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The Finite Spectrum of Sturm-Liouville Operator With δ-Interactions

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Abstract

The goal of this paper is to study the finite spectrum of Sturm-Liouville operator with δ-interactions. Such an equation gives us a Sturm-Liouville boundary value problem which has n transmission conditions. We show that for any positive numbers m_j $(j = 0, 1, ..., n)$ that are related to number of partition of the intervals between two successive interaction points, we can construct a Sturm-Liouville equations with δ-interactions, which have exactly d eigenvalues. Where d is the sum of m_j's.

Keywords: Sturm-Liouville operator; finite spectrum; point interactions

MSC 2010 No.: 34B24, 47A10, 47A75

1. Introduction

In their article, Kong at al. have been constructed a self-adjoint and non self-adjoint Sturm-Liouville problems with exactly n eigenvalues, Kong et al. (2001). According to this paper, for every given positive integer n we can construct a Sturm-Liouville problem (SLP) which has exactly n eigenvalues. Then, this problem has been expanded to various Sturm-Liouville problems such as in Ao et al. (2011) and Ao et al. (2013). In recent decades, fourth order boundary value problems with finite spectrum has been studied in Ao et al. (2014) and Bo et al. (2014). Then, the first equivalence relation between boundary value problems with finite spectrum and matrix eigenvalue problems was found in Kong et al. (2009). For the studies about this relations please see Ao et al. (2012)-Kablan et al. (2016). The purpose of this paper is to study the Sturm-Liouville equation with δ-interaction which is formally defined by

$$-(py')' + \sum_{j=1}^{n} \alpha_j \delta(x - x_j) y + qy = \lambda wy, \text{ on } J, \tag{1}$$

where $J = (a, x_1) \cup (x_1, x_2) \cup ... \cup (x_n, b)$, $x_1, ..., x_n \in (a, b)$, with $-\infty < a < b < \infty$, α_j's are real numbers, $\delta(x)$ is the Dirac delta function and $\lambda \in \mathbb{C}$ is a spectral parameter. And we shall construct some special type Sturm-Liouville problems with δ-interactions, which have finitely many eigenvalues. Equation (1) comes from the time-independent one-dimensional Schrödinger equation. Schrödinger operators with point interactions in one or more dimensions are widely used in applications to quantum and atomic physics because they can be used as exactly solvable models in many situations Albeverio et al. (1988)-Manafov et al. (2013).

This paper consists, besides this introductory section, of three sections. Section 2 is auxiliary and presents the statement of problem and some known results. In Section 3, we construct a Sturm-Liouville equations with δ-interaction which has finitely many eigenvalues and finaly, Section 4 is devoted to some examples.

2. Statement of Problem and Notations

The equation (1) is equivalent to the following many-point boundary value problem, Albeverio et al. (1988). So we can understand problem (1) as studying the equation

$$-(py')' + qy = \lambda wy, \text{ on } J, \tag{2}$$

and n transmission conditions

$$C_j Y(x_j-) = Y(x_j+), \; Y = \begin{bmatrix} y \\ py' \end{bmatrix}, \quad j = 1, 2, ..., n, \tag{3}$$

where x_j's are inner discontinuity points and

$$C_j = \begin{bmatrix} 1 & 0 \\ \alpha_j & 1 \end{bmatrix}.$$

Additionally, let's consider the boundary condition of the form

$$AY(a) + BY(b) = 0, \quad A, B \in M_2(\mathbb{C}), \tag{4}$$

where $A = (a_{ij})_{2\times 2},\ B = (b_{ij})_{2\times 2}$ are complex valued 2×2 matrices and $M_2(\mathbb{C})$ denotes the set of square matrices of order 2 over $\mathbb{C}$. Here, the coefficients satisfy the minimal conditions

$$r = \frac{1}{p},\ q,\ w \in L(J,\ \mathbb{C}), \tag{5}$$

where $L(J,\ \mathbb{C})$ denotes the complex valued functions which are Lebesgue integrable on J. (5) is necessary and sufficiently condition for the uniqueness of the solution of the initial value problem, (see Zettl (2005)).

Let $u_1 = y,\ u_2 = py'$. Then, we have the system representation of Eq. (2)

$$u_1' = ru_2,\ \ u_2' = (q - \lambda w)u_1, \text{ on } J. \tag{6}$$

Definition 2.1.

By a trivial solution of Eq. (2) on some interval we mean a solution y which is identically zero and whose quasi-derivative $u_2 = py'$ is also identically zero on this interval.

Lemma 2.2.

Let (5) holds and let $\Phi(x, \lambda) = [\phi_{ij}(x, \lambda)]$ be the fundamental matrix solution of the system (6) determined by the initial condition $\Phi(a, \lambda) = I$. Then, $\lambda \in \mathbb{C}$ is an eigenvalue of the Sturm-Liouville problem with δ-interactions (1), (4) or equivalently the Sturm-Liouville problem with transmission conditions (2)-(4) if and only if

$$\Delta(\lambda) = \det\left[A + B\Phi(b, \lambda)\right] = 0. \tag{7}$$

Then, $\Delta(\lambda)$ can be written as

$$\begin{aligned}\Delta(\lambda) = &\det(A) + \det(B) + h_{11}\phi_{11}(b, \lambda) + h_{12}\phi_{12}(b, \lambda)\\ &+ h_{21}\phi_{21}(b, \lambda) + h_{22}\phi_{22}(b, \lambda),\end{aligned} \tag{8}$$

where

$$H = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} a_{22}b_{11} - a_{12}b_{21} & a_{11}b_{21} - a_{21}b_{11} \\ a_{22}b_{12} - a_{12}b_{22} & a_{11}b_{22} - a_{21}b_{12} \end{bmatrix}.$$

Proof:

Let's consider the linear algebra system

$$\left[A + B\Phi(b, \lambda)\right] C = 0 \tag{9}$$

and assume that $\Delta(\lambda) = 0$. Then, the system (9) has a nontrivial vector solution. If we solve the following initial value problem

$$Y' = \begin{bmatrix} 0 & \frac{1}{p} \\ q - \lambda w & 0 \end{bmatrix},\ Y = \begin{bmatrix} y \\ py' \end{bmatrix} \text{ on } J,\ Y(a) = C, \tag{10}$$

we obtain $Y(b) = \Phi(b, \lambda)Y(a)$ and $\left[A + B\Phi(b, \lambda)\right] Y(a) = 0$. So we conclude that y which is the top component of Y is an eigenfunction of the problem (2)-(3) and λ is an eigenvalue of this problem. Conversely, if λ is an eigenvalue corresponding to eigenfunction y, then Y defined in (10) satisfies

$Y(b) = \Phi(b,\lambda)Y(a)$ and consequently $[A + B\Phi(b,\lambda)]\, Y(a) = 0$. Since $Y(a)$ is an eigenfunction, it can never be zero so we have that $\det [A + B\Phi(b,\lambda)] = 0$.

On the other hand, for any $A = (a_{ij})$, $B = (b_{ij}) \in M_2(\mathbb{C})$, we know that

$$\det(A + B) = \det(A) + \det(B) + P(A, B),$$

where $P(A, B)$ denotes the sum of the possible products of the elements belonging to different rows and different columns in matrices A and B. So we have

$$\begin{aligned}\Delta(\lambda) &= \det [A + B\Phi(b,\lambda)]\\ &= \det(A) + \det(B\Phi(b,\lambda)) + P(A, B\Phi(b,\lambda)).\end{aligned}$$

Since $\Phi(a,\lambda) = I$, then $\det(\Phi(b,\lambda)) = 1$, and $P(A, B\Phi(b,\lambda))$ can be written in the form

$$P(A, B\Phi(b,\lambda)) = h_{11}\phi_{11}(b,\lambda) + h_{12}\phi_{12}(b,\lambda) + h_{21}\phi_{21}(b,\lambda) + h_{22}\phi_{22}(b,\lambda),$$

where h_{11}, h_{12}, h_{21} and h_{22} are constants which depend only on the matrices A and B. Then, we can conclude that (8) is followed. ■

Definition 2.3.

The SLP with transmission conditions (2)-(4), or equivalently (6), (3), (4) is said to be degenerate if in (8) either $\Delta(\lambda) \equiv 0$ for all $\lambda \in \mathbb{C}$ or $\Delta(\lambda) \neq 0$ for any $\lambda \in \mathbb{C}$.

3. The finite spectrum of SLP's with $\delta-$interactions

Throughout this section we assume (5) holds and there exists a partition of subintervals of J

$$\begin{aligned}a = x_0 = x_{00} < x_{01} < x_{02} < \dots < x_{0,2m_0+1} = x_1,\\ x_1 = x_{10} < x_{11} < x_{12} < \dots < x_{1,2m_1+1} = x_2,\\ \vdots\\ x_{n-1} = x_{n-1,0} < x_{n-1,1} < x_{n-1,2} < \dots < x_{n-1,2m_{n-1}+1} = x_n,\\ x_n = x_{n0} < x_{n1} < x_{n2} < \dots < x_{n,2m_n+1} = x_{n+1} = b,\end{aligned} \tag{11}$$

for some integers $m_j,\ j = 0, 1, ..., n$. Then, for each $j \in \{0, 1, ..., n\}$ we suppose that

$$r = \frac{1}{p} = 0 \text{ on } (x_{j,2k}\ ,\ x_{j,2k+1}), \qquad \int\limits_{x_{j,2k}}^{x_{j,2k+1}} w \neq 0, \quad k = 0, 1, ..., m_j, \tag{12}$$

and

$$q = w = 0 \text{ on } (x_{j,2k+1}\ ,\ x_{j,2k+2}), \qquad \int\limits_{x_{j,2k+1}}^{x_{j,2k+2}} r \neq 0, \quad k = 0, 1, ..., m_j - 1. \tag{13}$$

Further, we assign some notations we will use later. For each $j \in \{0, 1, ..., n\}$ and given (11)-(13), let

$$r_{jk} = \int_{x_{j,2k+1}}^{x_{j,2k+2}} r, \quad k = 0, 1, ..., m_j - 1, \tag{14}$$

$$q_{jk} = \int_{x_{j,2k}}^{x_{j,2k+1}} q, \quad w_{jk} = \int_{x_{j,2k}}^{x_{j,2k+1}} w, \quad k = 0, 1, ..., m_j.$$

Before stating the main theorem of this paper, we determine the structure of the principal fundamental matrix of the system (6).

Lemma 3.1.

Let (5) and (11)-(13) hold. Let $\Phi(x,\lambda) = [\phi_{st}(x,\lambda)]$ be the fundamental matrix solution of the system (6) determined by the initial condition $\Phi(x_{00},\lambda) = I$ (here $\Phi(x_{00},\lambda) = \Phi(x_{00}+,\lambda)$ denote the right limit at point x_{00}) for each $\lambda \in \mathbb{C}$. Then, we have that

$$\Phi(x_{01},\lambda) = \begin{bmatrix} 1 & 0 \\ q_{00} - \lambda w_{00} & 1 \end{bmatrix}, \tag{15}$$

$$\Phi(x_{03},\lambda) = \begin{bmatrix} 1 + (q_{00} - \lambda w_{00})r_{00} & r_{00} \\ \phi_{21}(x_{03},\lambda) & 1 + (q_{01} - \lambda w_{01})r_{00} \end{bmatrix}, \tag{16}$$

where

$$\phi_{21}(x_{03},\lambda) = (q_{00} - \lambda w_{00}) + (q_{01} - \lambda w_{01}) + (q_{00} - \lambda w_{00})(q_{01} - \lambda w_{01})r_{00}.$$

Then, in general, for $k = 1,\ 2,\ ...,\ m_0$,

$$\Phi(x_{0,2k+1},\lambda) = \begin{bmatrix} 1 & r_{0,k-1} \\ q_{0k} - \lambda w_{0k} & 1 + (q_{0k} - \lambda w_{0k})r_{0,k-1} \end{bmatrix} \Phi(x_{0,2k-1},\lambda). \tag{17}$$

Proof:

Observe from (6) that u_1 is constant on each subinterval of (x_0, x_1) where r is identically zero and u_2 is constant on each subinterval of (x_0, x_1) where both q and w are identically zero. So we obtain the result from repeated applications of (6). ■

Lemma 3.2.

Let (5) and (11)-(13) hold. Let $\Psi_j(x,\lambda) = \left[\psi_{st}^j(x,\lambda)\right]$ be the fundamental matrix solution of the system (6) determined by the initial condition $\Psi_j(x_j,\lambda) = I$ for each $j \in \{1, ..., n\}$ (here $\Psi_j(x_j,\lambda) = \Psi_j(x_j+,\lambda)$ denote the right limit at point x_j) for each $\lambda \in \mathbb{C}$. Then, for each $j \in \{1, 2, ..., n\}$ we have that

$$\Psi_j(x_{j1},\lambda) = \begin{bmatrix} 1 & 0 \\ q_{j0} - \lambda w_{j0} & 1 \end{bmatrix}, \tag{18}$$

$$\Psi_j(x_{j3},\lambda)=\begin{bmatrix} 1+(q_{j0}-\lambda w_{j0})r_{j0} & r_{j0} \\ \psi^j_{21}(x_{j3},\lambda) & 1+(q_{j1}-\lambda w_{j1})r_{j0} \end{bmatrix}, \tag{19}$$

where

$$\psi^j_{21}(x_{j3},\lambda)=(q_{j0}-\lambda w_{j0})+(q_{j1}-\lambda w_{j1})+(q_{j0}-\lambda w_{j0})(q_{j1}-\lambda w_{j1})r_{j0}.$$

Then, in general, for $k=1,2,...,m_j$,

$$\Psi_j(x_{j,2k+1},\lambda)=\begin{bmatrix} 1 & r_{j,k-1} \\ q_{jk}-\lambda w_{jk} & 1+(q_{jk}-\lambda w_{jk})r_{j,k-1} \end{bmatrix}\Psi_j(x_{j,2k-1},\lambda). \tag{20}$$

Proof:

For each $j\in\{1,2,...,n\}$ on the intervals (x_j,x_{j+1}) since the proof is similar to the proof of Lemma 3.1 we ommited it. ■

Lemma 3.3.

Let (5) and (11)-(13) hold. Let $\Phi(x,\lambda)=[\phi_{st}(x,\lambda)]$ be the fundamental matrix solution of the system (6) determined by the initial condition $\Phi(x_{00},\lambda)=I$, and $\Psi_j(x,\lambda)=\left[\psi^j_{st}(x,\lambda)\right]$ be the fundamental matrix solution of the system (6) determined by the initial condition $\Psi_j(x_j,\lambda)=I$, for each $j\in\{1,...,n\}$ and $\lambda\in\mathbb{C}$. Then, we have that

$$\Phi(b,\lambda)=\prod_{j=0}^{n}\Psi_{n-j}(x_{n-j+1},\lambda)C_{n-j} \tag{21}$$

where $C_0=I$ and $\Psi_j(x_{j+1},\lambda)=\Psi_j(x_{j+1}-,\lambda)$ denotes the left limit at point x_{j+1} for $j=1,2,...,n$.

Proof:

In this proof for the sake of simplicity we will show $\Phi(x,\lambda)$ with $\Psi_0(x,\lambda)$. From the transmission condition (3) for $j=1$ and the initial condition we have that

$$\Phi(x_1+,\lambda)=C_1\Psi_0(x_1-,\lambda).$$

Additionally, from the definition of the fundamental matrix solution we can write that

$$\Phi(x_1+,\lambda)=\left(\Psi_1(x_2,\lambda)\right)^{-1}\Phi(x_2-,\lambda).$$

Hence,

$$\Phi(x_2-,\lambda)=\Psi_1(x_2,\lambda)C_1\Psi_0(x_1,\lambda). \tag{22}$$

Now plugging (22) into the transmission condition (3) for $j=2$ and using the initial condition and definition of the fundamental matrix solution again, we arrive at the following equality

$$\Phi(x_3-,\lambda)=\Psi_2(x_3,\lambda)C_2\Psi_1(x_2,\lambda)C_1\Psi_0(x_1,\lambda).$$

After repeating these processes we obtain the result. ■

The structure of Φ given in Lemma 3.1 and mathematical induction yield the following.

Corollary 3.4.

For the fundamental matrix Φ we have that

$$\phi_{11}(b,\lambda)=R_{11}\lambda^{d}+\widetilde{\phi}_{11}(\lambda), \tag{23}$$

$$\phi_{12}(b,\lambda)=R_{12}\lambda^{d-1}+\widetilde{\phi}_{12}(\lambda), \tag{24}$$

$$\phi_{21}(b,\lambda)=R_{21}\lambda^{d+1}+\widetilde{\phi}_{21}(\lambda), \tag{25}$$

$$\phi_{22}(b,\lambda)=R_{22}\lambda^{d}+\widetilde{\phi}_{22}(\lambda), \tag{26}$$

where

$$d=\sum_{j=0}^{n}m_j, \tag{27}$$

and R_{11}, R_{12}, R_{21} and R_{22} are related to α, for each $j\in\{0,1,...,n\}$ r_{jk}, $k=0,1,...,m_j-1$; w_{jk} and q_{jk}, $k=0,1,...,m_j$. $\widetilde{\phi}_{11}(\lambda)$, $\widetilde{\phi}_{12}(\lambda)$, $\widetilde{\phi}_{21}(\lambda)$ and $\widetilde{\phi}_{22}(\lambda)$ are functions of λ, in which the degrees of λ are smaller than d, $d-1$, $d+1$ and d respectively.

Now we construct Sturm-Liouville problems with δ-interactions which have exactly d eigenvalues for each $d\in\mathbb{N}$.

Theorem 3.5.

For each $j=0,1,...,n$, let $m_j\in\mathbb{N}$ and let (5) and (11)-(13) hold. Let $H=(h_{ij})_{2\times 2}$ be defined as in Lemma 2.2 and d be defined as in (27), then:

(1) If $h_{21}\neq 0,$ then the SLP with δ-interactions (1) has exactly $d+1$ eigenvalues λ_k, $k=0,1,...,d$.

(2) If $h_{21}=0,\ h_{11}\neq 0,$ and $h_{22}\neq 0,$ then the SLP with δ-interactions (1) has exactly d eigenvalues $\lambda_k,\ k=0,1,...,d-1$.

(3) If $h_{21}=h_{11}=h_{22}=0,$ but $h_{12}\neq 0,$ then the SLP with δ-interactions (1) has exactly $d-1$ eigenvalues $\lambda_k,\ k=0,1,...,d-2$.

(4) If none of the above conditions holds, then the SLP with δ-interactions (1) either has l eigenvalues for $l\in\{1,2,...,d-2\}$ or is degenerate.

Proof:

We prove the case (1), and the other cases can be proved in the same way. From (12) that the degrees of $\phi_{11}(b,\lambda)$, $\phi_{12}(b,\lambda)$, $\phi_{21}(b,\lambda)$, and $\phi_{22}(b,\lambda)$ in λ are d, $d-1$, $d+1$, and d respectively. Thus when $h_{21}\neq 0,$ we can conclude from Corollary 3.4 amd Lemma 2.2 that the degree of the characteristic polynomial function $\Delta(\lambda)$ is $d+1$, hence from the fundamental theorem of algebra we find that $\Delta(\lambda)$ has exactly $d+1$ roots. Then, the case (1) is achieved. ■

4. Examples

We now work out a simple examples to illustrate the above study.

Example 4.1.

Consider the SLP with δ-interactions on $J = (-6, -3) \cup (-3, 2) \cup (2, 5)$,

$$-(py')' + (2\delta(x+3) + \delta(x-2))y + qy = \lambda wy. \tag{28}$$

We know that, this equation is equivalent to the following SLP

$$-(py')' + qy = \lambda wy \tag{29}$$

with transmission conditions

$$\begin{cases} y(-3-) - y(-3+) = 0 \\ 2y(-3-) + py'(-3-) - py'(-3+) = 0 \end{cases} \tag{30}$$

and

$$\begin{cases} y(2-) - y(2+) = 0 \\ y(2-) + py'(2-) - py'(2+) = 0. \end{cases} \tag{31}$$

Then, let's consider the boundary conditions

$$\begin{cases} y(-6) = 0 \\ y(5) = 0. \end{cases} \tag{32}$$

Let choose $m_0 = 1$, $m_1 = 2$ and $m_2 = 1$ and p, q, w are piecewise constant functions defined as follows:

$$p(x) = \begin{cases} \infty, & (-6,-5) \\ \frac{1}{2}, & (-5,-4) \\ \infty, & (-4,-3) \\ \infty, & (-3,-2) \\ \frac{1}{4}, & (-2,-1) \\ \infty, & (-1,0) \\ 1, & (0,1) \\ \infty, & (1,2) \\ \infty, & (2,3) \\ \frac{1}{3}, & (3,4) \\ \infty, & (4,5) \end{cases} \quad q(x) = \begin{cases} 4, & (-6,-5) \\ 0, & (-5,-4) \\ 1, & (-4,-3) \\ \frac{1}{2}, & (-3,-2) \\ 0, & (-2,-1) \\ 2, & (-1,0) \\ 0, & (0,1) \\ 1, & (1,2) \\ \frac{1}{4}, & (2,3) \\ 0, & (3,4) \\ 3, & (4,5) \end{cases} \quad w(x) = \begin{cases} 1, & (-6,-5) \\ 0, & (-5,-4) \\ 2, & (-4,-3) \\ 1, & (-3,-2) \\ 0, & (-2,-1) \\ 3, & (-1,0) \\ 0, & (0,1) \\ \frac{1}{2}, & (1,2) \\ \frac{1}{8}, & (2,3) \\ 0, & (3,4) \\ 1, & (4,5) \end{cases}$$

From condition (32)

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{33}$$

and from transmission conditions (30) and (31)

$$C_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

It follows from (21) that

$$\Phi(5,\lambda) = \Psi_2(5,\lambda)C_2\Psi_1(2,\lambda)C_1\Psi_0(-3,\lambda). \tag{34}$$

By using the matrices (33) in (8) and taking account (34) we arrive at

$$\begin{aligned}\Delta(\lambda) &= \phi_{12}(5,\lambda)\\ &= \left\{\left[\psi_{11}^2(5,\lambda)+\psi_{12}^2(5,\lambda)\right]\psi_{11}^1(2,\lambda)+\psi_{12}^2(5,\lambda)\,\psi_{21}^1(2,\lambda)\right.\\ &\quad\left.+2\left[\psi_{11}^2(5,\lambda)+\psi_{12}^2(5,\lambda)\right]\psi_{12}^1(2,\lambda)+2\psi_{12}^2(5,\lambda)\,\psi_{22}^1(2,\lambda)\right\}\psi_{12}^0(-3,\lambda)\\ &\quad+\left\{\left[\psi_{11}^2(5,\lambda)+\psi_{12}^2(5,\lambda)\right]\psi_{12}^1(2,\lambda)+\psi_{12}^2(5,\lambda)\,\psi_{22}^1(2,\lambda)\right\}\psi_{22}^0(-3,\lambda)\end{aligned}$$

After some long calculations we find that

$$\Delta(\lambda) = -135\lambda^3+\frac{2223}{2}\lambda^2-\frac{8277}{4}\lambda+\frac{2161}{2}=0.$$

As a result the SLP with δ-interactions (28) has exactly $m_0+m_1+m_2-1=d-1=3$ eigenvalues which are

$$\lambda_1 = 0.95658, \qquad \lambda_2 = 1.43138, \qquad \lambda_3 = 5.84537.$$

Example 4.2.

Consider the SLP with δ-interactions on $J=(-3,0)\cup(0,4)\cup(4,9)\cup(9,12)$,

$$-(py')'+(\delta(x-0)+\delta(x-4)+\delta(x-9))y+qy=\lambda wy. \tag{35}$$

As in the first example, this equation is equivalent to the following SLP

$$-(py')'+qy=\lambda wy \tag{36}$$

with transmission conditions

$$\begin{cases} y(0-)-y(0+)=0\\ y(0-)+py'(0-)-py'(0+)=0\end{cases}, \tag{37}$$

$$\begin{cases} y(4-)-y(4+)=0\\ y(4-)+py'(4-)-py'(4+)=0\end{cases} \tag{38}$$

and

$$\begin{cases} y(9-)-y(9+)=0\\ y(9-)+py'(9-)-py'(9+)=0.\end{cases} \tag{39}$$

Then, we can consider the following boundary conditions

$$\begin{cases} py'(-3)+py'(12)=0\\ py'(-3)=0.\end{cases} \tag{40}$$

By selecting $m_0 = 1$, $m_1 = 1$, $m_2 = 2$ and $m_3 = 1$; let's define the piecewise constant functions p, q, w are as follows:

$$p(x) = \begin{cases} \infty, (-3,-2) \\ 1, \ (-2,-1) \\ \infty, \ (-1,0) \\ \infty, \ (0,2) \\ 2, \ (2,3) \\ \infty, \ (3,4) \\ \infty, \ (4,5) \\ \frac{1}{2}, \ (5,6) \\ \infty, \ (6,7) \\ 3, \ (7,8) \\ \infty, \ (8,9) \\ \infty \ (9,10) \\ \frac{1}{3} \ (10,11) \\ \infty \ (11,12) \end{cases} \quad q(x) = \begin{cases} 1, (-3,-2) \\ 0, (-2,-1) \\ 2, \ (-1,0) \\ \frac{1}{4}, \ (0,2) \\ 0, \ (2,3) \\ 4, \ (3,4) \\ 1, \ (4,5) \\ 0, \ (5,6) \\ 3, \ (6,7) \\ 0, \ (7,8) \\ 0, \ (8,9) \\ \frac{1}{2}, \ (9,10) \\ 0, \ (10,11) \\ 1, \ (11,12) \end{cases} \quad w(x) = \begin{cases} \frac{1}{2}, (-3,-2) \\ 0, (-2,-1) \\ 1, \ (-1,0) \\ \frac{1}{2}, \ (0,2) \\ 0, \ (2,3) \\ 2, \ (3,4) \\ 2, \ (4,5) \\ 0, \ (5,6) \\ 5, \ (6,7) \\ 0, \ (7,8) \\ 1, \ (8,9) \\ 3, \ (9,10) \\ 0, \ (10,11) \\ \frac{1}{5}, \ (11,12) \end{cases}$$

From the boundary condition (40)

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{41}$$

and from the transmission conditions (37), (38) and (39)

$$C_1 = C_2 = C_3 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

It follows from (21) that

$$\Phi(12,\lambda) = \Psi_3(12,\lambda)C_3\Psi_2(9,\lambda)C_2\Psi_1(4,\lambda)C_1\Psi_0(0,\lambda). \tag{42}$$

By using the matrices (41) in (8) and taking account (42) we arrive at

$$\begin{aligned} \Delta(\lambda) &= \phi_{21}(12,\lambda) \\ &= (E+F)\,\psi^0_{11}(0,\lambda) + F\psi^0_{21}(0,\lambda). \end{aligned}$$

Here,

$$\begin{aligned} E = \big\{ &\left[\psi^3_{21}(12,\lambda) + \psi^3_{22}(12,\lambda)\right]\left[\psi^2_{11}(9,\lambda) + \psi^2_{12}(9,\lambda)\right] \\ &+\psi^3_{22}(12,\lambda)\psi^2_{21}(9,\lambda) + \psi^3_{22}(12,\lambda)\psi^2_{22}(9,\lambda)\big\}\,\psi^1_{11}(4,\lambda) \\ &+ \big\{\left[\psi^3_{21}(12,\lambda) + \psi^3_{22}(12,\lambda)\right]\psi^2_{12}(9,\lambda) + \psi^3_{22}(12,\lambda)\psi^2_{22}(9,\lambda)\big\}\,\psi^1_{21}(4,\lambda) \end{aligned}$$

and

$$\begin{aligned} F = \big\{ &\left[\psi^3_{21}(12,\lambda) + \psi^3_{22}(12,\lambda)\right]\left[\psi^2_{11}(9,\lambda) + \psi^2_{12}(9,\lambda)\right] \\ &+\psi^3_{22}(12,\lambda)\psi^2_{21}(9,\lambda) + \psi^3_{22}(12,\lambda)\psi^2_{22}(9,\lambda)\big\}\,\psi^1_{12}(4,\lambda) \\ &+ \big\{\left[\psi^3_{21}(12,\lambda) + \psi^3_{22}(12,\lambda)\right]\psi^2_{12}(9,\lambda) + \psi^3_{22}(12,\lambda)\psi^2_{22}(9,\lambda)\big\}\,\psi^1_{22}(4,\lambda). \end{aligned}$$

After the end of lengthy calculations we find that

$$\Delta(\lambda) = 16\lambda^6 - \frac{1484}{5}\lambda^5 + \frac{125389}{60}\lambda^4 - \frac{340565}{48}\lambda^3 + \frac{291595}{24}\lambda^2 - \frac{195309}{20}\lambda + \frac{8168}{3} = 0.$$

Consequently the SLP with δ-interactions (35) has exactly $m_0 + m_1 + m_2 + m_3 + 1 = d + 1 = 6$ eigenvalues which are

$$\lambda_1 = 0.56528, \quad \lambda_2 = 1.66858, \quad \lambda_3 = 1.93014,$$
$$\lambda_4 = 2.93511, \quad \lambda_5 = 4.75851, \quad \lambda_6 = 6.69238.$$

5. Conclusion

In this paper we have enlarged the scope of the Sturm-Liouville problems with finite spectrum which was devised initially for second and fourth order problems. We have extended the concept " finite spectrum" to the Sturm-Liouville operator with δ-interactions. We have presented an example to illustrate the discussion above.

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