On Refinements of Hermite-Hadamard-Fejér Type Inequalities for Fractional Integral Operators

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Abstract

In this paper, utilizing convex functions, we first establish new refinements of Hermite-Hadamard-Fejer type inequalities via Riemann-Liouville fractional integral operators. A generalized refinements of Hermite-Hadamard-Fejer type inequalities for fractional integral operators with exponential kernel is also obtained. The results given in this paper would provide extensions of those presented in earlier studies.

Keywords: Hermite-Hadamard-Fejér inequality; Fractional integral operators; Convex function

MSC 2010 No.: 26D15, 26B25, 26D10

1. Introduction

The Hermite-Hadamard inequality, which is the first fundamental result for convex mappings with a natural geometrical interpretation and many applications, has drawn attention much interest in elementary mathematics.

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature (see, e.g., Pecaric et al. (1992, p137), Dragomir and Pearce (2000) These inequalities state that if $f : I \rightarrow \mathsf{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction if f is concave. Fejér (1906) obtained the following inequality which is the weighted generalization of Hermite-Hadamard inequality (1):

Let $f : [a,b] \to \mathbb{R}$ be convex function. Then the inequality

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x) \le \int_a^b f(x)g(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)dx$$

holds, where $g : [a,b] \to \mathbb{R}$ is nonnegative, integrable and symmetric to $(a+b)/2$.

A number of mathematicians have devoted their efforts to generalise, refine, counterpart and extend these two inqualities for different classes of functions, (see, for example, Azpeitia (1994)-Farissi (2010), Hwang et al. (2014)-İşcan (2015), Kirane and Torebek (2017), Latif (2012), Noor et al. (2016), Sarikaya and Yıldırım (2016)-Yang and Hong (1997)) and the references cited therein.

2. Preliminaries

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. More details, one can consult (Gorenflo and Mainardi (1997), Kilbas et al. (2006), Miller and Ross (1993), Podlubni (1999)).

2.1. Riemann-Liouville fractional integral operators

Definition 1.1.

Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha > 0$ with $a \ge 0$ are defined by

$$I_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \quad x > a$$

and

$$I_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \quad x < b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $I_{a+}^{0}f(x) = I_{b-}^{0}f(x) = f(x)$.

It is remarkable that Sarikaya et al. (2013) first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 1.

Let $f : [a,b] \rightarrow \mathsf{R}$ be a positive function with $0 \le a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[I_{a+}^{\alpha}f(b)+I_{b-}^{\alpha}f(a)\right] \le \frac{f(a)+f(b)}{2}, \qquad (2)$$

with $\alpha > 0$.

Hermite-Hadamard-Fejér inequality for Riemann-Liouville fractional integral operators was given by İşcan (2015), as follows:

Theorem 2.

Let $f : [a,b] \rightarrow \mathsf{R}$ be convex function with with $a < b$ and $f \in L[a,b]$. If $g : [a,b] \rightarrow \mathsf{R}$ is nonnegative, integrable and symmetric with respect to $\frac{a+b}{2}$ i.e. $g(a+b-x) = g(x)$, then the following inequalities hold

$$f\left(\frac{a+b}{2}\right)\left[I_{a+}^{\alpha}(g)(b)+I_{b-}^{\alpha}(g)(a)\right] \le I_{a+}^{\alpha}(fg)(b)+I_{b-}^{\alpha}(fg)(a)$$

$$\le \frac{f(a)+f(b)}{2}\left[I_{a+}^{\alpha}(g)(b)+I_{b-}^{\alpha}(g)(a)\right].$$

The following Lemma will be frequently used to prove our results.

Lemma 1.1. (Xiang (2015), Yang and Tseng (1999))

Let $f : [a,b] \rightarrow \mathsf{R}$ be a convex function and h be defined by

$$h(t) = \frac{1}{2}\left[f\left(\frac{a+b}{2}-\frac{t}{2}\right)+f\left(\frac{a+b}{2}+\frac{t}{2}\right)\right].$$

Then, h is convex, increasing on $[0, b-a]$ and for all $t \in [0, b-a]$,

$$f\left(\frac{a+b}{2}\right) \le h(t) \le \frac{f(a)+f(b)}{2}.$$

Xiang (2015) obtained following important inequalities for the Riemann-Liouville fractional integrals utilizing the Lemma 1.1:

Theorem 3.

Let $f : [a,b] \to \mathbb{R}$ be a positive function with $a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then WH is convex and monotonically increasing on $[0,1]$ and

$$f\left(\frac{a+b}{2}\right) = WH(0) \le WH(t) \le WH(1)$$

$$= \frac{\Gamma(1+\alpha)}{2(b-a)^\alpha}\left[\left(I_{a^+}^\alpha f\right)(b) + \left(I_{b^-}^\alpha f\right)(a)\right], \tag{3}$$

with $\alpha > 0$ where

$$WH(t) = \frac{\alpha}{2(b-a)^\alpha}\int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right)\left((b-x)^{\alpha-1} + (x-a)^{\alpha-1}\right)dx.$$

Theorem 4.

Let $f : [a,b] \to \mathbb{R}$ be a positive function with $a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then WP is convex and monotonically increasing on $[0,1]$ and

$$\frac{\Gamma(1+\alpha)}{2(b-a)^\alpha}\left[\left(I_{a^+}^\alpha f\right)(b) + \left(I_{b^-}^\alpha f\right)(a)\right] = WP(0) \le WP(t) \le WP(1)$$

$$= \frac{f(a) + f(b)}{2}, \tag{4}$$

with $\alpha > 0$ where

$$WP(t) = \frac{\alpha}{4(b-a)^\alpha}\int_a^b \left[f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x\right)\left(\left(\frac{2b-a-x}{2}\right)^{\alpha-1} + \left(\frac{x-a}{2}\right)^{\alpha-1}\right)\right.$$

$$\left. + f\left(\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x\right)\left(\left(\frac{b-x}{2}\right)^{\alpha-1} + \left(\frac{x+b-2a}{2}\right)^{\alpha-1}\right)\right]dx.$$

2.2. Fractional integral operators with exponential kernel

Recently, Kirane and Torebek (2017) have introduce a new class of fractional integrals:

Definition 2.

Let $f \in L_1(a,b)$. The fractional integrals $\mathbf{I}_{a+}^\alpha$ and $\mathbf{I}_{b-}^\alpha$ of order $\alpha \in (0,1)$ are defined by

$$\mathbf{I}_{a+}^{\alpha}(f)(x) = \frac{1}{\alpha}\int_{a}^{x}\exp\left\{-\frac{1-\alpha}{\alpha}(x-t)\right\}f(t)dt, \quad x > a$$

and

$$\mathbf{I}_{b-}^{\alpha}(f)(x) = \frac{1}{\alpha}\int_{x}^{b}\exp\left\{-\frac{1-\alpha}{\alpha}(t-x)\right\}f(t)dt, \quad x < b,$$

respectively.

For $\alpha = 1$, then

$$\lim_{\alpha\to 1}\mathbf{I}_{a+}^{\alpha}(f)(x) = \int_{a}^{x}f(t)dt \text{ and } \lim_{\alpha\to 1}\mathbf{I}_{b-}^{\alpha}(f)(x) = \int_{a}^{x}f(t)dt.$$

The authors also proved the following Hermite-Hadamard and Hermite-Hadamard-Fejér inequalities for fractional integral operators with exponential kernel:

Theorem 5.

Let $f : [a,b] \to \mathbb{R}$ be a positive function with $a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold with exponential kernel:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1-\alpha}{2\left[1-\exp\{-A\}\right]}\left[\mathbf{I}_{a+}^{\alpha}(f)(b) + \mathbf{I}_{b-}^{\alpha}(f)(a)\right] \leq \frac{f(a)+f(b)}{2},$$

where $A = \frac{1-\alpha}{\alpha}(b-a)$.

Theorem 6.

Let $f : [a,b] \to \mathbb{R}$ be convex and integrable function with $a < b$. If $g : [a,b] \to \mathbb{R}$ is nonnegative, integrable and symmetric with respect to $\frac{a+b}{2}$ i.e. $g(a+b-x) = g(x)$, then the following inequalities hold

$$f\left(\frac{a+b}{2}\right)\left[\mathbf{I}_{a+}^{\alpha}(g)(b) + \mathbf{I}_{b-}^{\alpha}(g)(a)\right] \leq \mathbf{I}_{a+}^{\alpha}(fg)(b) + \mathbf{I}_{b-}^{\alpha}(fg)(a)$$

$$\leq \frac{f(a)+f(b)}{2}\left[\mathbf{I}_{a+}^{\alpha}(g)(b) + \mathbf{I}_{b-}^{\alpha}(g)(a)\right].$$

In this sutudy, we establish some Hermite-Hadamard-Fejér type inequalities via fractional integrals summarised in the above.

3. Refinements of Hermite-Hadamard-Fejér Type Inequalities for Riemann-Liouville Fractional Integral Operators

In this section, we will present refirements of Hermite-Hadamard-Fejér type inequalities via Riemann-Liouville fractional integral operators .

Theorem 7.

Let $f : [a,b] \to \mathsf{R}$ be convex function with $a < b$ and $f \in L[a,b]$. If $g : [a,b] \to \mathsf{R}$ is nonnegative,integrable and symmetric to $(a+b)/2$, then A_g is convex and monotonically increasing on $[0,1]$, then the following inequalities for Riemann-Liouville fractional integrals hold

$$f\left(\frac{a+b}{2}\right)\left[\left(I_{a^+}^\alpha g\right)(b)+\left(I_{b^-}^\alpha g\right)(a)\right] = A_g(0) \le A_g(t) \le A_g(1)$$

$$= \left[\left(I_{a^+}^\alpha fg\right)(b)+\left(I_{b^-}^\alpha fg\right)(a)\right]. \tag{5}$$

with $\alpha > 0$, where A_g is defined by

$$A_g(t) = \frac{1}{\Gamma(\alpha)}\int_a^b f(tx+(1-t)\frac{a+b}{2})\left[(b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right]g(x)dx.$$

Proof :

Firstly, t_1, t_2, $\beta \in [0,1]$, then

$$A_g((1-\beta)t_1 + \beta t_2)$$

$$= \frac{1}{\Gamma(\alpha)}\int_a^b f\left(\left(x-\frac{a+b}{2}\right)[(1-\beta)t_1+\beta t_2]+\frac{a+b}{2}\right)\left[(b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right]g(x)dx$$

$$= \frac{1}{\Gamma(\alpha)}\int_a^b f\left((1-\beta)\left[\left(x-\frac{a+b}{2}\right)t_1+\frac{a+b}{2}\right]+\beta\left(x-\frac{a+b}{2}\right)t_2+\frac{a+b}{2}\right)$$

$$\times\left[(b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right]g(x)dx.$$

Since f is convex, we have

$$A_g\left((1-\beta)t_1+\beta t_2\right)$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_a^b\left[(1-\beta)f\left(\left(x-\frac{a+b}{2}\right)t_1+\frac{a+b}{2}\right)+\beta f\left(\left(x-\frac{a+b}{2}\right)t_2+\frac{a+b}{2}\right)\right]$$

$$\times\left[(b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right]g(x)\,dx$$

$$=\frac{1-\beta}{\Gamma(\alpha)}\int_a^b\left(f(x-\frac{a+b}{2})t_1+\frac{a+b}{2}\right)\left[(b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right]g(x)\,dx$$

$$+\frac{\beta}{\Gamma(\alpha)}\int_a^b\left(f(x-\frac{a+b}{2})t_2+\frac{a+b}{2}\right)\left[(b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right]g(x)\,dx$$

$$=(1-\beta)A_g(t_1)+\beta A_g(t_2).$$

Hence, we have

$$A_g\left((1-\beta)t_1+\beta t_2\right)\leq(1-\beta)A_g(t_1)+\beta A_g(t_2).$$

then we get A_g is convex on $[0,1]$. Then, by elementary calculus and symmetricity of g, we have

$$A_g(t)=\frac{1}{\Gamma(\alpha)}\int_a^b f(tx+(1-t)\frac{a+b}{2})\left[(b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right]g(x)\,dx$$

$$=\frac{1}{\Gamma(\alpha)}\int_a^{\frac{a+b}{2}} f(tx+(1-t)\frac{a+b}{2})\left[(b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right]g(x)\,dx$$

$$+\frac{1}{\Gamma(\alpha)}\int_{\frac{a+b}{2}}^b f(tx+(1-t)\frac{a+b}{2})\left[(b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right]g(x)\,dx$$

$$=\frac{1}{2\Gamma(\alpha)}\int_0^{b-a} f\left(\frac{a+b}{2}-\frac{ut}{2}\right)\left[\left(\frac{b-a}{2}-\frac{u}{2}\right)^{\alpha-1}+\left(\frac{b-a}{2}+\frac{u}{2}\right)^{\alpha-1}\right]g\left(\frac{a+b}{2}-\frac{u}{2}\right)du$$

$$+\frac{1}{2\Gamma(\alpha)}\int_0^{b-a} f\left(\frac{a+b}{2}+\frac{ut}{2}\right)\left[\left(\frac{b-a}{2}-\frac{u}{2}\right)^{\alpha-1}+\left(\frac{b-a}{2}+\frac{u}{2}\right)^{\alpha-1}\right]g\left(\frac{a+b}{2}+\frac{u}{2}\right)du$$

$$=\frac{1}{2\Gamma(\alpha)}\int_0^{b-a} f\left[\left(\frac{a+b}{2}-\frac{ut}{2}\right)+f\left(\frac{a+b}{2}+\frac{ut}{2}\right)\right]$$

$$\times\left[\left(\frac{b-a}{2}-\frac{u}{2}\right)^{\alpha-1}+\left(\frac{b-a}{2}+\frac{u}{2}\right)^{\alpha-1}\right]g\left(\frac{a+b}{2}+\frac{u}{2}\right)du.$$

From Lemma 1.1, we have $h(u) = \frac{1}{2}\left[f\left(\frac{a+b}{2} - \frac{ut}{2}\right) + f\left(\frac{a+b}{2} + \frac{ut}{2}\right)\right]$ is increasing on $\left[0, b-a\right]$. Since $\left(\frac{b-a}{2} - \frac{u}{2}\right)^{\alpha-1} + \left(\frac{b-a}{2} + \frac{u}{2}\right)^{\alpha-1}$ is nonnegative, then $A_g(t)$ is increasing on $\left[0,1\right]$. Thus, using the facts that

$$A_g(0) = f\left(\frac{a+b}{2}\right)\left[\left(I_{a^+}^\alpha g\right)(b) + \left(I_{b^-}^\alpha g\right)(a)\right]$$

and

$$A_g(1) = \frac{1}{\Gamma(\alpha)}\int_a^b f\left(\frac{a+b}{2}\right)\left[(b-x)^{\alpha-1} + (x-a)^{\alpha-1}\right]g(x)dx$$
$$= f\left(\frac{a+b}{2}\right)\left[\left(I_{a^+}^\alpha g\right)(b) + \left(I_{b^-}^\alpha g\right)(a)\right],$$

we obtain the desired result.

Remark 1.

If we take $g(t) = 1, \quad t \in \left[a,b\right]$ in Theorem 7, then the inequalities (5) reduce to the inequalities (3).

Remark 2.

Under assumptions of Theorem 7 with $\alpha = 1$, then the mapping

$$P(t) = \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right)g(x)dx$$

is convex and monotonically increasing on $\left[0,1\right]$ and we have the following refirement of Hermite-Hadamard-Fejér inequality

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \le P(t) \le \int_a^b f(x)g(x)dx,$$

which was given by Yang and Tseng (1999).

Theorem 8.

Let $f : \left[a,b\right] \to \mathsf{R}$ be convex function with $a < b$ and $f \in L\left[a,b\right]$. If $g : \left[a,b\right] \to \mathsf{R}$ is nonnegative,integrable and symmetric to $(a+b)/2$, then B_g is convex and monotonically increasing on $\left[0,1\right]$, Then, the following inequalities for Riemann-Liouville fractional integral

holds:

$$\left[\left(I_{a^+}^\alpha fg\right)(b)+\left(I_{b^-}^\alpha fg\right)(a)\right]=B_g(0)\le B_g(t)\le B_g(1)$$

$$=\frac{f(a)+f(b)}{2}\left[\left(I_{a^+}^\alpha g\right)(b)+\left(I_{b^-}^\alpha g\right)(a)\right],\tag{6}$$

where B_g is defined by

$$B_g(t)=\frac{1}{2\Gamma(\alpha)}\int_a^b f\left((\frac{1+t}{2})a+\left(\frac{1-t}{2}\right)x\right)\left(\left(\frac{2b-a-x}{2}\right)^{\alpha-1}+\left(\frac{x-a}{2}\right)^{\alpha-1}\right)g\left(\frac{a+x}{2}\right)dx$$

$$+\frac{1}{2\Gamma(\alpha)}\int_a^b f\left((\frac{1+t}{2})b+\left(\frac{1-t}{2}\right)x\right)\left(\left(\frac{b-x}{2}\right)^{\alpha-1}+\left(\frac{x+b-2a}{2}\right)^{\alpha-1}\right)g\left(\frac{x+b}{2}\right)dx.$$

Proof :

We note that if f is convex and v is linear, then the composition $f\circ v$ is convex. Moreover, we note that a positive constant multiple of a convex function and sum of two convex functions are convex. Therefore,

$$\left((\frac{1+t}{2})a+\left(\frac{1-t}{2}\right)x\right)\left(\left(\frac{2b-a-x}{2}\right)^{\alpha-1}+\left(\frac{x-a}{2}\right)^{\alpha-1}\right)g\left(\frac{a+x}{2}\right)$$

$$+\left((\frac{1+t}{2})b+\left(\frac{1-t}{2}\right)x\right)\left(\left(\frac{b-x}{2}\right)^{\alpha-1}+\left(\frac{x+b-2a}{2}\right)^{\alpha-1}\right)g\left(\frac{x+b}{2}\right)$$

is convex. Hence, we get that $B_g(t)$ is convex. Next, by elementary calculus and symmetricity of g, we have

$$B_g(t)=\frac{1}{2\Gamma(\alpha)}\int_a^b f\left((\frac{1+t}{2})a+\left(\frac{1-t}{2}\right)x\right)\left(\left(\frac{2b-a-x}{2}\right)^{\alpha-1}+\left(\frac{x-a}{2}\right)^{\alpha-1}\right)g\left(\frac{a+x}{2}\right)dx$$

$$+\frac{1}{2\Gamma(\alpha)}\int_a^b f\left((\frac{1+t}{2})b+\left(\frac{1-t}{2}\right)x\right)\left(\left(\frac{b-x}{2}\right)^{\alpha-1}+\left(\frac{x+b-2a}{2}\right)^{\alpha-1}\right)g\left(\frac{x+b}{2}\right)dx$$

$$=\frac{1}{2\Gamma(\alpha)}\int_0^{b-a} f\left(a+\left(\frac{1-t}{2}\right)u\right)\left(\left(\frac{2b-2a-u}{2}\right)^{\alpha-1}+\left(\frac{u}{2}\right)^{\alpha-1}\right)g\left(\frac{2a+u}{2}\right)du$$

$$+\frac{1}{2\Gamma(\alpha)}\int_0^{b-a} f\left(b-\left(\frac{1-t}{2}\right)u\right)\left(\left(\frac{2b-2a-u}{2}\right)^{\alpha-1}+\left(\frac{u}{2}\right)^{\alpha-1}\right)g\left(\frac{2b+u}{2}\right)du.$$

As g is symmetric to $(a+b)/2$,

$$B_g(t) = \frac{1}{2\Gamma(\alpha)} \int_0^{b-a} \left[f\left(a + \left(\frac{1-t}{2} \right)u \right) + f\left(b - \left(\frac{1-t}{2} \right)u \right) \right]$$
$$\times \left[\left(\frac{2b-2a-u}{2} \right)^{\alpha-1} + \left(\frac{u}{2} \right)^{\alpha-1} \right] g\left(\frac{2a+u}{2} \right) du.$$

It follows that from Lemma 1.1 that $h(t) = \frac{1}{2}\left[f\left(\frac{a+b}{2} - \frac{t}{2} \right) + f\left(\frac{a+b}{2} + \frac{t}{2} \right) \right]$ and $k(t) = b - a - (1-t)u$ are increasing on $[0,b]$ and $[0,1]$, respectively. Thus, $h(k(t)) = f\left(a + \left(\frac{1-t}{2} \right)u \right) + f\left(b - \left(\frac{1-t}{2} \right)u \right)$ is increasing on $[0,1]$. Since $\left[\left(\frac{2b-2a-u}{2} \right)^{\alpha-1} + \left(\frac{u}{2} \right)^{\alpha-1} \right]$ and g are nonnegative and , then we deduce that B_g is monotonically increasing on $[0,1]$. Then,

$$B_g(0) = \frac{1}{2\Gamma(\alpha)} \int_a^b f\left(\frac{x+a}{2} \right) \left(\left(\frac{2b-a-x}{2} \right)^{\alpha-1} + \left(\frac{x-a}{2} \right)^{\alpha-1} \right) g\left(\frac{a+x}{2} \right) dx$$
$$+ \frac{1}{2\Gamma(\alpha)} \int_a^b f\left(\frac{x+b}{2} \right) \left(\left(\frac{x+b-2a}{2} \right)^{\alpha-1} + \left(\frac{b-x}{2} \right)^{\alpha-1} \right) g\left(\frac{x+b}{2} \right) dx$$
$$= \frac{1}{\Gamma(\alpha)} \int_a^{\frac{a+b}{2}} f(u)g(u)\left[(b-u)^{\alpha-1} + (u-a)^{\alpha-1} \right] du$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{\frac{a+b}{2}}^b f(u)g(u)\left[(b-u)^{\alpha-1} + (u-a)^{\alpha-1} \right] du$$
$$= J_{a^+}^\alpha fg(b) + J_{b^-}^\alpha fg(a)$$

and

$$B_g(1) = \frac{f(a)}{2\Gamma(\alpha)} \int_a^b \left[f\left(\frac{2b-a-x}{2} \right)^{\alpha-1} + \left(\frac{x-a}{2} \right)^{\alpha-1} \right] g\left(\frac{a+x}{2} \right) dx$$
$$+ \frac{f(a)}{2\Gamma(\alpha)} \int_a^b \left[f\left(\frac{x+b-2a}{2} \right)^{\alpha-1} + \left(\frac{b-x}{2} \right)^{\alpha-1} \right] g\left(\frac{x+b}{2} \right) dx$$

$$= \frac{f(a)}{\Gamma(\alpha)} \int_{a}^{\frac{a+b}{2}} \left[(b-u)^{\alpha-1} + (u-a)^{\alpha-1} \right] g(u) du$$

$$+ \frac{f(b)}{\Gamma(\alpha)} \int_{\frac{a+b}{2}}^{b} \left[(b-u)^{\alpha-1} + (u-a)^{\alpha-1} \right] g(u) du$$

$$= f(a) \left[\frac{J_{a^+}^{\alpha} g(b) + J_{b^-}^{\alpha} g(a)}{2} \right] + f(b) \left[\frac{J_{a^+}^{\alpha} g(b) + J_{b^-}^{\alpha} g(a)}{2} \right]$$

$$= \frac{f(a) + f(b)}{2} \left[J_{a^+}^{\alpha} g(b) + J_{b^-}^{\alpha} g(a) \right].$$

Thus, we obtain the required result.

Remark 3.

If we take $g(t) = 1$, $t \in [a,b]$ in Theorem 8, then the inequalities (6) reduce to the inequalities (4).

Remark 4.

Under assumptions of Theorem 8 with $\alpha = 1$, then the mapping

$$Q(t) = \frac{1}{2} \int_{a}^{b} f\left((\frac{1+t}{2})a + \left(\frac{1-t}{2} \right) x \right) g\left(\frac{a+x}{2} \right) dx + \frac{1}{2} \int_{a}^{b} f\left((\frac{1+t}{2})b + \left(\frac{1-t}{2} \right) x \right) g\left(\frac{x+b}{2} \right) dx$$

is convex and monotonically increasing on $[0,1]$ and we have the following refinement of Hermite-Hadamard-Fejér inequality

$$\int_{a}^{b} f(x) g(x) dx \leq Q(t) \leq \frac{f(a) + f(b)}{2} \int_{a}^{b} g(x) dx,$$

which was proves by Yang and Tseng (1999).

4. Refinements of Hermite-Hadamard-Fejér Type Inequalities for Fractional Integral Operators with Exponential Kernel

Throughout this section, we denote $A = \frac{1-\alpha}{\alpha}(b-a)$, $\theta_a = \frac{1-\alpha}{\alpha}(x-a)$ and $\theta_b = \frac{1-\alpha}{\alpha}(b-x)$ for $\alpha \in (0,1)$.

In this section, we will give two theorems for Hermite-Hadamard-Fejér type inequalities via fractional integral operators with exponential kernel.

Theorem 9.

Let $f : [a,b] \to \mathbb{R}$ be a positive function with $a < b$, $f \in L_1[a,b]$ and let $g : [a,b] \to \mathbb{R}$ be a nonnegative, integrable and symmetric to $(a+b)/2$. If f is a convex function on $[a,b]$, then C_g is convex and monotonically increasing on $[0,1]$ and we have the following inequalities for fractional integral operators with exponential kernel:

$$f\left(\frac{a+b}{2}\right)\left[\mathbf{I}_{a+}^{\alpha}(g)(b) + \mathbf{I}_{b-}^{\alpha}(g)(a)\right] = C_g(0) \le C_g(t) \le C_g(1)$$

$$= \left[\mathbf{I}_{a+}^{\alpha}(fg)(b) + \mathbf{I}_{b-}^{\alpha}(fg)(a)\right],$$

where

$$C_g(t) = \frac{1}{\alpha}\int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right)\left[\exp(-\theta_b) + \exp(-\theta_a)\right]g(x)dx.$$

Proof :

Convexity of C_g can be proven similar to in Theorem 7. Then, using the change of variables and symmetricity of the function g, we have

$$C_g(t) = \frac{1}{\alpha}\int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right)\left[\exp(-\theta_b) + \exp(-\theta_a)\right]g(x)dx$$

$$= \frac{1}{\alpha}\int_a^{\frac{a+b}{2}} f\left(tx + (1-t)\frac{a+b}{2}\right)\left[\exp(-\theta_b) + \exp(-\theta_a)\right]g(x)dx$$

$$+ \frac{1}{\alpha}\int_{\frac{a+b}{2}}^b f\left(tx + (1-t)\frac{a+b}{2}\right)\left[\exp(-\theta_b) + \exp(-\theta_a)\right]g(x)dx$$

$$= \frac{1}{2\alpha}\int_0^{b-a} f\left(\frac{a+b}{2} - \frac{ut}{2}\right)g\left(\frac{b+a}{2} - \frac{u}{2}\right)$$

$$\times\left[\exp\left(-\frac{1-\alpha}{\alpha}\left(\frac{b-a}{2} + \frac{u}{2}\right)\right) + \exp\left(-\frac{1-\alpha}{\alpha}\left(\frac{b-a}{2} - \frac{u}{2}\right)\right)\right]du$$

$$+ \frac{1}{2\alpha}\int_0^{b-a} f\left(\frac{a+b}{2} + \frac{ut}{2}\right)g\left(\frac{u}{2} + \frac{b+a}{2}\right)$$

$$\times\left[\exp\left(-\frac{1-\alpha}{\alpha}\left(\frac{b-a}{2} + \frac{u}{2}\right)\right) + \exp\left(-\frac{1-\alpha}{\alpha}\left(\frac{b-a}{2} - \frac{u}{2}\right)\right)\right]du$$

$$
= \frac{1}{2\alpha} \int_0^{b-a} \left[f\left(\frac{a+b}{2} - \frac{ut}{2} \right) + f\left(\frac{a+b}{2} + \frac{ut}{2} \right) \right] g\left(\frac{u}{2} + \frac{b+a}{2} \right)
$$

$$
\times \left[\exp\left(-\frac{1-\alpha}{\alpha} \left(\frac{b-a}{2} + \frac{u}{2} \right) \right) + \exp\left(-\frac{1-\alpha}{\alpha} \left(\frac{b-a}{2} - \frac{u}{2} \right) \right) \right] du.
$$

Since $\left[\exp\left(-\frac{1-\alpha}{\alpha}\left(\frac{b-a}{2} + \frac{u}{2} \right) \right) + \exp\left(-\frac{1-\alpha}{\alpha}\left(\frac{b-a}{2} - \frac{u}{2} \right) \right) \right]$ and the function g are nonnegative, then $C_g(t)$ is increasing on $[0,1]$. Therefore, with the identities

$$
C_g(0) = f\left(\frac{a+b}{2} \right) \left[\mathbf{I}_{a+}^{\alpha}(g)(b) + \mathbf{I}_{b-}^{\alpha}(g)(a) \right].
$$

and

$$
C_g(1) = \left[\mathbf{I}_{a+}^{\alpha}(fg)(b) + \mathbf{I}_{b-}^{\alpha}(fg)(a) \right],
$$

we obtain the desired result.

Corollary 1.

If we choose $g(t)=1$, $t \in [a,b]$ in Theorem 9, then C_1 is convex and monotonically increasing on $[0,1]$ and we have the following refirement of Hermite-Hadamard inequality for the fractional integral operators with exponential kernel

$$
f\left(\frac{a+b}{2} \right) \frac{2\left[1 - \exp\{-A\} \right]}{1-\alpha} = C_1(0) \le C_1(t) \le C_1(1) = \mathbf{I}_{a+}^{\alpha}(f)(b) + \mathbf{I}_{b-}^{\alpha}(f)(a).
$$

Remark 5.

Under assumptions of Theorem 9 with $\alpha = 1$, then the Theorem 9 reduces to Remark 2.

Theorem 10.

Let $f : [a,b] \to \mathsf{R}$ be a positive function with $a < b$, $f \in L_1[a,b]$ and let $g : [a,b] \to \mathsf{R}$ be a nonnegative, integrable and symmetric to $(a+b)/2$. If f is a convex function on $[a,b]$, then D_g is convex and monotonically increasing on $[0,1]$ and we have the following inequalities for fractional integral operators with exponential kernel:

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$$\left[\mathbf{I}^{\alpha}_{a+}(fg)(b)+\mathbf{I}^{\alpha}_{b-}(fg)(a)\right]=D_g(0)\le D_g(t)\le D_g(1)$$

$$=\frac{f(a)+f(b)}{2}\left[\mathbf{I}^{\alpha}_{a+}(g)(b)+\mathbf{I}^{\alpha}_{b-}(g)(a)\right],$$

where

$$D_g(t)=\frac{1}{2\alpha}\int_a^b f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)x\right)g\left(\frac{a+x}{2}\right)$$

$$\times\left[\exp\left(-\frac{1-\alpha}{\alpha}\left(\frac{2b-a-x}{2}\right)\right)+\exp\left(-\frac{1-\alpha}{\alpha}\left(\frac{x-a}{2}\right)\right)\right]dx$$

$$+\frac{1}{2\alpha}\int_a^b f\left(\left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)x\right)g\left(\frac{x+b}{2}\right)$$

$$\times\left[\exp\left(-\frac{1-\alpha}{\alpha}\left(\frac{b-x}{2}\right)\right)+\exp\left(-\frac{1-\alpha}{\alpha}\left(\frac{x+b-2a}{2}\right)\right)\right]dx.$$

Proof :

Convexity of C_g can be proven similar to in Theorem 8. Then, using the change of variables and symmetricity of g, we get

$$D_g(t)$$

$$=\frac{1}{2\alpha}\int_a^b f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)x\right)g\left(\frac{a+x}{2}\right)$$

$$\times\left[\exp\left(-\frac{1-\alpha}{\alpha}\left(\frac{2b-a-x}{2}\right)\right)+\exp\left(-\frac{1-\alpha}{\alpha}\left(\frac{x-a}{2}\right)\right)\right]dx$$

$$+\frac{1}{2\alpha}\int_a^b f\left(\left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)x\right)g\left(\frac{x+b}{2}\right)$$

$$\times\left[\exp\left(-\frac{1-\alpha}{\alpha}\left(\frac{b-x}{2}\right)\right)+\exp\left(-\frac{1-\alpha}{\alpha}\left(\frac{x+b-2a}{2}\right)\right)\right]dx$$

$$= \frac{1}{2\alpha} \int_0^{b-a} f\left(a + \left(\frac{1-t}{2}\right)u\right) \left[\exp\left(-\frac{1-\alpha}{\alpha}\left(\frac{2b-2a-u}{2}\right)\right) + \exp\left(-\frac{1-\alpha}{\alpha}\frac{u}{2}\right)\right] g\left(\frac{2a+u}{2}\right) du$$

$$+ \frac{1}{2\alpha} \int_0^{b-a} f\left(b - \left(\frac{1-t}{2}\right)u\right) \left[\exp\left(-\frac{1-\alpha}{\alpha}\frac{u}{2}\right) + \exp\left(-\frac{1-\alpha}{\alpha}\left(\frac{2b-2a-u}{2}\right)\right)\right] g\left(\frac{2b-u}{2}\right) du$$

$$= \frac{1}{2\alpha} \int_0^{b-a} \left[f\left(a + \left(\frac{1-t}{2}\right)u\right) + f\left(b - \left(\frac{1-t}{2}\right)u\right)\right]$$

$$\times \left[\exp\left(-\frac{1-\alpha}{\alpha}\left(\frac{2b-2a-u}{2}\right)\right) + \exp\left(-\frac{1-\alpha}{\alpha}\frac{u}{2}\right)\right] g\left(\frac{2a+u}{2}\right) du.$$

Considering that $\left[\exp\left(-\frac{1-\alpha}{\alpha}\left(\frac{2b-2a-u}{2}\right)\right) + \exp\left(-\frac{1-\alpha}{\alpha}\frac{u}{2}\right)\right]$ and the mapping g are nonnegative, we find that D_g is monotonically increasing on $[0,1]$. On the other hand, using change of variables we have

$$D_g(0) = \left[\mathbf{I}_{a+}^{\alpha}(fg)(b) + \mathbf{I}_{b-}^{\alpha}(fg)(a)\right]$$

and since g is symmetric with respect to $\frac{a+b}{2}$, then we have

$$D_g(1) = \frac{f(a) + f(b)}{2} \left[\mathbf{I}_{a+}^{\alpha}(g)(b) + \mathbf{I}_{b-}^{\alpha}(g)(a)\right].$$

Hence, the proof is completed.

Corollary 2.

If we choose $g(t) = 1$, $t \in [a,b]$ in Theorem 10, then D_1 is convex and monotonically increasing on $[0,1]$ and we have the following refirement of Hermite-Hadamard inequality for the fractional integral operators with exponential kernel

$$\mathbf{I}_{a+}^{\alpha}(f)(b) + \mathbf{I}_{b-}^{\alpha}(f)(a) = D_1(0) \leq D_1(t) \leq D_1(1)$$

$$= \frac{2\left[1 - \exp\{-A\}\right]}{1-\alpha} \frac{f(a) + f(b)}{2}.$$

Remark 6.

Under assumptions of Theorem 10 with $\alpha = 1$, then the Theorem 10 reduces to Remark 4.

5. Concluding Remarks

In this study, we consider the refinements of Hermite-Hadamard-Fejér type inequalities

involving Riemann-Liouville fractional integral operators and fractional integral operators with exponential kernel. The results presented in this study would provide generalizations of those given in earlier works.

REFERENCES

Azpeitia, A. G. (1994). Convex functions and the Hadamard inequality. Rev. Colombiana Math., 28, 7-12.

Budak, H., Usta, F., Sarikaya, M. Z. and Özdemir, M. E. (2017). On generalization of midpoint type inequalities with generalized fractional integral operators. ResearchGate Article, Available online at: https://www.researchgate.net/publication/312596723.

Chen, H. and Katugampola, U. N. (2017). Hermite-Hadamard and Hermite--Hadamard--Fejér type inequalities for generalized fractional integrals. J. Math. Anal. Appl. 446,1274-1291.

Dragomir, S. S. and Pearce, C. E. M. (2000). Selected topics on Hermite-Hadamard inequalities and applications. RGMIA Monographs, Victoria University, 2000.

Dragomir, S. S. (1992). Two mappings in connection to Hadamard's inequalities. J. Math. Anal. Appl., 167, 49-56.

Farissi, A. E. (2010). Simple proof and re nement of Hermite-Hadamard inequality. J. Math. Ineq., 4, 365-369.

Fejér, L. (1906). Uberdie Fourierreihen, II, Math., Naturwise. Anz Ungar. Akad. Wiss, 24, 369-390, (in Hungarian).

Gorenflo, R. and Mainardi, F. (1997). Fractional calculus: integral and differential equations of fractional order. Springer Verlag, Wien, 223-276.

Hwang, S. R., Yeh, S. Y. and Tseng, K. L. (2014). Refinements and similar extensions of Hermite--Hadamard inequality for fractional integrals and their applications. Applied Mathematics and Computation, 249, 103-113.

Iqbal, M., Qaisar, S. and Muddassar, M. (2016). A short note on integral inequality of type Hermite-Hadamard through convexity. J. Computational analaysis and applications, 21(5), 946-953.

İşcan, I. (2015) Hermite-Hadamard-Fejér type inequalities for convex functions via fractional integrals. Stud. Univ. Babeş-Bolyai Math. 60(3), 355-366.

Kilbas, A. A., Srivastava, H. M. and Trujillo, J. J.(2006). Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, 204, Elsevier Sci. B.V., Amsterdam.

Kirane, M. and Torebek,B. T. (2017). Hermite-Hadamard, Hermite-Hadamard-Fejer, Dragomir-Agarwal and Pachpatte type inequalities for convex functions via fractional integrals. arXiv:1701.00092.

Latif, M. A. (2012) .On some refinements of companions of Fejér's inequality via superquadratic functions. Proyecciones J. Math. 31(4), 309-332.

Miller, S. and Ross, B. (1993). An introduction to the Fractional Calculus and Fractional Differential Equations.John Wiley & Sons, USA, p.2.

Noor, M. A., Noor K. I. and Awan, M. U. (2016). New fractional estimates of Hermite-Hadamard inequalities and applications to means. Stud. Univ. Babe̦s-Bolyai Math. 61(2), 3-15

Pečarić, J. E., Proschan, F. and Tong, Y. L.(1992). Convex functions, partial orderings and statistical applications, Academic Press, Boston.

Podlubni, I. (1999). Fractional Differential Equations, Academic Press, San Diego.

Sarikaya, M. Z. and Yildirim, H. (2016). On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals, Miskolc Mathematical Notes, 17 (2), 1049-1059.

Sarikaya, M. Z., Set, E., Yaldiz, H. and Basak, N. (2013). Hermite -Hadamard's inequalities for fractional integrals and related fractional inequalities. Mathematical and Computer Modelling, 57,2403-2407.

Sarikaya, M. Z. and Budak, H. (2016). Generalized Hermite-Hadamard type integral inequalities for fractional integrals, Filomat 30(5), 1315-1326.

Tseng, K. L., Hwang, S. R. and Dragomir, S. S. (2010). Fejér-type inequalities (I). J. Inequal. Appl, 1-7, Article Id 531976.

Tseng, K. L., Hwang, S. R. and Dragomir, S. S. (2017). Fejér-type inequalities (II). Mathematica Slovaca, 67(1), 109-120.

Tseng, K. L., Hwang, S. R. and Dragomir, S. S. (2015). Some companions of Fejér's inequality for convex functions. RACSAM, 109, 645-656.

Tseng, K. L., Hwang, S. R. and Dragomir, S. S. (2012). Refinements of Fejér's inequality for convex functions. Period. Math. Hung., 65, 17-28.

Usta, F., Budak, H., Sarkaya, M.Z. and Set, E.(2017). On generalization of trapezoid type inequalities for s-convexfunctions with generalized fractional integral operators. Filomat, accepted.

Wang, J., Li, X. and Zhou, Y. (2016). Hermite-Hadamard Inequalities Involving Riemann-Liouville Fractional Integrals via s-convex functions and applications to special means, Filomat, 30(5), 1143-1150.

Xiang, R. (2015). Refinements of Hermite-Hadamard type inequalities for convex functions via fractional integrals. J. Appl. Math. & Informatics . 33(1-2), pp. 119-125.

Yaldiz, H. and Sarikaya, M. Z. (2017). On Hermite-Hadamard type inequalities for fractional integral operators. ResearchGate Article, Available online at: https://www.researchgate.net/publication/309824275.

Yang, G. S. and Tseng, K. L. (1999). On certain integral inequalities related to Hermite-Hadamard inequalities. J. Math. Anal. Appl., 239, 180-187.

Yang, G. S. and Hong M. C. (1997). A note on Hadamard's inequality. Tamkang J. Math., 28, 33-37.