar{Q}\big(\psi_1(z)\big)\right]\times \\ \left[\sum_{r=0}^{a-1} q_r z^r\left[\bar{F}\big(T_1(z)\big)-1\right] + \sum_{r=a}^{b-1}\omega_r(z^b - z^r)\right]\end{array}\right]}{T_1(z)\psi_1(z)\left[z^b - \bar{Q}\big(\psi_1(z)\big)\right]}, \tag{63}$$

$$R^s(z) = \frac{\begin{bmatrix} \eta\bar{L}\big(T_1(z)\big)\bar{R}^f\big(T_1(z)\big)\big[1-\bar{R}^s\big(T_1(z)\big)\big]\big[1-\bar{Q}\big(\psi_1(z)\big)\big] \times \\ \big[\sum_{r=0}^{a-1} q_r z^r\big[\bar{F}\big(T_1(z)\big)-1\big] + \sum_{r=a}^{b-1} \omega_r(z^b - z^r)\big] \end{bmatrix}}{T_1(z)\psi_1(z)\big[z^b - \bar{Q}\big(\psi_1(z)\big)\big]}. \tag{64}$$

Finally, the PGF of queue size in steady state is:

$$P(z) = Q(z) + F(z) + L(z) + R^f(z) + R^s(z) \tag{65}$$

Substituting equations (60) to (64) in equation (65), we get

$$P(z) = \frac{\begin{bmatrix} [\sum_{r=0}^{a-1} q_r z^r(\bar{F}(T_1(z)) - 1) + \sum_{r=a}^{b-1} \omega_r(z^b - z^r)][1-\bar{Q}(\psi_1(z))] \\ [T_1(z) + \eta(1-\bar{L}(T_1(z))) + \eta(1-\bar{R}^f(T_1(z)))\bar{L}(T_1(z)) \\ +\eta(1-\bar{R}^s(T_1(z)))\bar{L}(T_1(z))\bar{R}^f(T_1(z))] \\ -\sum_{r=0}^{a-1} q_r z^r(\bar{F}(T_1(z)) - 1)\psi_1(z)[z^b - \bar{Q}(\psi_1(z))] \end{bmatrix}}{T_1(z)\psi_1(z)[z^b - \bar{Q}(\psi_1(z))]}. \tag{66}$$

where ω_r, q_r, $\psi_1(z)$ and $T_1(z)$ are given in Appendix .

Equation (66) gives the PGF of the number of customers in the queue involving 'b' unknowns. By Rouche's theorem, $[z^b - \bar{Q}(\psi_1(z))]$ has b-1 zeroes inside and one on $|z| = 1$. Due to analyticity of $P(z)$, the numerator must vanish at these points, which gives 'b' equations with 'b' unknowns.

5. Stability condition

The probability generating function should satisfy the condition $P(1) = 1$. In order to check this condition, apply L'Hopital rule and equate the expression to 1.

$$E(B)(1+\eta E(S))[\lambda E(X)E(V)\sum_{r=0}^{a-1} q_r + \sum_{r=a}^{b-1} \omega_r(b-r)]$$

$$+E(V)\sum_{r=0}^{a-1} q_r[b - \lambda E(X)E(B)(1+\eta E(S))]$$

$$= \big[b - \lambda E(X)E(B)\big(1+\eta E(S)\big)\big], \tag{67}$$

since ω_r, q_r are probabilities of 'r' customers in the queue, the left hand side of the above expression must be positive. Thus $P(1) = 1$ is satisfied if $[\,z^b - \bar{Q}(\psi_1(z))] > 0$. If $\rho =$

$\lambda E(X)E(B)(1+\eta E(S))/b$, then $\rho < 1$ is the steady state condition to be satisfied for the model under consideration.

6. Particular cases

Case (i):

When there is no server breakdown, the equation (66) reduces to

$$P(z) = \frac{\left[\sum_{r=a}^{b-1} \omega_r(z^b - z^r)(\bar{Q}(T_1(z)) - 1) + \sum_{r=0}^{a-1} q_r z^r(\bar{F}(T_1(z)) - 1)(z^b - 1)\right]}{(-\lambda + \lambda C(z))[z^b - \bar{Q}(T_1(z))]}.$$

which is the PGF of Jeyakumar and Senthilnathan (2012) for an non-Markovian batch arrival bulk service queue with multiple vacations, without server breakdown and closedown time zero.

Case(ii):

If no bulk service, (i.e) *a=b=1* and there is no breakdown, then the equation (66) reduces to

$$P(z) = \frac{\left[\omega_0(z-1)(\bar{F}(T_1(z)) - 1)\right]}{(-\lambda + \lambda C(z))[z - \bar{Q}(T_1(z))]},$$

which exactly coincides with PGF of Ayyappan and Sathiya (2013) for a non-Markovian batch arrival queueing system with multiple vacation, without breakdown, no restricted admissibility and single type of service .

7. Performance Measures

7.1. Expected Queue Size

The expected queue size L_q at an arbitrary epoch is obtained by differentiating $P(z)$ at $z = 1$ and is given by

$$L_q = \left[\frac{D^{(III)}(1)N^{(IV)}(1) - N^{(III)}(1)D^{(IV)}(1)}{4(D^{(III)}(1))^2}\right], \quad (68)$$

where

$$N^{(III)}(1) = 6F_2[E(B)S_1(V_1 \sum_{r=0}^{a-1} q_r + \sum_{r=a}^{b-1} \omega_r(b-r)) + E(V) \sum_{r=0}^{a-1} q_r(b - B_1)],$$

$$D^{(III)}(1) = 6F_2(b - B_1),$$

$$N^{(IV)}(1) = 12[B_1F_4[V_1\sum_{r=0}^{a-1} q_r + \sum_{r=a}^{b-1} \omega_r(b-r)] + F_1(B_2+\eta B_3)$$

$$\times [V_1\sum_{r=0}^{a-1} q_r + \sum_{r=a}^{b-1} \omega_r(b-r)]$$

$$+B_4[V_2\sum_{r=0}^{a-1} q_r + 2V_1\sum_{r=0}^{a-1} q_r r + \sum_{r=a}^{b-1} \omega_r\big(b(b-1)-r(r-1)\big)]$$

$$+V_3\sum_{r=0}^{a-1} q_r(b(b-1)-B_2-\eta B_3) + V_1F_4(b-B_1)\sum_{r=0}^{a-1} q_r$$

$$+F_1(V_2\sum_{r=0}^{a-1} q_r + 2V_1\sum_{r=0}^{a-1} q_r r](b-B_1)],$$

$$D^{(IV)}(1) = 12[F_2(b(b-1)-B_2-\eta B_3) + (F_0F_4+F_3)(b-B_1)].$$

where $\omega_{r,}$ $q_r, B_1,\ B_2,\ B_3,\ B_4,\ V_1,\ V_2,\ V_3,\ S_1,\ F_0,\ F_1,\ F_2,\ F_3$ and F_4 are given in Appendix.

7.2. Expected waiting time in the queue

By Little's formula, the average waiting time of the batch of customers in the queue is given as,

$$W_q = \frac{L_q}{\lambda E(X)}, \tag{69}$$

where L_q is given in (68).

7.3. System state probabilities:

The system state probabilities can be obtained from equations (60) to (64).

$$\Pr[\text{the server is busy}] = Q(1) = \frac{E(B)[\sum_{r=0}^{a-1} q_r V_1 + \sum_{r=a}^{b-1} \omega_r\ (b-r)]}{b-B_1},$$

$$\Pr[\text{the server is on vacation}] = F(1) = E(V)\sum_{r=0}^{a-1} q_r,$$

$$\Pr[\text{the server is in delay time to repair}]$$

$$= L(1) = \frac{\eta E(B)E(D)[\sum_{r=0}^{a-1} q_r V_1 + \sum_{r=a}^{b-1} \omega_r \ (b-r)]}{b-B_1},$$

Pr[the server is under first phase of repair]

$$= R^f(1) = \frac{\eta E(B)E(R_1)[\sum_{r=0}^{a-1} q_r V_1 + \sum_{r=a}^{b-1} \omega_r \ (b-r)]}{b-B_1},$$

Prob[the server is under second phase of repair]

$$= R^s(1) = \frac{\eta E(B)E(R_2)[\sum_{r=0}^{a-1} q_r V_1 + \sum_{r=a}^{b-1} \omega_r \ (b-r)]}{b-B_1},$$

where ω_r, q_r,V_1 and B_1 are given in Appendix .

8. Departure Point Queue Size Distribution

In this section, we derive the PGF of the limiting distribution of queue size at the departure point. Let Q_i^+ be the steady state probability that 'i' customers are left behind in the system at a departure point. Thus, we have

$$Q_i^+ = E\int_0^\infty Q_i(u)\mu(u)du, i \geq 0,$$

where E is the nomalization constant.

Using this relationship, the PGF of Q_i^+ can be obtained.

$$Q^+(z) = E\int_0^\infty \sum_{r=0}^{\infty} Q_i(u)\mu(u)z^i du$$
$$= EQ(z,0)\bar{Q}\big(\psi_1(z)\big).$$

From $Q^+(1) = 1$, we have

$$E = \frac{E(B)(b-B_1)}{\begin{bmatrix} [(b-B_1) - E(V)\sum_{r=0}^{a-1} q_r(b-B_1) \\ -\eta E(B)[\sum_{r=0}^{a-1} q_r V_1 + \sum_{r=a}^{b-1} \omega_r(b-r)]E(S)] \end{bmatrix}}.$$

From the relation

$$Q(z) = \frac{Q(z,0)[1-\bar{Q}(\psi_1(\text{z})],}{\psi_1(\text{z})}$$

we get

$$Q^{+}(z)=\frac{E(B)(b-B_1)\psi_1(z)\bar{Q}(\psi_1(z))Q(z)}{\begin{bmatrix}[(b-B_1)-E(V)\sum_{r=0}^{a-1}q_r(b-B_1)\\ -\eta E(B)[\sum_{r=0}^{a-1}q_rV_1+\sum_{r=a}^{b-1}\omega_r(b-r)]E(S)][1-\bar{Q}(\psi_1(z))]\end{bmatrix}},$$

where $Q(z)$ is given in equation (60) and $\omega_r,\ q_r,\ V_1,\ B_1,\ E(S),\ \psi_1(z)$ are given in Appendix .

9. Special cases

In this section, some special cases of the proposed model by specifying vacation time random variable as hyper exponentially distributed and bulk service time random variable as K- Erlangian distributed are discussed.

Case (i):

Let the service time random variable be *Erlang-K* with probability density function

$$\frac{dQ(u)}{du}=\frac{(k\mu)^k u^{(k-1)}e^{(-k\mu u)}}{(k-1)!}, u>0, k\geq 0, \text{then,}$$

$$\bar{Q}(T_1(z))=[\frac{\mu k}{\mu k+T_1(z)}]^k.$$

Hence, the PGF of queue size distribution can be obtained by,

$$Q(z)=\frac{\begin{bmatrix}[\sum_{r=0}^{a-1}q_rz^r(\bar{F}(T_1(z))-1)+\sum_{r=a}^{b-1}\omega_r(z^b-z^r)][(\mu k+T_1(z))^k-(\mu k)^k]\\ [T_1(z)+\eta(1-\bar{L}(T_1(z)))+\eta(1-\bar{R}^f(T_1(z)))\bar{L}(T_1(z))\\ +\eta(1-\bar{R}^s(T_1(z))\bar{L}(T_1(z))\bar{R}^f(T_1(z))]\\ -\sum_{r=0}^{a-1}q_rz^r(\bar{F}(T_1(z))-1)\psi_1(z)[z^b(\mu k+T_1(z))^k-(\mu k)^k]\end{bmatrix}}{T_1(z)\psi_1(z)[z^b(\mu k+T_1(z))^k-(\mu k)^k]}.$$

Case (ii)

Let the vacation time random variable be *hyper exponential* with probability density function

$$v(u)=cge^{-gu}+(1-c)he^{-hu},\text{ where '}g\text{' and '}h\text{' are parameters.}$$

Then,

$$\bar{F}(T_1(z))=\frac{gc}{g+T_1(z)}+\frac{h(1-c)}{h+T_1(z)}.$$

Hence, the PGF of queue size distribution can be obtained by,

$$P(z) = \frac{\left[\begin{array}{l} [\sum_{r=0}^{a-1} q_r z^r [(\frac{gc}{g+T_1(z)} + \frac{h(1-c)}{h+T_1(z)}) - 1] + \sum_{r=a}^{b-1} \omega_r (z^b - z^r)](1 - \bar{Q}(\psi_1(z))) \\ [T_1(z) + \eta(1 - \bar{L}(T_1(z))) + \eta(1 - \bar{R}^f(T_1(z)))\bar{L}(T_1(z)) \\ +\eta(1 - \bar{R}^s(T_1(z))\bar{L}(T_1(z))\bar{R}^f(T_1(z))] \\ -\sum_{r=0}^{a-1} q_r z^r [(\frac{gc}{g+T_1(z)} + \frac{h(1-c)}{h+T_1(z)}) - 1]\psi_1(z)[z^b - \bar{Q}(\psi_1(z))] \end{array}\right]}{T_1(z)\psi_1(z)[z^b - \bar{Q}(\psi_1(z))]}.$$

10. Numerical Illustration

The unknown probabilities of the queue size distribution are deliberated using numerical techniques. The zeroes of the function $[z^b - \bar{Q}(\psi_1(z))]$ are obtained and simultaneous equations are solved.

A numerical example is analyzed with the following assumptions and notations:

1. Service time follows Erlang-2 distribution .
2. Batch size of the arrival follows geometric distribution with mean 2.
3. Vacation time and Delay time are exponential with parameter ν and β.
4. First and second phases of repair time follows exponential with parameter ζ_1 and ζ_2.
5. Minimum service capacity $a = 3$ and maximum service capacity $b = 5$.

The input parameters are chosen such that they satisfy the stability condition. The utilization factor ρ, expected queue length L_q, expected waiting time in the queue W_q, are calculated for various arrival rates, service rates, repair rates, breakdown rates and the results are tabulated.

From Table 1 to Table 4 and Figure 1 to Figure 4 the following observations are made.

1. As arrival rate λ increases, the utilization factor ρ, expected queue length L_q and expected waiting time W_q increase.
2. As service rate μ increases, the utilization factor ρ, expected queue length L_q and expected waiting time W_q decrease.
3. As repair rate ζ_1 increases, the utilization factor ρ, expected queue length L_q and

expected waiting time W_q decrease.

4. As breakdown rate η increases, the utilization factor ρ, expected queue length L_q and expected waiting time W_q increase.

Table 1: Arrival rate vs Performance measure
$a=3, b=5, \mu=5, \zeta_1=3, \zeta_2=2, \nu=10, \eta=0.5$ and $\beta=1.5$

λ	ρ	L_q	W_q
2.00	0.2800	3.1498	0.7874
2.25	0.3150	3.7947	0.8433
2.50	0.3500	4.5537	0.9107
2.75	0.3850	5.4453	0.9901
3.00	0.4200	6.4925	1.0821
3.25	0.4550	7.7240	1.1883
3.50	0.4900	9.1769	1.3110
3.75	0.5250	10.898	1.4532
4.00	0.5600	12.9525	1.6191

Table 2: Service rate *vs* Performance measure
$a=3, b=5, \lambda=3, \zeta_1=3, \zeta_2=2, \nu=10, \eta=0.5$ and $\beta=1.5$

μ	ρ	L_q	W_q
4.00	0.5250	10.010	1.6684
4.25	0.4941	8.7969	1.4661
4.50	0.4667	7.8541	1.3090
4.75	0.4421	7.1030	1.1838
5.00	0.4200	6.4925	1.0821
5.25	0.4000	5.9876	0.9979
5.50	0.3818	5.5642	0.9274
5.75	0.3652	5.2048	0.8675
6.00	0.3500	4.8963	0.8160

Table 3: Phase 1 repair rate *vs* Performance measure
$a=3, b=5, \mu=5, \lambda=2, \nu=10, \zeta_2=4, \eta=0.5$ and $\beta=1.5$

ζ_1	ρ	L_q	W_q
1	0.3133	4.0732	1.0183
2	0.2733	3.0108	0.7527
3	0.2600	2.7427	0.6857
4	0.2533	2.6235	0.6559
5	0.2493	2.5566	0.6392
6	0.2467	2.5139	0.6285
7	0.2448	2.4844	0.6211
8	0.2433	2.4627	0.6157
9	0.2422	2.4461	0.6115

Table 4: Breakdown rate *vs* Performance measure
$a = 3, b = 5, \mu = 5, \lambda = 2, \nu = 10, \zeta_1 = 3, \zeta_2 = 2$ and $\beta = 1.5$

η	ρ	L_q	W_q
0.25	0.2200	2.2661	0.5665
0.50	0.2800	3.1498	0.7874
0.75	0.3400	4.2677	1.0669
1.00	0.4000	5.6863	1.4216
1.25	0.4600	7.5017	1.8754
1.50	0.5200	9.8581	2.4645
1.75	0.5800	2.9819	3.2455
2.00	0.6400	17.2506	4.3127
2.25	0.7000	23.3441	5.8360

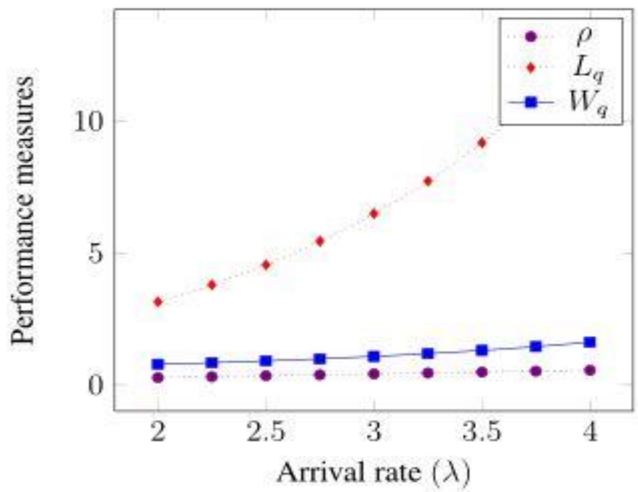

Figure 1: Arrival rate *(Vs)* Performance measures

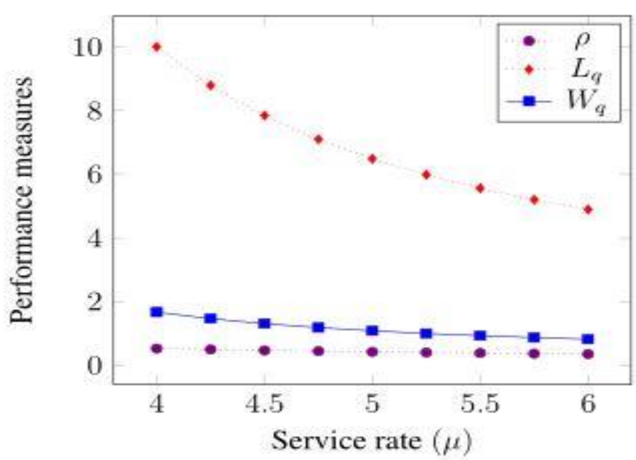

Figure 2: Service rate *(Vs)* Performance measures

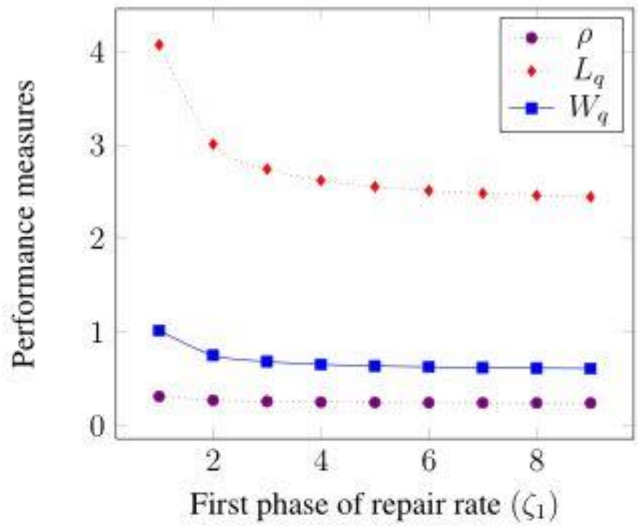

Figure 3: First phase repair rate *(Vs)* Performance measures

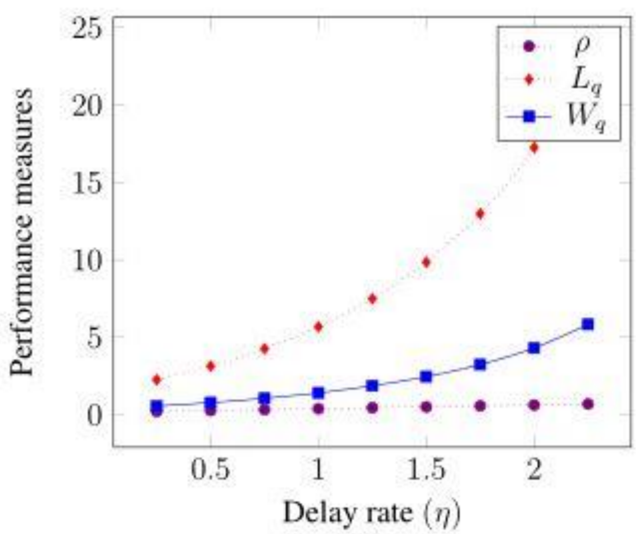

Figure 4: Delay rate *(Vs)* Performance measures

11. Conclusion

In this work, we analyzed the transient and steady state behaviour of $M^X/G(a,b)/1$ queue with breakdown and two phases of repair with delay under multiple vacation policy. We obtained the PGF of the queue length at an arbitrary and a departure epoch. Various performance measures such as the expected queue length, expected waiting time of the batch of customers spend in the queue and some particular cases were obtained. Numerical computation was performed to study the effects of the system parameters on the system characteristics. From the numerical results, it is observed that if the arrival rate, breakdown rate increase the expected queue length and waiting time of the batch of customers also increase. It is also observed that when the service rate and repair rate increase, the expected queue length and waiting time of the batch of customers decrease.

Acknowledgments

The authors are thankful to both the referees and the Editor-in-Chief Professor Aliakbar Montazer Haghighi for their valuable comments and suggestions for the improvement of the paper.

REFERENCES

Ayyappan, G. and Sathiya, K. (2013). $M^{[X]}/G/1$ queue with two phase service subject to random breakdowns, multiple vacation and restricted admissibility, *Applied Mathematical Sciences(Ruse),* Vol. 7, Issue 53, pp. 2599-2611.

Ayyappan, G. and Shyamala, S. (2014). $M^X/G_1,G_2)/1$ with Setup time, Bernoulli vacations, Breakdown and Delayed Repair, *International Journal of Stochastic Analysis,* Vol. 2014, pp. 1-12.

Ben-Israel, A. and Naor, P. (1960). A problem of delayed-service - I, *Journal of the Royal* Statistical Society, Series B (Methodological), Vol. 22, pp. 245-269.

Doshi, B. T. (1986). *Queueing systems with vacations* : *a survey*. Queueing systems 1, pp. 29-66.

Foley, R. D. and Disney, R. L. (1983). Queues with delayed feedback, *Adv. Appl. Prob.*, Vol. 15, pp.162-182.

Gautam Choudhury and Jau-Chuan Ke. (2012). A batch arrival retrial queue with general retrial times under Bernoulli vacation schedule for unreliable server and delaying repair, *Applied Mathematical Modeling,* Vol. 36, pp. 255-269.

Gautam Choudhury and Lotfi Tadj (2009). An M/G/1 queue with two phases of service subject to the server breakdown and delayed repair, *Applied Mathematical Modelling,* Vol. 33, pp. 2699-2709.

Haghighi, Aliakbar Montazer and Mishev, Dimitar P. (2016). Busy period of a single-server Poisson queueing system with splitting and batch delayed-feedback, *Int. J.Mathematics in Operational Research,* Vol. 8, Issue 2, pp. 239-257.

Haghighi, Aliakbar Montazer and Mishev, Dimitar P. (2016). *Delayed and Network Queues,* John Wiley and Sons Inc., New Jersey.

Haghighi, Aliakbar Montazer and Mishev, Dimitar P. (2013). Stochastic Three-stage Hiring Model as a Tandem Queueing Process with Bulk Arrivals and Erlang Phase-Type Selection, *International Journal of Mathematics in Operations Research (IJMOR),* Vol. 5, Issue 5,pp. 571-603.

Haghighi, Aliakbar Montazer, Chukova, Stefanka and Mishev, Dimitar P. (2011). Single Server Poisson queueing system with delayed-feedback: Part 1, *International Journal Mathematics in Operational Research (IJMOR),* Vol. 3, Issue 1. pp. 363-383.

Jeyakumar, S. and Senthilnathan, B. (2012). A study on the behaviour of the server breakdown without interruption in a $M^{[X]}/G(a,b)/1$ queueing system with multiple vacations and closedown time, *Applied Mathematics and Computation*, Vol. 219, pp. 2618-2633.

Lee, H. W., Lee, S. S., Park, J.O. and Chae, K. C. (1994). Analysis of $M^X/G/1$ queue with N-policy and Multiple vacations, *J. Appl. Prob.*, Vol. 31, pp. 467-496.

Madan, K. C. (2003). An $M/G/1$ queue with Time-Homogeneous Breakdowns and Deterministic Repair Times, *Soochow Journal of Mathematics*, Vol. 29, Issue 1, pp. 103-110.

Madan, K. C. and Ebrahim Malalla. (2015). A single server bulk input queue with Random failures and two phase repairs with delay, *Revista Investigation Operacional*, Vol. 36, Issue 1, pp. 45-59.

Manoharan, P. and Shakir Majid (2017). Stationary Analysis of a Multiserver queue with multiple working vacation and impatient customers, *Applications and Applied* Mathematics: An International Journal (AAM), Vol. 12, Issue 2, pp. 658-670.

Rajadurai, P., Saravanarajan, M. C. and Chandrasekaran, V. M. (2015). Analysis of $M^X/G/1$ retrial queue with two phases service under Bernoulli vacation schedule and random breakdown, *Mathematics in Operational Research*, Vol. 7, Issue 1, 19-31.

Rajadurai, P., Saravanarajan, M. C. and Chandrasekaran, V. M. (2017). A study on $M/G/1$ feedback retrial queue with subject to server breakdown and repair under multiple vacation policy, *Alexandria engineering Journal.*

Tadj, L. and Benkherouf, L. (1998). On a delayed-service queueing system with random server capacity and impatient customers, *Applied Statistical Science, III Commack, NY: Nova Sci. Publ.*, pp. 309-328.

Takagi, H. (1991). Queueing Analysis, A Foundation of Performance Evaluation, Vol. I, Vacation and Priority Systems, Part I, North -Holland.

APPENDIX

The following expressions are used throughout this work

$$\psi(z) = s + T(z) + \eta[1 - \bar{L}(T(z))\bar{R}^f(T(z))\bar{R}^s(T(z))],$$
$$T(z) = s + \lambda(1 - C(z)),$$
$$\psi_1(z) = T_1(z) + \eta[1 - \bar{L}(T_1(z))\bar{R}^f(T_1(z))\bar{R}^s(T_1(z))],$$
$$T_1(z) = \lambda(1 - C(z)),$$
$$q_r = \int_0^\infty [Q_r(u)\mu(u) + F_r(u)\nu(u)]du,\ 0 \le r \le a-1,$$
$$\omega_r = \int_0^\infty [Q_r(u)\mu(u) + F_r(u)\nu(u)]du,\ a \le r \le b-1,$$
$$V_1 = \lambda E(X)E(V),$$
$$V_2 = \lambda E(X^2)E(V) + \lambda^2 E(V^2)(E(X))^2,$$
$$V_3 = \lambda^2(E(X))^2 E(V)S_1,$$
$$B_1 = \lambda E(X)E(B)S_1,$$
$$B_2 = \lambda E(X^2)E(B)S_1 + \lambda^2(E(X))^2 E(B^2)S_1^2,$$
$$B_3 = \lambda^2(E(X))^2 E(B)E(S^2),$$
$$B_4 = \lambda^2(E(X))^2 E(B)S_1^2,$$
$$F_0 = \lambda E(X),$$

$$F_1 = \lambda E(X) S_1,$$
$$F_2 = \lambda^2 (E(X))^2 S_1,$$
$$F_3 = \lambda^2 E(X^2) E(X) S_1,$$
$$F_4 = \lambda E(X^2) S_1 + \eta \lambda^2 (E(X))^2 E(S^2),$$
$$S_1 = 1 + \eta\, E(S),$$
$$E(S) = E(D) + E(R^1) + E(R^2),$$
$$E(S^2) = E(D^2) + E(R^1)^2 + E(R^2)^2 + 2E(D)E(R^1)$$
$$+2E(D)E(R^2) + 2E(R^1)E(R^2),$$
$$C'(1) = E(X) \text{ and } C''(1) = E(X^2).$$