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A numerical method for functional Hammerstein integro-differential equations

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Abstract

In this paper, a numerical method is presented to solve functional Hammerstein integro-differential equations. The presented method combines the successive approximations method with trapezoidal quadrature rule and natural cubic spline interpolation to solve the mentioned equations. The existence and uniqueness of the problem is also investigated. The convergence and numerical stability of the problem are proved, and finally, the accuracy of the method is verified by presenting some numerical computations.

Keywords: Successive approximations; Functional Hammerstein; Integro-differential equation; Trapezoidal quadrature; Spline interpolation; Nonlinear; Numerical method

MSC 2010 No.: 65R20, 97N40

1. Introduction

The field of integral and integro-differential equations is a very important subject in applied mathematics, because mathematical formulation of many physical phenomena contains integro-differential equations (Jerry (1999)). Since only a few of these equations can be solved explicitly, it is often necessary to use of numerical techniques which are appropriate combination of numerical integration and interpolation ((Linz (1985)).

There are various numerical and analytical methods to solve such problems, for example, see Adomian decomposition method (Deniz and Bildik (2014) and Wazwaz (1999)), perturbation iteration method (Bildik and Deniz (2017)) and variational iteration method (Bildik and Deniz (2015), Wang and Hi (2007), Biazar et al. (2011) and Konuralp and Sorkun (2014)). Bica et al. (2012) used Walsh functions operational matrix with Newton-Cotes nodes for solving Fredholm-Hemmerstein integro-differential equations. A collocation method based on Bessel polynomials was used for solution of the nonlinear Fredholm-Volterra Hammerstein integro-differential equations under mixed conditions by Ordokhani (2013). Brunner (1992), applied a collocation-type method for nonlinear Volterra-Hammerstein integral equations and integro-differential equations, and discussed its connection with the iterated collocation method. Turkyilmazoglu (2014) solved high-order linear Fredholm integro-differential equations by means of an elegant and accurate effective technique. A numerical method was developed by Mirzaee and Fathi (2014) to solve the Hammerstein integral equations using the least-squares approximation schemes based on Legendre-Bernstein basis. The approximate solution for the nonlinear Volterra-Fredholm-Hammerstein integral equations was obtained by using the Tau-collocation method by Gouyandeh et al. (2016). Parand and Rad (2012) presented a numerical technique based on the spectral method to solve the nonlinear Volterra-Fredholm-Hammerstein integral equations. Numerical solution of Hammerstein integral equations of mixed type by means of sinc collocation method was presented by Hashemizadeh and Rostami (2015). Rationalized Haar functions were developed to approximate the solution of the nonlinear Volterra-Fredholm Hammerstein integral equations in Ordokhania and Razzaghi (2008).

For other methods for numerical solution of Hammerstein equations, see Han (1995), Kumar and Sloan (1987), Kumar (1988), Kaneko et al. (2003), Micula (2015), Chidume and Djitt (2013), Rashidinia and Zarebnia (2007), Babolian et al. (2007), Ghoreishi and Hadizadeh (2009), Elnagar and Kazemi (1996) and Ganesh and Josm (1991).

But, only in a few papers, are the numerical solution of functional Hammerstein integral and integro-differential equations studied. In this paper, we are concerned with the functional Hammerstein integro-differential equations (FHIDEs)

$$x'(\tau) = g_1(\tau) + \int_a^b H(\tau,s).f(s,x(s),x(\varphi(s)))ds, \qquad \tau \in [a,b] \tag{1.1}$$

with supplementary condition

$$x(a) = x_0, \tag{1.2}$$

where $a,b \in \mathbb{R},\ a < b,\ H:[a,b] \times [a,b] \to \mathbb{R},\ a \le \varphi(t) \le b,\ \forall t \in [a,b]$, and $\varphi,\ g_1$ and H are continuous.

Here, we develop the numerical method of successive interpolations of Bica et al. (2012) for solving FHIDEs (1.1). To this end, we convert Equation (1.1) to a functional Hammerstein integral equation (FHIE) and apply the numerical method of successive interpolations to solve it. The rest of this paper is organized as follows.

In Section 2, some preliminary results are presented. In Section 3, we investigate the existence and uniqueness of the solution of the problem (1.1) - (1.2). Trapezoidal rule for double integrals is investigated in Section 4. We formulate the problem and present the construction of the numerical algorithm, in Section 5. In Section 6, the convergence of the method is proved. In Section 7, we study the numerical stability of the method and in Section 8, we give some examples to show the accuracy and efficiency of the presented method.

2. Preliminaries

Assuming that $f \in \mathbb{C}^2([a,b]\times\mathbb{R}\times\mathbb{R})$, $H \in \mathbb{C}^2([a,b]\times[a,b])$, $\varphi, g \in \mathbb{C}^2[a,b]$ and $\forall t \in [a,b]$: $a \leq \varphi(t) \leq b$, consider the following conditions:

(i) there exist $\alpha, \beta > 0$ such that

$$|f(s,u,v) - f(s,u',v')| \leq \alpha|u-u'| + \beta|v-v'|,$$

for all $s \in [a,b]$, $(u,v), (u',v') \in \mathbb{R}\times\mathbb{R}$,

(ii) $(b-a)^2 < \frac{1}{K(\alpha+\beta)}$ where $K \geq 0$ is such that $|H(t,s)| \leq K$ for all $(t,s) \in [a,b]\times[a,b]$.

Let $f_0: [a,b] \to \mathbb{R}$, $f_0(s) = f(s, g(s), g(\varphi(s)))$, then f_0 is continuous on the compact set $[a,b]$ due to continuity of f, g and φ and so $M_0 \geq 0$ exists, such that $|f_0(s)| \leq M_0$ for all $s \in [a,b]$.

For solving FHIDE by using of numerical method of successive interpolations, we first convert Equation (1.1) to a FHIE. To this end, by integrating of Equation (1.1) and using condition (1.2), we obtain

$$x(t) = x_0 + \int_a^t g_1(\tau)d\tau + \int_a^t \int_a^b H(\tau,s).f(s,x(s),x(\varphi(s)))dsd\tau.$$

The assumption that $g(t) = x_0 + \int_a^t g_1(\tau)d\tau$, gives

$$x(t) = g(t) + \int_a^t \int_a^b H(\tau,s).f(s,x(s),x(\varphi(s)))dsd\tau, \qquad t \in [a,b]. \tag{2.1}$$

The sequence of successive approximations for Equation (2.1) is in the following form

$$x_0(t) = g(t), \qquad \forall t \in [a,b],$$

$$x_m(t) = g(t) + \int_a^t \int_a^b H(\tau,s).f(s,x_{m-1}(s),x_{m-1}(\varphi(s)))dsd\tau, \tag{2.2}$$

$$\forall t \in [a,b], \qquad m \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}.$$

.

Theorem 2.1.

Consider the sequence of successive approximations (2.2) and let $x^* \in \mathbb{C}[a,b]$ be the exact solution of (2.1). Then the following error estimation holds

$$|x^*(t) - x_m(t)| \le \frac{[(b-a)^2 K(\alpha+\beta]^m}{1-K(b-a)(\alpha+\beta)} \cdot KM_0(b-a)^2. \tag{2.3}$$

Proof:

By (2.2), we have

$$|x_1(t) - x_0(\mathrm{t})| = \left|\int_a^t \int_a^b H(\tau,s).f(s,x_0(s),x_0(\varphi(s)))dsd\tau\right| \le KM(b-a)^2, \tag{2.4}$$

and

$$\begin{aligned}|x_m(t) - x_{m-1}(\mathrm{t})| &= |\int_a^t \int_a^b H(\tau,s).f(s,x_{m-1}(s),x_{m-1}(\varphi(s)))dsd\tau \\ &\quad - \int_a^t \int_a^b H(\tau,s).f(s,x_{m-2}(s),x_{m-2}(\varphi(s)))dsd\tau| \\ &\le K\alpha \int_a^t \int_a^b |x_{m-1}(s) - x_{m-2}(s)|dsd\tau \\ &\quad + K\beta \int_a^t \int_a^b |x_{m-1}(\varphi(s)) - x_{m-2}(\varphi(s))|dsd\tau \\ &\le K(\alpha+\beta)[K\alpha \int_a^t \int_a^b \int_a^t \int_a^b |x_{m-2}(s) - x_{m-3}(s)|dsd\tau dsd\tau \\ &\quad + K\beta \int_a^t \int_a^b \int_a^t \int_a^b |x_{m-2}(\varphi(s)) - x_{m-3}(\varphi(s))|dsd\tau dsd\tau].\end{aligned}$$

Repeating the above process yields

$$\begin{aligned}|x_m(t) - x_{m-1}(\mathrm{t})| &\le \left(K(\alpha+\beta)\right)^{m-2}[K\alpha \int_a^t \int_a^b \cdots \int_a^t \int_a^b |x_1(s) - x_0(s)|dsd\tau \cdots dsd\tau \\ &\quad + K\beta \int_a^t \int_a^b \cdots \int_a^t \int_a^b |x_1(\varphi(s)) - x_0(\varphi(s))|dsd\tau \cdots dsd\tau] \\ &\le (K(\alpha+\beta))^{m-2}[K\alpha \cdot KM_0(b-a)^2 \int_a^t \int_a^b \cdots \int_a^t \int_a^b dsd\tau \cdots dsd\tau \\ &\quad + K\beta \cdot KM_0(b-a)^2 \int_a^t \int_a^b \cdots \int_a^t \int_a^b dsd\tau \cdots dsd\tau] \\ &\le (K(\alpha+\beta))^{m-1} \cdot KM_0(b-a)^2 \cdot ((b-a)^2)^{m-1}\end{aligned} \tag{2.5}$$

Now suppose that $n > m$,

$$|x_n(t) - x_m(\mathrm{t})| \le |x_n(t) - x_{n-1}(\mathrm{t})| + |x_{mn-1}(t) - x_{n-2}(\mathrm{t})| + \cdots + |x_{m+1}(t) - x_m(\mathrm{t})|$$

and by using (2.5)

$$\begin{aligned}|x_n(t) - x_m(\mathrm{t})| &\le (K(\alpha+\beta))^{n-1} \cdot KM_0(b-a)^2 \cdot ((b-a)^2)^{n-1} \\ &\quad + (K(\alpha+\beta))^{n-2} \cdot KM_0(b-a)^2 \cdot ((b-a)^2)^{n-2} \\ &\quad + \cdots + (K(\alpha+\beta))^m \cdot KM_0(b-a)^2 \cdot ((b-a)^2)^m \\ &\le (K(b-a)^2(\alpha+\beta))^m \cdot KM_0(b-a)^2[1 + (b-a)^2 K(\alpha+\beta) \\ &\quad + \cdots + (K(b-a)^2(\alpha+\beta))^{n-1-m}],\end{aligned}$$

so

$$|x_n(t) - x_m(\text{t})| \leq \frac{1-(K(b-a)^2(\alpha+\beta))^{n-m}}{1-K(b-a)^2(\alpha+\beta)} \cdot (K(b-a)^2(\alpha+\beta))^m \cdot KM_0(b-a)^2,$$

where m is constant and $K(b-a)^2(\alpha+\beta) < 1.$ Therefore, if $n \to \infty$, we obtain

$$|x^*(t) - x_m(\text{t})| \leq \frac{(K(b-a)^2(\alpha+\beta))^m}{1-K(b-a)^2(\alpha+\beta)} \cdot KM_0(b-a)^2, \qquad \forall t \in [a,b], \qquad m \in \mathbb{N}_0, \tag{2.6}$$

which is the required bound.

3. Existence and uniqueness of the solution

In this section, the existence and uniqueness of the solution for (2.1) is presented. To this end, we extend the existence and uniqueness analysis of Linz (1985) to the presented problem.

We first, recall a useful transformation formula from Jerry (1999) which converts a multiple integral to a single integral.

Proposition 3.1.

A multiple integral can be reduced to a single integral by:

$$\int_0^t \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{n-1}} f(t_n) dt_n \cdots dt_1 = \frac{1}{(n-1)!} \int_0^t (t-x)^{n-1} f(x) dx.$$

Theorem 3.2.

Assume that in (2.1), $[a,b]$ is a finite interval and the functions g, H, φ and f are continuous on their corresponding domains and f satisfies the condition (i). Then (2.1) has a unique continuous solution.

Proof:

Consider the Equation (2.1) as

$$x_n(t) = g(t) + \int_a^t \int_a^b H(\tau, s). f(s, x_{n-1}(s), x_{n-1}(\varphi(s))) ds d\tau \tag{3.1}$$

with $x_0(t) = g(t)$. Subtracting a similar equation with n replaced by $n-1$ from (3.1), yields

$$x_n(t) - x_{n-1}(t) = \int_a^t \int_a^b H(\tau, s). [f(s, x_{n-1}(s), x_{n-1}(\varphi(s)))$$

$$-f(s,x_{n-2}(s),x_{n-2}(\varphi(s)))]dsd\tau. \tag{3.2}$$

By introducing $\psi_n(t)$ as

$$\psi_n(t)=x_n(t)-x_{n-1}(t), \tag{3.3}$$

with $\psi_0(t)=g(t)$, we have $x_n(t)=\sum_{i=0}^{n}\psi_i(t)$ and

$$\begin{aligned}|\psi_n(t)|=|x_n(t)-x_{n-1}(t)| &\leq K\int_a^t\int_a^b \alpha|x_{n-1}(t)-x_{n-2}(t)|dsd\tau\\ &+K\int_a^t\int_{\varphi(a)}^{\varphi(b)}\beta|x_{n-1}(\varphi(s))-x_{n-2}(\varphi(s))|dsd\tau\\ &\leq K(\alpha+\beta)\int_a^t\int_a^b \max_{s\in[a,b]}|x_{n-1}(s)-x_{n-2}(s)|\,dsd\tau\\ &=K(\alpha+\beta)\int_a^t\int_a^b \max_{s\in[a,b]}|\psi_{n-1}(s)|\,dsd\tau.\end{aligned}$$

By repeating this process and using Proposition 3.1, we obtain

$$|\psi_n(t)| \leq \frac{[K(\alpha+\beta)(t-a)(b-a)]^n G}{n!},$$

where $G=\max_{a\leq t\leq b}|g(t)|$. Therefore,

$$x(t)=\sum_{i=0}^{\infty}\psi_i(t) \tag{3.5}$$

exists and is a continuous function. To prove that $x(t)$ defined by (3.5) satisfies the equation (2.1), set

$$x(t)=x_n(t)+\Delta_n(t), \tag{3.6}$$

and by using (3.1), we have

$$x(t)-\Delta_n(t)= \quad g(t)+\int_a^t\int_a^b H(\tau,s).f(s,x(s)-\Delta_{n-1}(s),x(\varphi(s))-\Delta_{n-1}(\varphi(s)))dsd\tau, \tag{3.7}$$

so that

$$\begin{aligned}&x(t)-g(t)-\int_a^t\int_a^b H(\tau,s).f(s,x(s),x(\varphi(s)))dsd\tau=\Delta_n(t)\\ &+\int_a^t\int_a^b H(\tau,s).[f(s,x(s)-\Delta_{n-1}(s),x(\varphi(s))-\Delta_{n-1}(\varphi(s)))-f(s,x(s),x(\varphi(s)))]dsd\tau.\end{aligned}$$

Applying condition (i) gives

$$\left|x(t)-g(t)-\int_a^t\int_a^b H(\tau,s).f(s,x(s),x(\varphi(s)))dsd\tau\right|$$

$$\leq |\Delta_n(t)| + K\left[\alpha \int_a^t \int_a^b |\Delta_{n-1}(s)|dsd\tau + \beta \int_a^t \int_a^b |\Delta_{n-1}(\varphi(s))|dsd\tau\right]$$
$$\leq |\Delta_n(t)| + K(t-a)(b-a)(\alpha+\beta)\|\Delta_{n-1}\|,$$

where $\|\Delta_{n-1}\| = \max_{a\leq s\leq t}|\Delta_{n-1}(s)|$. But $\lim_{n\to\infty}|\Delta_{n-1}| = 0$, so that by taking n large enough, the right-hand side of the above inequality can be made as small as desired. It follows that the function $x(t)$ defined by (3.5) satisfies in Equation (2.1) and therefore is a solution of (2.1).

To show uniqueness, let $y(t)$ be another continuous solution of (2.1), then

$$|x(t)-y(t)| \leq K\int_a^t \int_a^b \alpha|x(s)-y(s)|dsd\tau + K\int_a^t \int_a^b \beta\left|x(\varphi(s)) - y(\varphi(s))\right|dsd\tau,$$

and by inserting the bound B for $|x(t)-y(t)|$ in the right hand side:

$$|x(t)-y(t)| \leq KB(t-a)(b-a)(\alpha+\beta).$$

Repeating this argument leads to

$$|x(t)-y(t)| \leq \frac{B[K(t-a)(b-a)(\alpha+\beta)]^n}{n!},$$

for any n, which yields $x(t) = y(t)$.

4. Trapezoidal rule for double integrals

In this section, we present the trapezoidal rule for double integrals with its error analysis. For this purpose, consider the integral

$$I = \int_c^d \int_a^b f(x,t)dxdt \tag{4.1}$$

and the uniform partitions of $[a,b]$, $[c,d]$ as:

$$x_i = a + ih, \qquad i = \overline{0,n}, \qquad\qquad t_j = c + jh, \qquad j = \overline{0,m},$$

with $h = \frac{(b-a)}{n}$, $k = \frac{(d-c)}{m}$ where m and n are positive integers. The trapezoidal rule for integral (4.1) is the following:

$$\int_c^d \int_a^b f(x,t)dxdt = \frac{hk}{4}\cdot \sum_{j=0}^{m-1}\sum_{i=0}^{n-1}[f(x_i,t_j) + f(x_i,t_{j+1}) + f(x_{i+1},t_j) + f(x_{i+1},t_{j+1})]$$
$$+R_{m,n}(f) = T(h,k)+R_{m,n}(f), \tag{4.2}$$

where $R_{m,n}(f)$ is error of the rule.

Theorem 4.1.

Let $f: \mathbb{C}^2([a,b] \times [c,d]) \to \mathbb{R}$ be a Lipschitzian mapping, that is

$$|f(x,t) - f(x',t)| \le L_1 |x - x'|, \tag{4.3}$$

$$|f(x,t) - f(x,t')| \le L_2 |t - t'|, \tag{4.4}$$

with $L_1, L_2 > 0$. Then the error bound for the above trapezoidal rule is as follows:

$$|R_{m,n}(f)| \le \frac{1}{6}(b-a)(d-c)(L_1 h + L_2 k). \tag{4.5}$$

Proof:

We first obtain error of the trapezoidal rule for intervals $[t_j, t_{j+1}]$, $[x_i, x_{i+1}]$. To this end, we define

$$r_{m,n}(f) = \int_{t_j}^{t_{j+1}} \int_{x_i}^{x_{i+1}} f(x,t)dxdt \\ - \frac{hk}{4} \cdot \sum_{j=0}^{m-1} \sum_{i=0}^{n-1} \left[f(x_i, t_j) + f(x_i, t_{j+1}) + f(x_{i+1}, t_j) + f(x_{i+1}, t_{j+1}) \right]. \tag{4.6}$$

We write the Taylor expansion of $f(x,t)$ at (x_i, t_j) as:

$$f(x,t) = f(x_i, t_j) + \frac{(x - x_i)}{1!} f_x(x_i, t_j) + \frac{(t - t_j)}{1!} f_t(x_i, t_j) \\ + \frac{(x - x_i)^2}{2!} f_{xx}(x_i, t_j) \\ + \frac{2(x - x_i)(t - t_j)}{2!} f_{xt}(x_i, t_j) + \frac{(t - t_j)^2}{2!} f_{tt}(x_i, t_j) \\ + \cdots \tag{4.7}$$

so

$$\int_{t_j}^{t_{j+1}} \int_{x_i}^{x_{i+1}} f(x,t)dxdt = hkf(x_i, t_j) + \frac{h^2 k}{2!} f_x(x_i, t_j) + \frac{hk^2}{2!} f_t(x_i, t_j) + \frac{h^3 k}{3!} f_{xx}(x_i, t_j) \\ + \frac{h^2 k^2}{4} f_{xt}(x_i, t_j) + \frac{hk^3}{3!} f_{tt}(x_i, t_j) + \cdots \tag{4.8}$$

Also, Taylor expansions of $f(x_i, t_{j+1})$, $f(x_{i+1}, t_j)$ and $f(x_{i+1}, t_{j+1})$ at (x_i, t_j) are as follows

$$f(x_i, t_{j+1}) = f(x_i, t_j) + kf_t(x_i, t_j) + \frac{k^2}{2!} f_{tt}(x_i, t_j) + \cdots$$

$$f(x_{i+1}, t_j) = f(x_i, t_j) + hf_x(x_i, t_j) + \frac{h^2}{2!} f_{xx}(x_i, t_j) + \cdots$$

$$f(x_{i+1},t_{j+1}) = f(x_i,t_j) + hf_x(x_i,t_j) + kf_t(x_i,t_j) + \frac{h^2}{2!}f_{xx}(x_i,t_j) + \frac{2hk}{2!}f_{xt}(x_i,t_j) + \frac{k^2}{2!}f_{tt}(x_i,t_j) + \cdots \tag{4.9}$$

and substituting (4.8) and (4.9) into (4.6), yields

$$r_{m,n}(f) = -\frac{h^3k}{12}f_{xx}(x_i,t_j) - \frac{hk^3}{12}f_{tt}(x_i,t_j) + \cdots \tag{4.10}$$

and it can be written as

$$r_{m,n}(f) = -\frac{h^3k}{12}f_{xx}(\xi_i,\theta_j) - \frac{hk^3}{12}f_{tt}(\xi_i,\theta_j),$$

where $\xi_i \in (x_i, x_{i+1})$, $\theta_j \in (t_j, t_{j+1})$.

To obtain total error, we have

$$\begin{aligned} R_{m,n}(f) &= \int_{t_0}^{t_m}\int_{x_0}^{x_n} f(x,t)dxdt - T(h,k) = \sum_{j=0}^{m-1}\sum_{i=0}^{n-1}\Big[\int_{t_j}^{t_{j+1}}\int_{x_i}^{x_{i+1}} f(x,t)dxdt \\ &\quad - \frac{hk}{4}\Big(f(x_i,t_j) + f(x_i,t_{j+1}) + f(x_{i+1},t_j) + f(x_{i+1},t_{j+1})\Big)\Big] \\ &= \sum_{j=0}^{m-1}\sum_{i=0}^{n-1}\left[-\frac{h^3k}{12}f_{xx}(\xi_i,\theta_j) - \frac{hk^3}{12}f_{tt}(\xi_i,\theta_j)\right], \end{aligned}$$

and it can be written by Intermediate Value Theorem as:

$$\begin{aligned} R_{m,n} &= -m.n\frac{h^3k}{12}f_{xx}(\xi,\theta) - m.n\frac{hk^3}{12}f_{tt}(\xi,\theta) \\ &= -\frac{(nh)(mk).h^2}{12}f_{xx}(\xi,\theta) - \frac{(nh)(mk).k^2}{12}f_{tt}(\xi,\theta) \\ &= -\frac{(b-a)(d-c)}{12}\big(h^2 f_{xx}(\xi,\theta) + k^2 f_{tt}(\xi,\theta)\big), \end{aligned} \tag{4.11}$$

where $\xi \in (x_0, x_n)$, $\theta \in (t_0, t_n)$. We can also write

$$f_{xx}(\xi,\theta) \simeq \frac{1}{h^2}[f(\xi+h,\theta) - 2f(\xi,\theta) + f(\xi-h,\theta)], \tag{4.12}$$

$$f_{tt}(\xi,\theta) \simeq \frac{1}{k^2}[f(\xi,\theta+k) - 2f(\xi,\theta) + f(\xi,\theta-k)]. \tag{4.13}$$

Therefore,

$$\begin{aligned} |f_{xx}(\xi,\theta)| &\le \frac{1}{h^2}(|f(\xi+h,\theta) - f(\xi,\theta)| + |f(\xi,\theta) + f(\xi-h,\theta)|) \\ &\le \frac{1}{h^2}(L_1h + L_1h) = \frac{2L_1}{h}, \end{aligned} \tag{4.14}$$

and

$$|f_{tt}(\xi,\theta)| \le \frac{1}{k^2}\left(|f(\xi,\theta+k)-f(\xi,\theta)|+|f(\xi,\theta)+f(\xi,\theta-k)|\right)$$
$$\le \frac{1}{k^2}(L_2k+L_2k)=\frac{2L_2}{k}, \tag{4.15}$$

and by substituting (4.14) and (4.15) into (4.11), the proof is completed.

5. Formulation of the Problem

By assuming that $[c,d]=[a,b]$, setting $t=t_i$ and applying the trapezoidal rule (4.2) to Equation (2.2), we obtain

$$\begin{aligned} x_m(t_i) = g(t_i) + \frac{h^2}{4}\cdot\sum_{k=1}^{i}\sum_{j=1}^{n}[&H(t_{k-1},t_{j-1}).f(t_{j-1},x_{m-1}(t_{j-1}),x_{m-1}(\varphi(t_{j-1})))\\ &+H(t_k,t_{j-1}).f(t_{j-1},x_{m-1}(t_{j-1}),x_{m-1}(\varphi(t_{j-1})))\\ &+H(t_{k-1},t_j).f(t_j,x_{m-1}(t_j),x_{m-1}(\varphi(t_j)))\\ &+H(t_k,t_j).f(t_j,x_{m-1}(t_j),x_{m-1}(\varphi(t_j)))]+R_{m,i}, \qquad i=\overline{0,n}, \qquad m\in\mathbb{N}. \end{aligned} \tag{5.1}$$

So, the following numerical algorithm can be presented:

$$x_0(t_i)=g(t_i), \qquad i=\overline{0,n},$$
$$\begin{aligned} x_1(t_i) = g(t_i) + \frac{h^2}{4}\cdot\sum_{k=1}^{i}\sum_{j=1}^{n}[&H(t_{k-1},t_{j-1}).f(t_{j-1},g(t_{j-1}),g(\varphi(t_{j-1})))\\ &+H(t_k,t_{j-1}).f(t_{j-1},g(t_{j-1}),g(\varphi(t_{j-1})))\\ &+H(t_{k-1},t_j).f(t_j,g(t_j),g(\varphi(t_j)))\\ &+H(t_k,t_j).f(t_j,g(t_j),g(\varphi(t_j)))]+R_{1,i}\\ =\ &\overline{x_1(t_\iota)}+R_{1,i}, \qquad i=\overline{0,n}, \end{aligned} \tag{5.2}$$

$$\begin{aligned} x_2(t_i) = g(t_i) + \frac{h^2}{4}\cdot\sum_{k=1}^{i}\sum_{j=1}^{n}[&H(t_{k-1},t_{j-1}).f(t_{j-1},\overline{x_1(t_{j-1})}+R_{1,j-1},x_1(\varphi(t_{j-1})))\\ &+H(t_k,t_{j-1}).f(t_{j-1},\overline{x_1(t_{j-1})}+R_{1,j-1},x_1(\varphi(t_{j-1})))\\ &+H(t_{k-1},t_j).f(t_j,\overline{x_1(t_j)}+R_{1,j},x_1(\varphi(t_j)))\\ &+H(t_k,t_j).f(t_j,\overline{x_1(t_j)}+R_{1,j},x_1(\varphi(t_j)))]+R_{2,i}, \qquad i=\overline{0,n}, \end{aligned} \tag{5.3}$$

by replacing $x_1(t_i)$ by $s_1(t_i)$ (where s_1 is the natural cubic spline interpolation of x_1), we obtain

$$\begin{aligned} x_2(t_i) = g(t_i) + \frac{h^2}{4}\cdot\sum_{k=1}^{i}\sum_{j=1}^{n}[&H(t_{k-1},t_{j-1}).f(t_{j-1},\overline{x_1(t_{j-1})},s_1(\varphi(t_{j-1})))\\ &+H(t_k,t_{j-1}).f(t_{j-1},\overline{x_1(t_{j-1})},s_1(\varphi(t_{j-1})))\\ &+H(t_{k-1},t_j).f(t_j,\overline{x_1(t_j)},s_1(\varphi(t_j)))\\ &+H(t_k,t_j).f(t_j,\overline{x_1(t_j)},s_1(\varphi(t_j)))]+\overline{R_{2,\iota}},\\ =\ &\overline{x_2(t_\iota)}+\overline{R_{2,\iota}}, \qquad i=\overline{0,n}, \end{aligned} \tag{5.4}$$

and for $m\ge 3$, we have

$$x_m(t_i) = g(t_i) + \frac{h^2}{4} \cdot \sum_{k=1}^{i} \sum_{j=1}^{n} [H(t_{k-1}, t_{j-1}).f(t_{j-1}, \overline{x_{m-1}(t_{j-1})} + \overline{R_{m-1,j-1}}, x_{m-1}(\varphi(t_{j-1})))$$
$$+H(t_k, t_{j-1}).f(t_{j-1}, \overline{x_{m-1}(t_{j-1})} + \overline{R_{m-1,j-1}}, x_{m-1}(\varphi(t_{j-1})))$$
$$+H(t_{k-1}, t_j).f(t_j, \overline{x_{m-1}(t_j)} + \overline{R_{m-1,j}}, x_{m-1}(\varphi(t_j)))$$
$$+H(t_k, t_j).f(t_j, \overline{x_{m-1}(t_j)} + \overline{R_{m-1,j}}, x_{m-1}(\varphi(t_j)))] + R_{m,i}, \qquad i = \overline{0,n}, \tag{5.5}$$

and by replacing $x_{m-1}(t_i)$ by $s_{m-1}(t_i)$ for $m \geq 3$ in Equation (5.5), we obtain the following algorithm:

$$x_0(t_i) = g(t_i), \qquad i = \overline{0,n},$$
$$x_m(t_i) = g(t_i) + \frac{h^2}{4} \cdot \sum_{k=1}^{i} \sum_{j=1}^{n} [H(t_{k-1}, t_{j-1}).f(t_{j-1}, \overline{x_{m-1}(t_{j-1})}, s_{m-1}(\varphi(t_{j-1})))$$
$$+H(t_k, t_{j-1}).f(t_{j-1}, \overline{x_{m-1}(t_{j-1})}, s_{m-1}(\varphi(t_{j-1})))$$
$$+H(t_{k-1}, t_j).f(t_j, \overline{x_{m-1}(t_j)}, s_{m-1}(\varphi(t_j)))$$
$$+H(t_k, t_j).f(t_j, \overline{x_{m-1}(t_j)}, s_{m-1}(\varphi(t_j)))] + \overline{R_{m,i}},$$
$$= \overline{x_m(t_i)} + \overline{R_{m,i}}, \qquad i = \overline{0,n}, \qquad m \geq 1, \tag{5.6}$$

where $s_{m-1}: [a,b] \to \mathbb{R}$, is the natural cubic spline interpolation, which interpolates the values $\overline{x_{m-1}(t_i)}$, $i = \overline{0,n}$ and having the restrictions to the intervals $[t_{i-1}, t_i]$, $i = \overline{1,n}$ (Iancu (1981) and Ahlberg et al. (1967)):

$$s_{m-1}^{(i)}(t) = [\frac{(t-t_{i-1})^2}{2} - \frac{(t-t_{i-1})^3}{6h} - \frac{h(t-t_{i-1})}{3}] \cdot M_{m-1}^{(i-1)}$$
$$+\left[\frac{(t-t_{i-1})^3}{6h} - \frac{h(t-t_{i-1})}{6}\right] \cdot M_{m-1}^{(i)} + \frac{t-t_{i-1}}{h} \cdot \overline{x_{m-1}(t_i)} + \frac{t_i - t}{h} \cdot \overline{x_{m-1}(t_{i-1})}, \tag{5.7}$$

where $M_{m-1}^{(0)} = M_{m-1}^{(n)} = 0$. And to find $M_{m-1}^{(i)}, i = \overline{1, n-1}$ we use the following algorithm:

$a_i = 2, \qquad b_i = c_i = \frac{1}{2}, \qquad d_i = \frac{3}{h^2} \cdot [\ \overline{x_{m-1}(t_{i+1})} - 2\overline{x_{m-1}(t_i)} + \overline{x_{m-1}(t_{i-1})}\ ], \qquad i = \overline{1, n-1}$

$\alpha_1 = \frac{c_1}{a_1}, \qquad \omega_i = a_i - \alpha_{i-1} \cdot b_i, \qquad \alpha_i = \frac{c_i}{\omega_i}, \qquad i = \overline{2, n-2},$

$\omega_{n-1} = a_{n-1} - \alpha_{n-2} \cdot b_{n-1}, \qquad z_1 = \frac{d_1}{2}, \qquad z_i = \frac{d_i - b_i \cdot z_{i-1}}{\omega_i}, \qquad i = \overline{2, n-1},$

and by using of the backward recurrence, we have

$$M_{m-1}^{(n-1)} = z_{n-1}, \qquad M_{m-1}^{(i)} = z_i - \alpha_i \cdot M_{m-1}^{(i+1)}, \qquad i = \overline{\mathrm{n}-2,1}.$$

Lemma 5.1. (Bica et al. (2012))

If $f: [a,b] \to \mathbb{R}$ is a uniformly continuous function and $s \in \mathbb{C}^2[a,b]$ is the cubic spline of interpolation generated by initial conditions, with natural boundary conditions $s''(a) = s''(b) =$

0, such that $s(t_i) = f(t_i) = f_i, \forall i = \overline{0,n}$, then the following error estimation holds:

$$\max_{t\in[a,b]} |s(t) - f(t)| \le \frac{7}{4}\omega(f,h), \tag{5.8}$$

where $\omega(f,h) = sup\{|f(t) - f(t')|: t, t' \in [a,b], |t - t'| \le h\}$ is the uniform modulus of continuity.

6. The convergence analysis

In this section, we extend the convergence analysis of Bica et al. (2012) to new case.

Theorem 6.1.

Under conditions (i) and (ii), the sequence $(\overline{x_m(t_i)})_{m\in\mathbb{N}_0}$ approximates the solution $x^*(t_i)$ on the knots $t_i = a + ih, i = \overline{0,n}$, with an error bound as:

$$\left|x^*(t_i) - \overline{x_m(t_i)}\right| \le \frac{(K(b-a)^2(\alpha+\beta))^m}{1-K(b-a)^2(\alpha+\beta)} \cdot KM_0(b-a)^2 + \frac{h(b-a)^2(L_1+L_2)}{6[1-K(b-a)^2(\alpha+\beta)]}$$
$$+ \frac{7K\beta(b-a)^2\omega(V_{m-1},h)}{4[1-K(b-a)^2(\alpha+\beta)]}, \qquad \forall t \in [a,b], \qquad m \in \mathbb{N}_0. \tag{6.1}$$

where V_{m-1} is defined below in (6.2).

Proof:

By using of numerical algorithm, we get

$$|x^*(t_i) - \overline{x_m(t_i)}| \le |x^*(t_i) - x_m(t_i)| + |x_m(t_i) - \overline{x_m(t_i)}|$$
$$= |x^*(t_i) - x_m(t_i)| + \left|\overline{R_{m,i}}\right|, \qquad \forall m \in \mathbb{N}, \qquad i = \overline{0,n},$$

and by (2.6), we have

$$|x^*(t) - x_m(t)| \le \frac{(K(b-a)^2(\alpha+\beta))^m}{1-K(b-a)^2(\alpha+\beta)} \cdot KM_0(b-a)^2.$$

Since $x_m(t_i) \ne \overline{x_m(t_i)}, \forall m \in \mathbb{N}_0, i = \overline{0,n}$, so s_m interpolates the values $\overline{x_m(t_i)}, i = \overline{0,n}$ but not the function x_m. Therefore, we define function $V_m: [a,b] \to \mathbb{R}, m \in \mathbb{N}$ given by its restrictions to the subinterval $[t_{i-1}, t_i], i = \overline{1,n}$, as follows

$$V_m(t) = x_m(t) + \left[\overline{x_m(t_i)} - x_m(t_i)\right] \cdot \frac{t-t_{i-1}}{h} + [\overline{x_m(t_{i-1})} - x_m(t_{i-1})] \cdot \frac{t_i-t}{h}. \tag{6.2}$$

We see that $V_m(t_i) = \overline{x_m(t_i)}, \forall i = \overline{0,n}$, that is, V_m interpolates the values $\overline{x_m(t_i)}, i = \overline{0,n}$, and it is continuous, so, s_m interpolates the function V_m for any $m \in \mathbb{N}_0$ and V_m is uniformly continuous on the compact set $[a,b]$. Then, from (5.8), we obtain

$$|V_m(t)-s_m(t)| \le \frac{7\omega(V_m,h)}{4}, \quad \forall t\in[a,b], \qquad m\in\mathbb{N}_0. \tag{6.3}$$

Now by using of numerical algorithm, we have

$$\begin{aligned}
|\overline{R_{2,i}}| &= |x_2(t_i)-\overline{x_2(t_i)}| \le |R_{2,i}| + \frac{h^2}{4}\cdot\sum_{k=1}^{i}\sum_{j=1}^{n}[\,|H(t_{k-1},t_{j-1})|\\
&.\,|f(t_{j-1},\overline{x_1(t_{j-1})}+R_{1,j-1},x_1(\varphi(t_{j-1}))) - f(t_{j-1},\overline{x_1(t_{j-1})},s_1(\varphi(t_{j-1})))|\\
&+|H(t_k,t_{j-1})|.\,|f(t_{j-1},\overline{x_1(t_{j-1})}+R_{1,j-1},x_1(\varphi(t_{j-1}))) - f(t_{j-1},\overline{x_1(t_{j-1})},s_1(\varphi(t_{j-1})))|\\
&+|H(t_{k-1},t_j)|.\,|f(t_j,\overline{x_1(t_j)}+R_{1,j},x_1(\varphi(t_j))) - f(t_j,\overline{x_1(t_j)},s_1(\varphi(t_j)))|\\
&+|H(t_k,t_j)|.\,|f(t_j,\overline{x_1(t_j)}+R_{1,j},x_1(\varphi(t_j))) - f(t_j,\overline{x_1(t_j)},s_1(\varphi(t_j)))|]\\
&\le |R_{2,i}| + \frac{h^2}{4}\cdot\sum_{k=1}^{i}\sum_{j=1}^{n}[\,K(\alpha|R_{1,j-1}| + \beta|x_1(\varphi(t_{j-1}))-s_1(\varphi(t_{j-1}))|)\\
&+K(\alpha|R_{1,j-1}| + \beta|x_1(\varphi(t_{j-1}))-s_1(\varphi(t_{j-1}))|)\\
&+K(\alpha|R_{1,j}| + \beta|x_1(\varphi(t_j))-s_1(\varphi(t_j))|)\\
&+K(\alpha|R_{1,j}| + \beta|x_1(\varphi(t_j))-s_1(\varphi(t_j))|)],
\end{aligned} \tag{6.4}$$

and for $m\ge 3$, we obtain

$$\begin{aligned}
|\overline{R_{m,i}}| &= |x_m(t_i)-\overline{x_m(t_i)}| \le |R_{m,i}| + \frac{h^2}{4}\cdot\sum_{k=1}^{i}\sum_{j=1}^{n}[\,|H(t_{k-1},t_{j-1})|\\
&.\,|f(t_{j-1},\overline{x_{m-1}(t_{j-1})}+R_{m-1,j-1},x_{m-1}(\varphi(t_{j-1}))) - f(t_{j-1},\overline{x_{m-1}(t_{j-1})},s_{m-1}(\varphi(t_{j-1})))|\\
&+|H(t_k,t_{j-1})|.\,|f(t_{j-1},\overline{x_{m-1}(t_{j-1})}+R_{m-1,j-1},x_{m-1}(\varphi(t_{j-1})))\\
&-f(t_{j-1},\overline{x_{m-1}(t_{j-1})},s_{m-1}(\varphi(t_{j-1})))|\\
&+|H(t_{k-1},t_j)|.\,|f(t_j,\overline{x_{m-1}(t_j)}+R_{m-1,j},x_{m-1}(\varphi(t_j))) - f(t_j,\overline{x_{m-1}(t_j)},s_{m-1}(\varphi(t_j)))|\\
&+|H(t_k,t_j)|.\,|f(t_j,\overline{x_{m-1}(t_j)}+R_{m-1,j},x_{m-1}(\varphi(t_j))) - f(t_j,\overline{x_{m-1}(t_j)},s_{m-1}(\varphi(t_j)))|]\\
&\le |R_{m,i}| + \frac{h^2}{4}\cdot\sum_{k=1}^{i}\sum_{j=1}^{n}[\,K(\alpha|R_{m-1,j-1}| + \beta|x_{m-1}(\varphi(t_{j-1}))-s_{m-1}(\varphi(t_{j-1}))|)\\
&+K(\alpha|R_{m-1,j-1}| + \beta|x_{m-1}(\varphi(t_{j-1}))-s_{m-1}(\varphi(t_{j-1}))|)\\
&+K(\alpha|R_{m-1,j}| + \beta|x_{m-1}(\varphi(t_j))-s_{m-1}(\varphi(t_j))|)\\
&+K(\alpha|R_{m-1,j}| + \beta|x_{m-1}(\varphi(t_j))-s_{m-1}(\varphi(t_j))|)].
\end{aligned}$$

.

Now we need to find an estimate for $|x_{m-1}(t) - s_{m-1}(t)|$. To this end, we have

$$\begin{aligned}
|x_{m-1}(t) - s_{m-1}(t)| &\leq |x_{m-1}(t) - V_{m-1}(t)| + |V_{m-1}(t) - s_{m-1}(t)| \\
&\leq \left|\frac{t-t_{i-1}}{h}\right| \cdot \left|\overline{R_{m-1,i}}\right| + \left|\frac{t_i - t}{h}\right| \cdot \left|\overline{R_{m-1,i-1}}\right| + |V_{m-1}(t) - s_{m-1}(t)| \\
&\leq max\left(\overline{R_{m-1,i-1}}, \overline{R_{m-1,i}}\right) + \frac{7}{4}\omega(V_{m-1}, h), \\
&\quad \forall t \in [t_{i-1}, t_i], \qquad \forall i = \overline{0,n}.
\end{aligned} \tag{6.5}$$

Now we can obtain a bound for $|\overline{R_{m,i}}|$. So, from (6.4) and (6.5), it follows that

$$\begin{aligned}
|\overline{R_{2,i}}| \leq |R_{2,i}| &+ \frac{h^2}{4} \cdot \textstyle\sum_{k=1}^{i} \sum_{j=1}^{n} [\\
& K\left(\alpha \cdot \frac{h}{6}(b-a)^2(L_1+L_2) + \beta(\frac{h}{6}(b-a)^2(L_1+L_2) + \frac{7}{4}\omega(V_1,h))\right) \\
& +K\left(\alpha \cdot \frac{h}{6}(b-a)^2(L_1+L_2) + \beta(\frac{h}{6}(b-a)^2(L_1+L_2) + \frac{7}{4}\omega(V_1,h))\right) \\
& +K\left(\alpha \cdot \frac{h}{6}(b-a)^2(L_1+L_2) + \beta(\frac{h}{6}(b-a)^2(L_1+L_2) + \frac{7}{4}\omega(V_1,h))\right) \\
& +K\left(\alpha \cdot \frac{h}{6}(b-a)^2(L_1+L_2) + \beta(\frac{h}{6}(b-a)^2(L_1+L_2) + \frac{7}{4}\omega(V_1,h))\right)] \\
& \leq \frac{h}{6}(b-a)^2(L_1+L_2) + \frac{(b-a)^2}{4n^2}.4n^2K \\
& .\left(\alpha \cdot \frac{h}{6}(b-a)^2(L_1+L_2) + \beta(\frac{h}{6}(b-a)^2(L_1+L_2) + \frac{7}{4}\omega(V_1,h))\right) \\
& \leq [1 + K(b-a)^2(\alpha+\beta)] \cdot \frac{h}{6}(b-a)^2(L_1+L_2) + \beta(b-a)^2 \cdot \frac{7K\omega(V_1,h)}{4}, \qquad \forall i = \overline{0,n}.
\end{aligned}$$

By induction, for $m \geq 3$, we have

$$\begin{aligned}
|\overline{R_{m,i}}| &\leq [1 + K(b-a)^2(\alpha+\beta) + \cdots + (K(b-a)^2(\alpha+\beta))^{m-1}] \\
&\cdot \frac{h}{6}(b-a)^2(L_1+L_2) + \beta(b-a)^2 \cdot \frac{7K\omega(V_{m-1},h)}{4} \\
&\cdot [1 + K(b-a)^2(\alpha+\beta) + \cdots + (K(b-a)^2(\alpha+\beta))^{m-2}] \\
&\leq \frac{h(b-a)^2(L_1+L_2)}{6[1-K(b-a)^2(\alpha+\beta)]} + \frac{7K\beta(b-a)^2\omega(V_{m-1},h)}{4[1-K(b-a)^2(\alpha+\beta)]} \ ,
\end{aligned} \tag{6.6}$$

for all $i = \overline{0,n}$. According to the contraction condition (ii), inequality (6.1) follows.

Corollary 6.2.

Under assumption of Theorem 6.1, for $n \to \infty, m \to \infty$, it follows that

$$|x^*(t_i) - \overline{x_m(t_i)}| \to 0,$$

for any $i = \overline{0,n}$, which yields the convergence of the proposed method.

Proof:

Since $\lim_{h\to 0}\omega(V_{m-1}, h) = 0$ and α, β, K, M_0 and $(b-a)$ are fixed numbers, we have

$$\lim_{h\to 0}\frac{7K\beta(b-a)^2\omega(V_{m-1},h)}{4[1-K(b-a)^2(\alpha+\beta)]} = 0, \qquad \lim_{h\to 0} \frac{h(b-a)^2(L_1+L_2)}{6[1-K(b-a)^2(\alpha+\beta)]} = 0,$$

also according to condition (ii), $((b-a)^2K(\alpha+\beta)) \leq 1$. Therefore,

$$\lim_{m\to\infty} \frac{(b-a)^{2m}[K(\alpha+\beta)^m]}{1-K(b-a)^2(\alpha+\beta)} \cdot KM_0(b-a)^2 = 0$$

7. The stability analysis

In order to prove numerical stability of the algorithm, we consider a small perturbation in the first iteration $x_0 = g$, so we consider the Equation (2.1) as

$$y(t) = h(t) + \int_a^t \int_a^b H(\tau, s). f(s, y(s), y(\varphi(s)))dsd\tau, \qquad t \in [a, b], \tag{7.1}$$

where $h \in \mathbb{C}[a, b]$ and for small $\varepsilon > 0$, $|g(t) - h(t)| < \varepsilon$. Let ρ' and M_0' be such that

$$|h(t) - h(t')| \leq \rho'|t - t'|, \qquad \forall t, t' \in [a, b],$$

$$M_0' = max\{|f(s, h(s), h(\varphi(s)))|: s \in [a, b]\}.$$

By using the presented method for (7.1), we obtain the sequence of successive approximations on the knots $t_i = a + ih$, $i = \overline{0,n}$ as:

$$y_0(t) = h(t), \qquad \forall t \in [a, b],$$
$$y_0(t_i) = h(t_i), \qquad i = \overline{0,n},$$
$$y_m(t_i) = h(t_i) + \int_a^{t_i} \int_a^b H(\tau, s) \cdot f(s, y_{m-1}(s), y_{m-1}(\varphi(s)))dsd\tau, \;\; i = \overline{0,n}\;, \;\; m \in \mathbb{N}_0,$$

the calculated values are as follows

$$y_0(t_i) = h(t_i), \qquad i = \overline{0,n}$$
$$y_m(t_i) = \overline{y_m(t_i)} + \overline{R'_{m,i}}\;, \qquad i = \overline{0,n}\;, \qquad m \in \mathbb{N}_0.$$

Therefore,

$$|x_0(t) - y_0(t)| < \varepsilon, \qquad \forall t \in [a, b].$$

Definition 7.1. (Bica et al. (2.12))

The numerical method constructed above is numerically stable, if there exist $p \in \mathbb{N}_0$, a sequence of continuous functions $\mu_m: [0, b-a] \to [0, \infty)$, $m \in \mathbb{N}_0$ with the property $lim_{h\to 0}\mu_m(h) = 0$, $\forall m \in \mathbb{N}_0$ and the constants $K_1, K_2, K_3 > 0$ independent of h, such that

$$\left|\overline{x_m(t_i)} - \overline{y_m(t_i)}\right| \le K_1\varepsilon + K_2 \cdot h^p + K_3 \cdot \mu_m(h), \qquad i = \overline{0, n}\ , \qquad m \in \mathbb{N}_0.$$

Similar to Bica et al. (2012), we have the following theorem.

Theorem 7.2.

Under conditions of Theorem 6.1, the presented method is numerically stable.

Proof:

We have

$$\begin{aligned} |\overline{x_m(t_i)} - \overline{y_m(t_i)}| &\le |\overline{x_m(t_i)} - x_m(t_i)| + |x_m(t_i) - y_m(t_i)| + |y_m(t_i) - \overline{y_m(t_i)}| \\ &\le \left|\overline{R_{m,i}}\right| + |x_m(t_i) - y_m(t_i)| + \left|\overline{R'_{m,\iota}}\right|, \end{aligned} \qquad (7.2)$$

and according to (6.6),

$$\left|\overline{R_{m,i}}\right| \le \frac{h(b-a)^2(L_1+L_2)}{6[1-K(b-a)^2(\alpha+\beta)]} + \frac{7K\beta(b-a)^2\omega(V_{m-1},h)}{4[1-K(b-a)^2(\alpha+\beta)]}\ , \qquad (7.3)$$

$$\left|\overline{R'_{m,\iota}}\right| \le \frac{h(b-a)^2(L'_1+L'_2)}{6[1-K(b-a)^2(\alpha+\beta)]} + \frac{7K\beta(b-a)^2\omega(V_{m-1},h)}{4[1-K(b-a)^2(\alpha+\beta)]}\ , \qquad (7.4)$$

where $L'_1,\ L'_2 \ge 0$ are Lipschitz constants similar as (4.3) and (4.4). Since $|x_0(t) - y_0(t)| \le \varepsilon$, $\forall t \in [a, b]$, therefore,

$$\begin{aligned} |x_1(t) - y_1(t)| \le |x_0(t) - y_0(t)| + \int_a^t \int_a^b |H(\tau, s)|.|f(s, x_0(s), x_0(\varphi(s))) \\ -f(s, y_0(s), y_0(\varphi(s)))|ds d\tau \le [1 + K(b-a)^2(\alpha+\beta)] \cdot \varepsilon, \end{aligned}$$

and for $m \ge 2$

$$|x_m(t)-y_m(t)| \le |x_0(t)-y_0(t)| + \int_a^t\int_a^b |H(\tau,s)|.|f(s,x_{m-1}(s),x_{m-1}(\varphi(s)))$$
$$-f(s,y_{m-1}(s),y_{m-1}(\varphi(s)))|dsd\tau$$
$$\le [1+K(b-a)^2(\alpha+\beta)+\left(K(b-a)^2(\alpha+\beta)\right)^2+\cdots+(K(b-a)^2(\alpha+\beta))^m]\cdot\varepsilon$$
$$=\frac{1-(K(b-a)^2(\alpha+\beta))^{m+1}}{1-K(b-a)^2(\alpha+\beta)}\cdot\varepsilon \le \frac{\varepsilon}{1-K(b-a)^2(\alpha+\beta)}\ . \tag{7.5}$$

Substituting (7.3), (7.4) and (7.5) into (7.2) gives

$$\left|\overline{x_m(t_i)}-\overline{y_m(t_i)}\right| \le \frac{\varepsilon}{1-K(b-a)^2(\alpha+\beta)}+\frac{h(b-a)^2\left(L_1+L_2+L_1'+L_2'\right)}{6[1-K(b-a)^2(\alpha+\beta)]}+\frac{7K\beta(b-a)^2\omega(V_{m-1},h)}{2[1-K(b-a)^2(\alpha+\beta)]}$$
$$\le K_1\varepsilon+K_2\cdot h+K_3\cdot\mu_m(h),$$

where

$$K_1=\frac{1}{1-K(b-a)^2(\alpha+\beta)}\ ,\qquad K_2=\frac{(b-a)^2\left(L_1+L_2+L_1'+L_2'\right)}{6[1-K(b-a)^2(\alpha+\beta)]}\ ,\qquad K_3=\frac{7K\beta(b-a)^2}{2[1-K(b-a)^2(\alpha+\beta)]}\ ,$$

with $p=1$ and $\mu_m(h)=\omega(V_{m-1},h)$.

8. Numerical examples

In this section, some examples are investigated to test the accuracy of the proposed method.

Example 1.

Consider the following FHIDE

$$x'(t)=-\sin(t)-\frac{4}{3}-\frac{\pi}{24}+\frac{7\sqrt{3}}{16}+\int_0^{\frac{\pi}{3}}\sin(s)\left(sx(s)+x(\tfrac{s}{2})\right)ds, \tag{8.1}$$

with initial condition $x(0)=1$ and exact solution $x(t)=cos(t)$. To solve this example, as mentioned before, we convert FHIDE (8.1) to FHIE as:

$$x(t)=\cos(t)-\left(\frac{4}{3}+\frac{\pi}{24}-\frac{7\sqrt{3}}{16}\right)t+\int_0^t\int_0^{\frac{\pi}{3}}\sin(s)\left(sx(s)+x(\tfrac{s}{2})\right)dsd\tau \tag{8.2}$$

We apply the proposed method to solve (8.2). Computational results which include the absolute error at t_i (e_i), are reported in Table 1 for $n=10, n=60$ and $n=120$.

Table 1. Numerical results of Example 1

n	$n=10$	$n=60$	$n=120$
t_i	e_i	e_i	e_i
0	0	0	0
0.1047197551	1.4968e-4	4.1258e-6	1.0312e-6
0.2094395102	2.9936e-4	8.2516e-6	2.0625e-6
0.3141592653	4.4904e-4	1.2377e-5	3.0938e-6
0.4188790204	5.9873e-4	1.6503e-5	4.1251e-6
0.5235987755	7.4841e-4	2.0629e-5	5.1564e-6
0.6283185307	8.9809e-4	2.4754e-5	6.1876e-6
0.7330382858	1.0477e-3	2.8881e-5	7.2189e-6
0.8377580409	1.1974e-3	3.3006e-5	8.2502e-6
0.9424777960	1.3471e-3	3.7132e-5	9.2815e-6
1.0471975511	1.4968e-3	4.1258e-5	1.0312e-5

Example 2.

As a second example, consider the FHIDE

$$x'(t) = -e^{-t} + \frac{3}{2}t - \frac{5}{6} + \int_0^1 (\tau - s)\left(s + e^{\frac{s}{2}}x(\tfrac{s}{2})\right)ds, \tag{8.3}$$

with initial condition $x(0) = 1$ whose exact solution is $x(t) = e^{-t}$. Similar to Example 1, we convert FHIDE (8.3) to FHIE

$$x(t) = e^{-t} + \frac{3}{4}t^2 - \frac{5}{6}t + \int_0^t \int_0^1 (\tau - s)\left(s + e^{\frac{s}{2}}x(\tfrac{s}{2})\right)ds d\tau. \tag{8.4}$$

Table 2 shows the computational results.

Example 3.

As the last example, consider the following FHIDE

$$x'(t) = \frac{73}{20}t^2 - \frac{41}{80}t - \int_0^1 \tau(\tau - s)(x(s) + s^2((x(s^2))^2 + 1))ds, \tag{8.5}$$

with initial condition $x(0) = 0$ and exact solution $x(t) = t^3$. Similar to previous examples, we convert (8.5) to

$$x(t) = \frac{73}{60}t^3 - \frac{41}{160}t^2 - \int_0^t \int_0^1 \tau(\tau - s)(x(s) + s^2((x(s^2))^2 + 1))ds d\tau. \tag{8.6}$$

Computational results are reported in Table 3.

Table 2. Numerical results of Example 2

n	$n=10$	$n=60$	$n=120$

t_i	e_i	e_i	e_i
0	0	0	0
0.1	2.1262e-4	3.0771e-5	1.1826e-6
0.2	4.1773e-4	5.7313e-5	2.3626e-6
0.3	6.1531e-4	7.9627e-5	3.5401e-6
0.4	8.0537e-4	9.7712e-5	4.7151e-6
0.5	9.8791e-4	1.1156e-4	5.8874e-6
0.6	1.1629e-3	1.2119e-4	7.0571e-6
0.7	1.3304e-3	1.2659e-4	8.2243e-6
0.8	1.4903e-3	1.2776e-4	9.3889e-6
0.9	1.6428e-3	1.2471e-4	1.0551e-5
1	1.7877e-3	1.1741e-4	1.1710e-5

Table 3. Numerical results of Example 3

n	$n=10$	$n=60$	$n=120$
t_i	e_i	e_i	e_i
0	0	0	0
0.1	2.0126e-5	4.3192e-7	1.0727e-7
0.2	1.1884e-4	3.6419e-6	9.1239e-7
0.3	3.8228e-4	1.1241e-5	2.8138e-6
0.4	7.3556e-4	2.1386e-5	5.3520e-6
0.5	1.1440e-3	3.3096e-5	8.2816e-6
0.6	1.5731e-3	4.5392e-5	1.1357e-5
0.7	1.9881e-3	5.7292e-5	1.4334e-5
0.8	2.3544e-3	6.7818e-5	1.6968e-5
0.9	2.6373e-3	7.5988e-5	1.9012e-5
1	2.8024e-3	8.0823e-5	2.0222e-5

9. Conclusion

In this paper, we extended the method of successive approximations to solve functional Fredholm Hammerstein integro-differential equations. We also investigated the convergence and stability analysis. The numerical results confirm convergency and stability, too, and show the accuracy of the method. It seems that this method can be developed to solve other type of functional equations.

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