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Ultimate boundedness and periodicity results for a certain system Of third-order nonlinear Vector delay differential equations

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Abstract

In the last years, there has been increasing interest in obtaining the sufficient conditions for stability, instability, boundedness, ultimately boundedness, convergence, etc. For instance, in applied sciences some practical problems concerning mechanics, engineering technique fields, economy, control theory, physical sciences and so on are associated with third, fourth and higher order nonlinear differential equations. The problem of the boundedness and stability of solutions of vector differential equations has been widely studied by many authors, who have provided many techniques especially for delay differential equations. In this work a class of third order nonlinear non-autonomous vector delay differential equations is considered by employing the direct technique of Lyapunov as basic tool, where a complete Lyapunov functional is constructed and used to obtain sufficient conditions that guarantee existence of solutions that are periodic, uniformly asymptotically stable, uniformly ultimately bounded and the behavior of solutions at infinity. In addition to being for a more general equation, the obtained results here are new even when our equation is specialized to the forms previously studied and include many recent results in the literature. Finally, an example is given to show the feasibility of our results.

Keywords: Stability; Lyapunov functional; Ultimate boundedness; Periodicity; third-order delay vector differential equations

MSC 2010 No.: 34D05, 34D20, 34C25

1. Introduction

In this paper, we are concerned with the uniform asymptotic stability of solutions of the equation

$$\begin{aligned}&\left[P(X(t))X'(t)\right]'' + A(t)(Q(X(t))X'(t))' + B(t)(R(X(t))X'(t))\\&+C(t)F(X(t-r(t))) = 0,\end{aligned}\tag{1}$$

and the ultimate boundedness and the existence of periodic solutions of the equation

$$\begin{aligned}&\left[P(X(t))X'(t)\right]'' + A(t)(Q(X(t))X'(t))' + B(t)(R(X(t))X'(t))\\&+C(t)F(X(t-r(t))) = H(t,X,X',X''),\end{aligned}\tag{2}$$

in which $X \in \mathbb{R}^n$, $F : \mathbb{R}^n \to \mathbb{R}^n$, $H : \mathbb{R}_+ \times \mathbb{R}^{3n} \to \mathbb{R}^n$, P, Q and $R : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, A, B and $C : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$, are continuous differentiable functions with P is twice differentiable and $F(0) = 0$, $0 \leq r(t) \leq \gamma$, γ is a positive constant, and $r'(t) \leq \beta_0$, $0 < \beta_0 < 1$ and the dots indicate differentiation with respect to t.

Numerous research activities are concerned with the stability and boundedness of solutions to different functional differential equations, for some related contributions, we refer the reader to Hale (1977) and Tunç (2006a, 2006b, 2006c, 2009, 2014a, 2014b, 2017).

Ezeilo and Tejumola (1966), Afuwape (1983), Meng (1993) studied the ultimately boundedness and existence of periodic solutions of the nonlinear vector differential equation

$$X''' + AX'' + BX' + H(X) = P(t,X,X',X''). \tag{3}$$

Afterward, Feng (1995) established sufficient conditions under which the nonlinear vector differential equation

$$X''' + A(t)X'' + B(t)X' + H(X) = P(t,X,X',X''), \tag{4}$$

has at least unique periodic solution.

Moreover, Omeike (2007) established some sufficient conditions for the ultimate boundedness of the equation (3).

Recently, Omeike (2015) investigated the asymptotic stability of solutions to the following nonlinear third order scalar differential equation with delay for $P \equiv 0$

$$X''' + AX'' + BX' + H(X(t-r(t))) = P(t). \tag{5}$$

Equation (5) is a particular case to our preceding non-autonomous vector differential equation with the deviating argument r if $P(X) = Q(X) = R(X) = C(t) = I$, $A(t) = A$ and $B(t) = B$. On the other hand, we can find the same result for the equation (2) without delay by putting $r = 0$, which is generalization of (3) and (4).

In the case $n = 1$, these problems have been investigated [see Graef et al. (2015a, 2015b), Oudjedi et al. (2014, 2017) and Remili et al. (2014a, 2014b, 2014c, 2016a, 2015, 2016b, 2016c, 2016d, 2016e, 2016f)] for a general scalar delay differential equation. Equation (2) have not been discussed in the literature, yet. The basic reason may be the difficulty to find a suitable Lyapunov function for differential systems of higher order.

The object of the present paper is to provide results for n-dimensional equation (2) following the arguments used in some of the papers mentioned above.

2. Preliminaries

The following notations (see Omeike (2015)) will be useful in subsequent sections. For $x \in \mathbb{R}^n$, $|x|$ is the norm of x. For a given $r > 0, t_1 \in \mathbb{R}$,

$$C(t_1) = \{\phi : [t_1 - r, t_1] \to \mathbb{R}^n / \phi \text{ is continuous}\}.$$

In particular, $C = C(0)$ denotes the space of continuous functions mapping the interval $[-r, 0]$ into $\mathbb{R}^n$ and for $\phi \in C, \phi = \sup_{-r \le \theta \le 0} |\phi(0)|$. C_H will denote the set of ϕ such that $\phi \le H$. For any continuous function $x(u)$ defined on $-h \le u < A$, where $A > 0$, and $0 \le t < A$, the symbol x_t will denote the restriction of $x(u)$ to the interval $[t - r, t]$, that is, x_t is an element of C defined by

$$x_t(\theta) = x(t + \theta), -r \le \theta \le 0.$$

The following results will be basic to the proofs of Theorems.

Lemma 2.1 (Afuwape (1983), Afuwape (2004), Ezeilio (1966), Tiryaki (1999)).

Let D be a real symmetric positive definite $n \times n$ matrix, then for any X in $\mathbb{R}^n$, we have

$$\delta_d \parallel X \parallel^2 \le \langle DX, X \rangle \le \Delta_d \parallel X \parallel^2,$$

where δ_d, Δ_d are the least and the greatest eigenvalues of D, respectively.

Lemma 2.2 (Afuwape (1983), Afuwape (2004), Ezeilio (1966), Tiryaki (1999)).

Let Q, D be any two real $n \times n$ commuting matrices, then

(i) The eigenvalues $\lambda_i (QD) \, (i = 1, 2..., n)$ of the product matrix QD are all real and satisfy

$$\min_{1 \le j,k \le n} \lambda_j (Q) \lambda_k (D) \le \lambda_i (QD) \le \max_{1 \le j,k \le n} \lambda_j (Q) \lambda_k (D).$$

(ii) The eigenvalues $\lambda_i (Q + D) \, (i = 1, 2..., n)$ of the sum of matrices Q and D are all real and satisfy

$$\left\{ \min_{1 \le j \le n} \lambda_j (Q) + \min_{1 \le k \le n} \lambda_k (D) \right\} \le \lambda_i (Q + D) \le \left\{ \max_{1 \le j \le n} \lambda_j (Q) + \max_{1 \le k \le n} \lambda_k (D) \right\}.$$

Lemma 2.3 (Ezeilio (1966), Mahmoud and Tunç (2016), Tiryaki (1999)).

Let $H(X)$ be a continuous vector function with $H(0) = 0$.

$$1) \quad \frac{d}{dt} \left(\int_0^1 \langle H(\sigma X), X \rangle d\sigma \right) = \langle H(X), X' \rangle.$$

$$2) \quad \int_0^1 \langle C(t) H(\sigma X), X \rangle d\sigma = \int_0^1 \int_0^1 \sigma [\langle C(t) J_H(\sigma \tau X) X, X \rangle] d\sigma d\tau.$$

Lemma 2.4 (Ezeilio (1966), Mahmoud and Tunç (2016), Tiryaki (1999)).

Let $H(X)$ be a continuous vector function with $H(0) = 0$.

$$1)\quad \langle H(X), H(X)\rangle = 2\int_0^1 \int_0^1 \sigma\langle J_H(\sigma X)J_H(\sigma\tau X)X, X\rangle d\sigma d\tau.$$

$$2)\quad \langle C(t)H(X), X\rangle = \int_0^1 \langle J_H(\sigma X)C(t)X, X\rangle d\sigma.$$

Lemma 2.5.

Let $H(X)$ be a continuous vector function and that $H(0) = 0$. Then,

$$\delta_h \parallel X \parallel^2 \leq \int_0^1 \langle H(\sigma X), X\rangle \, d\sigma \leq \Delta_h \parallel X \parallel^2,$$

where δ_h, Δ_h are the least and the greatest eigenvalues of $J_h(X)$ (Jacobian matrix of H), respectively.

Definition 2.6.

We definite the spectral radius $\rho(A)$ of a matrix A by

$$\rho(A) = \max\{\lambda / \ \lambda \text{ is eigenvalue of } A\}.$$

Lemma 2.7.

For any $A \in \mathbb{R}^{n\times n}$, we have the norm $\|A\| = \sqrt{\rho(A^T A)}$ if A is symmetric then,

$$\|A\| = \rho(A).$$

We shall note all the equivalents norms by the same notation $\|X\|$ for $X \in \mathbb{R}^n$ and $\|A\|$ for a matrix $A \in \mathbb{R}^{n\times n}$.

3. Stability

Consider the functional differential equation

$$x' = f(t, x_t), \quad x_t(\theta) = x(t+\theta), \quad -r \leq \theta \leq 0, \quad t \geq 0, \tag{6}$$

where $f : I \times C_H \to \mathbb{R}^n$ is a continuous mapping, $f(t, 0) = 0$,

$$f(t, 0) = 0, \ C_H := \{\phi \in (C[-r, 0], \ \mathbb{R}^n) : \|\phi\| \leq H\},$$

and for $H_1 < H$, there exists $L(H_1) > 0$, with $|f(t, \phi)| < L(H_1)$ when $\|\phi\| < H_1$.

Definition 3.1 (Burton (2005)).

An element $\psi \in C$ is in the $\omega - limit$ set of ϕ, say $\Omega(\phi)$, if $x(t, 0, \phi)$ is defined on $[0, +\infty)$ and there is a sequence $\{t_n\}, t_n \to \infty$, as $n \to \infty$, with $\|x_{t_n}(\phi) - \psi\| \to 0$ as $n \to \infty$ where $x_{t_n}(\phi) = x(t_n + \theta, 0, \phi) \ for \ -r \leq \theta \leq 0$.

Definition 3.2 (Burton (2005)).

A set $Q \subset C_H$ is an invariant set if for any $\phi \in Q$, the solution of (6), $x(t, 0, \phi)$, is defined on $[0, \infty)$ and $x_t(\phi) \in Q$ for $t \in [0, \infty)$.

Definition 3.3 (Burton (1985)).

If $\phi \in C_H$is such that the solution $x_t(\phi)$ of (6) with $x_0(\phi) = \phi$ is defined on $[0, \infty)$ and $\|x_t(\phi)\| \leq H_1 < H$ for $t \in [0, \infty)$, then $\Omega(\phi)$ is a non-empty, compact, invariant set and

$$dist(x_t(\phi), \Omega(\phi)) \to 0 \ \ as \ \ t \to \infty.$$

Definition 3.4 (Burton (1985)).

Let $V(t, \phi) : I \times C_H \to \mathbb{R}$ be a continuous functional satisfying a local Lipschitz condition. $V(t, 0) = 0$, and such that:

(i) $W_1(|\phi(0)|) \leq V(t, \phi) \leq W_2(|\phi(0)|) + W_3(\|\phi\|_2)$ where $\|\phi\|_2 = (\int_{t-r}^{t} \|\phi(s)\|^2 ds)^{\frac{1}{2}}$,

(ii) $\dot{V}_{(6)}(t, \phi) \leq -W_4(|\phi(0)|)$,

where W_i $(i = 1, 2, 3, 4)$ are wedges, then the zero solution of (6) is uniformly asymptotically stable.

4. Assumptions and main results

We shall state here some assumptions which will be used on the functions that appeared in equation (1), and suppose that there are constants $\delta_a, \delta_b, \delta_c$, $\delta_{a'}, \delta_{b'}$, $\delta_{c'}, \delta_p$, δ_f, δ_q, δ_r, Δ_a, Δ_b, Δ_c, $\Delta_{a'}, \Delta_{b'}$, $\Delta_{c'}, \Delta_p, \Delta_q$, Δ_r and Δ_f, such that the matrices A, B, C, P, Q, R and $J_F(X)$ (Jacobian matrix of $F(X)$) are symmetric and positive definite, and furthermore the eigenvalues $\lambda_i(A), \lambda_i(B)$, $\lambda_i(C), \lambda_i(A'), \lambda_i(B'), \lambda_i(C'), \lambda_i(P), \lambda_i(Q), \lambda_i(R)$ and $\lambda_i(J_F(X))(i = 1, 2, ..., n)$ of A, B, C, A', B', C', P, Q, R and $J_F(X)$, respectively satisfy,

$$\begin{aligned}
&0 < \delta_p \leq \lambda_i(P) \leq \Delta_p, \quad 0 < \delta_q \leq \lambda_i(Q) \leq \Delta_q, \quad 0 < \delta_r \leq \lambda_i(R) \leq \Delta_r,\\
&0 < \delta_a \leq \lambda_i(A) \leq \Delta_a, \quad 0 < \delta_c \leq \lambda_i(C) \leq \Delta_c, \quad 0 < \delta_b \leq \lambda_i(B) \leq \Delta_b,\\
&\quad \delta_{a'} \leq \lambda_i\left(A'\right) \leq \Delta_{a'}, \quad \delta_{c'} \leq \lambda_i\left(C'\right) \leq \Delta_{c'} \leq 0, \quad \delta_{b'} \leq \lambda_i\left(B'\right) \leq \Delta_{b'} \leq 0,\\
&0 < \delta_f \leq \lambda_i(J_F(X)) \leq \Delta_f.
\end{aligned}$$

Note that for any matrix M symmetric invertible, we have

$$\Delta_{M^{-1}} = \delta_M^{-1}, \quad \text{and} \quad \delta_{M^{-1}} = \Delta_M^{-1}.$$

For the sake of brevity, we define

$$\begin{aligned}
A_1 &= \frac{1}{2}(1 + \Delta_{p^{-1}}) + \delta_a\delta_q\Delta_{p^{-1}}^2 + \frac{1}{\delta_f\delta_c}\left\|(B(t)R(X) - \delta_b\delta_r I)P^{-1}(X)\right\|^2,\\
A_2 &= \frac{1}{2}(1 + \Delta_{p^{-1}}) + \frac{1}{\delta_f\delta_c}\left\|(A(t)Q(X) - \delta_a\delta_q I)P^{-1}(X)\right\|^2,\\
\Gamma(t) &= B(t)R(X)P^{-1}(X), \qquad \rho(t) = t - r(t),
\end{aligned}$$

and

$$\theta_1(t) = \frac{d}{dt}P^{-1}(X(t)) = -P^{-1}\big(X(t)\big)\big[\frac{d}{dt}P\big(X(t)\big)\big]P^{-1}\big(X(t)\big),$$
$$\theta_2(t) = \big[\frac{d}{dt}Q\big(X(t)\big)\big]P^{-1}\big(X(t)\big) + Q(X(t))\theta_1(t),$$
$$\theta_3(t) = \big[\frac{d}{dt}R\big(X(t)\big)\big]P^{-1}\big(X(t)\big) + R(X(t))\theta_1(t),$$
$$\mu(t) = \int_0^t (\|\theta_1(s)\| + \|\theta_2(s)\| + \|\theta_3(s)\|)ds.$$

Consider the equivalent system to (1) :

$$\begin{aligned} X' &= P^{-1}(X)Y, \\ Y' &= Z, \\ Z' &= -A(t)\theta_2(t)Y - A(t)Q(X)P^{-1}(X)Z - \Gamma(t)Y \\ &\quad - C(t)F(X) + C(t)\int_{\rho(t)}^t J_F\left(X(s)\right)P^{-1}\left(X(s)\right)Y(s)ds. \end{aligned} \tag{7}$$

The following result is introduced.

Theorem 4.1.

Suppose that $\Delta_c \leq \delta_b$, $\Delta_{b'} \leq \delta_{c'}$ and the assumptions

(i) $\dfrac{\Delta_p \Delta_f}{\delta_r} < \alpha < \delta_a\delta_q$,

(ii) $\dfrac{1}{2}(\alpha + \delta_a\delta_q)\Delta_{a'}\Delta_q\Delta_{p^{-1}} - \delta_b(\alpha\Delta_p^{-1}\delta_r - \Delta_f) < -\epsilon < 0$,

(iii) $\beta < \min\left\{\delta_b\delta_r, \delta_{p^{-1}}\delta_b(\delta_a\delta_q\delta_r\delta_{p^{-1}} - \Delta_f)A_1^{-1}, \dfrac{1}{2}(\delta_a\delta_q - \alpha)A_2^{-1}\right\}$,

(iv) $\displaystyle\int_0^{+\infty}\left\|\frac{d}{ds}\big(P(X(s)) + Q(X(s)) + R(X(s))\big)\right\|ds < +\infty$,

are satisfied, then the zero solution of (7) is uniformly asymptotically stable, if

$$\gamma < \min\left\{\frac{\delta_f\delta_c}{\Delta_f\Delta_c\delta_p^{-1}}, 2\epsilon\delta_{p^{-1}}(1-\beta_0)A_3^{-1}, \frac{(\delta_a\delta_q - \alpha)\delta_{p^{-1}}}{2\Delta_f\Delta_c\delta_p^{-1}}\right\},$$

where

$$A_3 = \Delta_f\Delta_c\delta_p^{-1}(2 + \delta_p^{-1}(\alpha + \delta_a\delta_q)(2-\beta_0) + \beta).$$

Proof:

Let a continuously differentiable Lyapunov functional U defined by

$$U(t, X_t, Y_t, Z_t) = e^{-\frac{\mu(t)}{\upsilon}}V(t, X_t, Y_t, Z_t) = e^{-\frac{\mu(t)}{\upsilon}}V, \tag{8}$$

where

$$V = (\alpha+\delta_a\delta_q)\int_0^1 < C(t)F(\sigma X), X > d\sigma + 2 < C(t)F(X), Y > + < \Gamma(t)Y, Y >$$
$$+ \frac{1}{2}(\alpha+\delta_a\delta_q) < A(t)Q(X)P^{-2}(X)Y, Y > + (\alpha+\delta_a\delta_q) < P^{-1}(X)Y, Z >$$
$$+ < Z, Z > + \beta\delta_a\delta_q < X, P^{-1}(X)Y > + \beta < X, Z > + \frac{1}{2}\beta\delta_b\delta_r < X, X >$$
$$+ \frac{1}{2}\beta < Y, Y > + \omega_0 \int_{-r(t)}^{0}\int_{t+s}^{t} \langle Y(\tau), Y(\tau)\rangle d\tau ds.$$

ω_0, υ are some positive constants which will be specified later in the proof. Since

$$\omega_0 \int_{-r(t)}^{0}\int_{t+s}^{t} \langle Y(\tau), Y(\tau)\rangle d\tau ds$$

is non-negative, and by Lemma 2.3, we have

$$V \geq (\alpha+\delta_a\delta_q)\int_0^1\int_0^1 \sigma < C(t)J_F(\tau\sigma X)X, X > d\tau d\sigma - \left\| C(t)\Gamma^{-\frac{1}{2}}(t)F(X)\right\|^2$$
$$+ \left\|\Gamma^{\frac{1}{2}}(t)Y + C(t)\Gamma^{-\frac{1}{2}}(t)F(X)\right\|^2 + \frac{1}{2}\left\|Z + \alpha P^{-1}(X)Y\right\|^2$$
$$+ \frac{1}{2} < \Big((\alpha+\delta_a\delta_q)A(t)Q(X) - (\alpha^2+\delta_a^2\delta_q^2)I\Big)P^{-2}(X)Y, Y >$$
$$+ \frac{1}{2}\beta\|Y\|^2 + \frac{1}{2}\beta(\delta_b\delta_r - \beta)\|X\|^2 + \frac{1}{2}\|\beta X + \delta_a\delta_q P^{-1}(X)Y + Z\|^2,$$

since

$$\left\|\Gamma^{\frac{1}{2}}(t)Y + C(t)\Gamma^{-\frac{1}{2}}(t)F(X)\right\|^2 \geq 0,$$

and by Lemma 2.4, we have

$$V \geq \int_0^1\int_0^1 \sigma \left\langle \Big[(\alpha+\delta_a\delta_q)C(t) - 2C^2(t)\Gamma^{-1}(t)J_F(\sigma X)\Big] J_F(\tau\sigma X)X, X\right\rangle d\tau d\sigma$$
$$+ \frac{1}{2}\left\langle \Big((\alpha+\delta_a\delta_q)A(t)Q(X) - (\alpha^2+\delta_a^2\delta_q^2)I\Big)P^{-2}(X)Y, Y\right\rangle$$
$$+ \frac{1}{2}\left\|Z + \alpha P^{-1}(X)Y\right\|^2 + \frac{1}{2}\|\beta X + \delta_a\delta_q P^{-1}(X)Y + Z\|^2$$
$$+ \frac{1}{2}\beta\|Y\|^2 + \frac{1}{2}\beta(\delta_b\delta_r - \beta)\|X\|^2,$$

under our hypothesis, we get

$$V \geq \frac{1}{2}\left[\delta_c\delta_f\Big((\alpha+\delta_a\delta_q) - 2\Delta_p\Delta_{r^{-1}}\Delta_f\Big) + \beta(\delta_b\delta_r - \beta)\right]\|X\|^2$$
$$+ \frac{1}{2}\left\|Z + \alpha P^{-1}(X)Y\right\|^2 + \frac{1}{2}\left[\alpha(\delta_a\delta_q - \alpha)\delta_{p^{-2}} + \beta\right]\|Y\|^2$$
$$+ \frac{1}{2}\|\beta X + \delta_a\delta_q P^{-1}(X)Y + Z\|^2,$$

from conditions (i) and (iii) of Theorem 4.1. We can find a constant k such that

$$V \geq k\left(\|X\|^2 + \|Y\|^2 + \|Z\|^2\right). \tag{9}$$

By (iv), we obtain

$$\begin{aligned}\mu(t) \leq \Delta_{p^{-1}}^2(1+\Delta_r+\Delta_q)\int_0^t \left\|\frac{d}{ds}P\big(X(s)\big)\right\| ds \\ + \Delta_{p^{-1}}\int_0^t (\left\|\frac{d}{ds}R\big(X(s)\big)\right\| + \left\|\frac{d}{ds}Q\big(X(s)\big)\right\|)ds \\ \leq N < \infty. \end{aligned} \tag{10}$$

This may be combined with (9) to obtain

$$U \geq K\big(\parallel X \parallel^2 + \parallel Y \parallel^2 + \parallel Z \parallel^2 \big), \tag{11}$$

where $K = k\exp\left(-\dfrac{N}{\upsilon}\right)$.

The derivative of V along the trajectories of the system (7) is given by

$$\begin{aligned}\frac{d}{dt}V = -< \Big[(\alpha+\delta_a\delta_q)P^{-1}(X)B(t)R(X) - 2C(t)J_F(X) \\ - \frac{1}{2}(\alpha+\delta_a\delta_q)A'(t)Q(X)P^{-1}(X)\Big]P^{-1}(X)Y, Y > \\ - < \Big(2A(t)Q(X) - (\alpha+\delta_a\delta_q)I\Big)P^{-1}(X)Z, Z > \\ - \beta\left[< X, \Gamma(t)Y > -\delta_b\delta_r < X, P^{-1}(X)Y >\right] \\ + \beta\delta_a\delta_q < P^{-1}(X)Y, P^{-1}(X)Y > +\beta < (I+P^{-1}(x))Y, Z > \\ - \beta < X, \Big(A(t)Q(X) - \delta_a\delta_q I\Big)P^{-1}(X)Z > -\beta < X, C(t)F(X) > \\ - \omega_0\left(1-r'(t)\right)\int_{\rho(t)}^t \langle Y(\tau), Y(\tau)\rangle d\tau + \omega_0 r(t)\langle Y, Y\rangle + \psi_1+\psi_2+\psi_3,\end{aligned}$$

where

$$\begin{aligned}\psi_1 = (\alpha+\delta_a\delta_q)\int_0^1 < C'(t)F(\sigma X), X > d\sigma + 2 < C'(t)F(X), Y > \\ + < B'(t)R(X)P^{-1}(X)Y, Y >,\end{aligned}$$

$$\begin{aligned}\psi_2 = (\alpha+\delta_a\delta_q) < \theta_1(t)Y, Z > + \frac{(\alpha+\delta_a\delta_q)}{2} < A(t)Q(X)P^{-1}(X)\theta_1(t)Y, Y > \\ + < B(t)\theta_3(t)Y, Y > - \frac{(\alpha+\delta_a\delta_q)}{2} < A(t)\theta_2(t)P^{-1}(X)Y, Y > \\ + \delta_a\delta_q\beta < X, \theta_1(t)Y > -2 < A(t)\theta_2(t)Y, Z > -\beta < X, A(t)\theta_2(t)Y >,\end{aligned}$$

and

$$\begin{aligned}\psi_3 = \int_{\rho(t)}^t \Big\langle\ C(t)J_F\left(X(s)\right)P^{-1}(X(s))Y(s), 2Z(t) \\ + (\alpha+\delta_a\delta_q)P^{-1}(X(t))Y(t) + \beta X(t)\Big\rangle ds.\end{aligned}$$

We claim that $\psi_1 < 0$, indeed

$$\begin{aligned}\psi_1 &\leq (\alpha+\delta_a\delta_q)\int_0^1 < C'F(\sigma X), X > d\sigma - \left\| C'^{\frac{1}{2}}Y - C'^{\frac{1}{2}}F(X)\right\|^2 \\ &+ \Delta_{c'}\Big(\parallel F(X) \parallel^2 + \parallel Y \parallel^2 \Big) + \Delta_{b'}\delta_r\delta_{p^{-1}} \parallel Y \parallel^2 \\ &\leq (\alpha+\delta_a\delta_q)\int_0^1 < C'F(\sigma X), X > d\sigma \\ &= (\alpha+\delta_a\delta_q)\int_0^1\int_0^1 \sigma < C'J_F(\tau\sigma X)X, X > d\sigma d\tau \\ &\leq \frac{(\alpha+\delta_a\delta_q)}{2}\Delta_{c'}\delta_f \parallel X \parallel^2 < 0.\end{aligned}$$

By the identity $2\left|\langle U,V\rangle\right| \leq \|U\|^2 + \|V\|^2$, we obtain the following estimates

$$\begin{aligned}\psi_2 &\leq \left[\left(\frac{(\alpha+\delta_a\delta_q)}{2}(1+\frac{\Delta_a\Delta_q}{\delta_p}) + \frac{\beta}{2}\delta_a\delta_q\right)\|\theta_1(t)\| \right. \\ &\left. + \Delta_a\left(1+\frac{\alpha+\delta_a\delta_q}{2\delta_p}+\frac{\beta}{2}\right)\|\theta_2(t)\| + \Delta_b\|\theta_3(t)\|\right]\frac{V}{k} \\ &\leq \frac{k_1}{k}\Big[\|\theta_1(t)\| + \|\theta_2(t)\| + \|\theta_3(t)\|\Big]V,\end{aligned}$$

and

$$\begin{aligned}\psi_3 &\leq \int_{t-r(t)}^t \Big(\|2Z(t)\| + (\alpha+\delta_a\delta_q)\left\|P^{-1}(X(t))Y(t)\right\| \\ &+\beta\|X(t)\|\Big)\left\|C(t)J_F\left(X(s)\right)P^{-1}\left(X(s)\right)Y(s)\right\| ds \\ &\leq \frac{1}{2}\Delta_f\Delta_c\delta_p^{-1}\int_{\rho(t)}^t \Big[2\|Z(t)\|^2 + (\alpha+\delta_a\delta_q)\delta_p^{-1}\|Y(t)\|^2 \\ &+\beta\|X(t)\|^2 + (2+(\alpha+\delta_a\delta_q)\delta_p^{-1}+\beta)\|Y(s)\|^2\Big] ds \\ &\leq \frac{1}{2}\Delta_f\Delta_c\delta_p^{-1}\Big[\gamma\left(2\|Z\|^2 + (\alpha+\delta_a\delta_q)\delta_p^{-1}\|Y\|^2 + \beta\|X\|^2\right) \\ &+(2+(\alpha+\delta_a\delta_q)\delta_p^{-1}+\beta)\int_{\rho(t)}^t \|Y(s)\|^2 ds\Big],\end{aligned}$$

where

$$k_1 = \max\left\{\frac{(\alpha+\delta_a\delta_q)}{2}(1+\frac{\Delta_a\Delta_q}{\delta_p}) + \frac{\beta}{2}\delta_a\delta_q, \Delta_a(1+\frac{\alpha+\delta_a\delta_q}{2\delta_p}+\frac{\beta}{2}), \Delta_b\right\}.$$

From (i) and (ii) of Theorem 4.1 and Lemma 2.4, we obtain

$$\begin{aligned}
\frac{d}{dt}V \leq & \frac{k_1}{k}\Big[\|\theta_1(t)\| + \|\theta_2(t)\| + \|\theta_3(t)\|\Big]V - \frac{\beta}{2}\big(\delta_f\delta_c - \Delta_f\Delta_c\delta_p^{-1}\gamma\big)\|X\|^2 \\
& - \left[\epsilon\delta_{p^{-1}} - \left(\omega_0 + \frac{1}{2}(\alpha+\delta_a\delta_q)\Delta_f\Delta_c\delta_p^{-2}\right)\gamma\right]\|Y\|^2 \\
& - \frac{1}{2}\left[(\delta_a\delta_q - \alpha)\delta_{p^{-1}} - 2\Delta_f\Delta_c\delta_p^{-1}\gamma\right]\|Z\|^2 \\
& - \left[\delta_b\delta_{p^{-1}}(\delta_a\delta_q\delta_r\delta_{p^{-1}} - \Delta_f) - \beta A_1\right]\|Y\|^2 - \left[\frac{\delta_{p^{-1}}}{2}(\delta_a\delta_q - \alpha) - \beta A_2\right]\|Z\|^2 \\
& - \frac{\beta}{4\delta_f\delta_c}\left[\delta_f\delta_c\|X\| + 2\left\|(B(t)R(X) - \delta_b\delta_r I)P^{-1}(X)Y\right\|\right]^2 \\
& - \frac{\beta}{4\delta_f\delta_c}\left[\delta_f\delta_c\|X\| + 2\left\|(A(t)Q(X) - \delta_a\delta_q I)P^{-1}(X)Z\right\|\right]^2 \\
& - \left[\omega_0(1-\beta_0) - \frac{1}{2}\Delta_f\Delta_c\delta_p^{-1}(2 + (\alpha+\delta_a\delta_q)\delta_p^{-1} + \beta)\right]\int_{\rho(t)}^{t}\|Y(s)\|^2\,ds.
\end{aligned}$$

Choosing

$$\omega_0 = \frac{\Delta_f\Delta_c\delta_p^{-1}(2 + (\alpha+\delta_a\delta_q)\delta_p^{-1} + \beta)}{2(1-\beta_0)},$$

and by (iii) we get

$$\begin{aligned}
\frac{d}{dt}V \leq & \frac{k_1}{k}\Big[\|\theta_1(t)\| + \|\theta_2(t)\| + \|\theta_3(t)\|\Big]V - \frac{\beta}{2}\big(\delta_f\delta_c - \Delta_f\Delta_c\delta_p^{-1}\gamma\big)\|X\|^2 \\
& - \left[\epsilon\delta_{p^{-1}} - \frac{1}{2}\Delta_f\Delta_c\delta_p^{-1}\gamma\left(\frac{2 + (\alpha+\delta_a\delta_q)\delta_p^{-1} + \beta}{1-\beta_0} + (\alpha+\delta_a\delta_q)\delta_p^{-1}\right)\right]\|Y\|^2 \\
& - \frac{1}{2}\left[(\delta_a\delta_q - \alpha)\delta_{p^{-1}} - 2\Delta_f\Delta_c\delta_p^{-1}\gamma\right]\|Z\|^2.
\end{aligned}$$

Using (8), (9), (10) and taking $\upsilon = \frac{k}{k_1}$ we see at once that

$$\begin{aligned}
\frac{d}{dt}U = & e^{-\frac{k_1\mu(t)}{k}}\left(\frac{d}{dt}V - \frac{k_1(||\theta_1(t)|| + ||\theta_2(t)|| + ||\theta_3(t)||)}{k}V\right) \\
\leq & e^{-\frac{k_1 N}{k}}\Bigg[-\frac{\beta}{2}\big(\delta_f\delta_c - \Delta_f\Delta_c\delta_p^{-1}\gamma\big)\|X\|^2 \\
& - \left\{\epsilon\delta_{p^{-1}} - \frac{1}{2}\Delta_f\Delta_c\delta_p^{-1}\gamma\left(\frac{2 + (\alpha+\delta_a\delta_q)\delta_p^{-1} + \beta}{1-\beta_0} + (\alpha+\delta_a\delta_q)\delta_p^{-1}\right)\right\}\|Y\|^2 \\
& - \left\{\frac{1}{2}\left((\delta_a\delta_q - \alpha)\delta_{p^{-1}} - 2\Delta_f\Delta_c\delta_p^{-1}\gamma\right)\right\}\|Z\|^2\Bigg].
\end{aligned}$$

To conclude, if we choose γ so that

$$\gamma < \min\left\{\frac{\delta_f\delta_c}{\Delta_f\Delta_c\delta_p^{-1}}, 2\epsilon\delta_{p^{-1}}(1-\beta_0)A_3^{-1}, \frac{(\delta_a\delta_q - \alpha)\delta_{p^{-1}}}{2\Delta_f\Delta_c\delta_p^{-1}}\right\},$$

we will have the desired inequality

$$\frac{d}{dt}U(t, X_t, Y_t, Z_t) \leq -\xi\Big(\|X\|^2 + \|Y\|^2 + \|Z\|^2\Big). \tag{12}$$

This shows that the zero solution of system (7) is uniformly asymptotically stable. ■

Example 4.2.

As a special case of the equation (1)

$$(P(X(t))X'(t))'' + A(t)(Q(X(t))X'(t))' + B(t)R(X(t))X'(t) + C(t)F(X(\rho(t))) = 0,$$

where

$$X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

and

$$F\big(X(\rho(t))\big) = \begin{pmatrix} \frac{1}{4}\arctan x(\rho(t)) + \frac{1}{4}x(\rho(t)) \\ 0.16y(\rho(t)) \end{pmatrix}, \quad J_F(X) = \begin{pmatrix} \frac{1}{4(1+x^2)} + \frac{1}{4} & 0 \\ 0 & 0.16 \end{pmatrix},$$

$$P(X(t)) = \begin{pmatrix} \frac{\sin(x(t))}{1+x^2(t)} + 2 & 0 \\ 0 & \frac{\cos(y(t))}{1+y^2(t)} + 2 \end{pmatrix}, \quad A(t) = \begin{pmatrix} \frac{e^{\sin t}}{4} + \frac{10}{4} & 0 \\ 0 & \frac{\cos t}{2} + 2 \end{pmatrix},$$

$$Q(X(t)) = \begin{pmatrix} \frac{e^{\sin(x(t))}}{1+x^2(t)} + 2 & 0 \\ 0 & \frac{9\cos(y(t))}{10+y^2(t)} + \frac{41}{10} \end{pmatrix}, \quad B(t) = \begin{pmatrix} \frac{e^{-t^2}+3}{2} & 0 \\ 0 & \frac{\sin t}{2} + 1 \end{pmatrix},$$

$$R(X(t)) = \begin{pmatrix} \frac{10e^{-x^2(t)}}{2+x^2(t)} + 75 & 0 \\ 0 & \frac{100\sin(y(t))}{4+y^2(t)} + 50 \end{pmatrix}, \quad C(t) = \begin{pmatrix} e^{-2t} + 5 & 0 \\ 0 & e^{-t} + 5 \end{pmatrix}.$$

Clearly, $P(X), Q(X), R(X), A, B, C$ and $J_F(X)$ are diagonal matrices, hence they are symmetric and commute pairwise. Then, by an easy calculation, we obtain eigenvalues of the matrices $P, Q, R,\ A,\ B,\ C$ and $J_F(X)$ as follows:

$$\delta_p = 1 \leq \lambda_1(P(X(t))) = \tfrac{\sin x}{1+x^2} + 2, \qquad \lambda_2(P(X(t))) = \tfrac{\cos x}{1+x^2} + 2 \leq 3 = \Delta_p,$$

$$\delta_q = 2 \leq \lambda_1(Q(X(t))) = \tfrac{e^{\sin x}}{1+x^2} + 2, \qquad \lambda_2(Q(X(t))) = \tfrac{9\cos y}{10+y^2} + \tfrac{41}{10} \leq 5 = \Delta_q,$$

$$\delta_r = 25 \leq \lambda_1(R(X(t))) = \tfrac{100\sin y}{4+y^2} + 50, \quad \lambda_2(R(X(t))) = \tfrac{10e^{-x^2}}{2+x^2} + 75 \leq 80 = \Delta_r,$$

$$\delta_a = 1.5 \leq \lambda_1(A(t)) = \tfrac{1}{2}\cos t + 2, \qquad \lambda_2(A(t)) = \tfrac{e^{\sin t}}{4} + \tfrac{10}{4} \leq 3.1795 = \Delta_a,$$

$$\delta_b = 0.5 \leq \lambda_1(B(t)) = \tfrac{\sin t}{2} + 1, \qquad \lambda_2(B(t)) = \tfrac{e^{-t^2}}{2} + \tfrac{3}{2} \leq 2 = \Delta_b,$$

$$\delta_c = 5 \leq \lambda_1(C(t)) = e^{-2t} + 5, \qquad \lambda_2(C(t)) = e^{-3t} + 5 \leq 6 = \Delta_c,$$

$$\delta_f = 0.16 = \lambda_1(J_F(X)), \qquad \lambda_2(J_F(X)) = \tfrac{1}{4(1+x^2)} + \tfrac{1}{4} \leq \tfrac{1}{2} = \Delta_f.$$

A simple computation gives

$$\lambda_1\left(A'(t)\right)=-\frac{1}{2}\sin t,\quad \lambda_2\left(A'(t)\right)=\frac{\cos t}{4}e^{\sin t},$$
$$\lambda_1\left(B'(t)\right)=-\frac{\cos t}{2},\quad \lambda_2\left(B'\right)=-te^{-t^2},$$
$$\lambda_1\left(C'(t)\right)=-2e^{-2t},\quad \lambda_2\left(C'(t)\right)=-e^{-t}.$$

A trivial verification shows that P,Q and R are nonsingular matrices and we have

$$\frac{d}{dt}P(X(t))=\begin{pmatrix}\left(\frac{\cos(x(t))}{1+x^2(t)}-\frac{2x(t)\sin(x(t))}{(1+x^2(t))^2}\right)x'(t) & 0\\ 0 & \left(\frac{-\sin(y(t))}{1+y^2(t)}-\frac{2y(t)\cos(y(t))}{(1+y^2(t))^2}\right)y'(t)\end{pmatrix},$$

$$\frac{d}{dt}Q(X(t))=\begin{pmatrix}(\frac{\cos(x(t))e^{\sin(x(t))}}{1+x^2(t)}-\frac{2x(t)e^{\sin(x(t))}}{(1+x^2(t))^2})x'(t) & 0\\ 0 & (\frac{-9\sin(y(t))}{10+y^2(t)}-\frac{18y(t)\cos(y(t))}{(10+y^2(t))^2})y'(t)\end{pmatrix},$$

and

$$\frac{d}{dt}R(X(t))=\begin{pmatrix}\frac{-20x(t)e^{-x^2(t)}(3+x^2(t))}{(2+x^2(t))^2}x'(t) & 0\\ 0 & (\frac{100\cos(y(t))}{4+y^2(t)}-\frac{200y(t)\sin(y(t))}{(4+y^2(t))^2})y'(t)\end{pmatrix}.$$

For $t\in[0,+\infty)$ a straightforward calculation give

$$\begin{aligned}\int_0^t\|\frac{d}{ds}P(X(s))\|ds&=\int_0^t\left|(\frac{\cos(x(s))}{1+x^2(s)}-\frac{2x(s)\sin(x(s))}{(1+x^2(s))^2})x'(s)\right|ds\\&\quad+\int_0^t\left|(\frac{-\sin(y(s))}{1+y^2(s)}-\frac{2y(s)\cos y(s)}{(1+y^2(s))^2})y'(s)\right|ds\\&\le\int_{\omega_1(t)}^{\omega_2(t)}\left|(\frac{\cos u}{1+u^2}-\frac{2u\sin u}{(1+u^2)^2})du\right|\\&\quad+\int_{\varphi_1(t)}^{\varphi_2(t)}\left|(\frac{-\sin v}{1+v^2}-\frac{2v\cos v}{(1+v^2)^2})dv\right|\\&<\int_{-\infty}^{+\infty}\frac{1+u^2+2\left|u\right|}{(1+u^2)^2}du+\int_{-\infty}^{+\infty}\frac{1+u^2+2\left|u\right|}{(1+u^2)^2}du\\&=2\pi,\end{aligned}$$

$$\begin{aligned}\int_0^t \|\frac{d}{ds}(Q(X(s)))\|ds &= \int_0^t \left|(\frac{\cos(x(s))e^{\sin(x(s))}}{1+x^2(s)} - \frac{2x(s)e^{\sin(x(s))}}{(1+x^2(s))^2})x'(s)\right| ds \\ &\quad + \int_0^t \left|(\frac{-9\sin(y(s))}{10+y^2(s)} - \frac{18y(s)\cos(y(s))}{(10+y^2(s))^2})y'(s)\right| ds \\ &\leq \int_{\omega_1(t)}^{\omega_2(t)} \left|(\frac{\cos u e^{\sin u}}{1+u^2} - \frac{2ue^{\sin u}}{(1+u^2)^2})du\right| \\ &\quad + \int_{\varphi_1(t)}^{\varphi_2(t)} \left|(\frac{-9\sin v}{10+v^2} - \frac{18v\cos v}{(10+v^2)^2})dv\right| \\ &< \int_{-\infty}^{+\infty}(\frac{e}{1+u^2} + \frac{2e\,|u|}{(1+u^2)^2})du \\ &\quad + \int_{-\infty}^{+\infty}(\frac{9}{10+v^2} + \frac{18\,|v|}{(10+v^2)^2})dv \\ &= (e + \frac{9}{\sqrt{10}})\pi,\end{aligned}$$

and

$$\begin{aligned}\int_0^t \|\frac{d}{ds}(R(X(s)))\|ds &= \int_0^t \left|\frac{-20x(s)e^{-x^2(s)}(3+x^2(s))}{(2+x^2(s))^2}x'(s)\right| ds \\ &\quad + \int_0^t \left|(\frac{100\cos(y(s))}{4+y^2(s)} - \frac{200y(s)\sin(y(s))}{(4+y^2(s))^2})y'(s)\right| ds \\ &\leq \int_{\omega_1(t)}^{\omega_2(t)} \left|(\frac{-20ue^{-u^2}(3+u^2)}{(2+u^2)^2})du\right| \\ &\quad + \int_{\varphi_1(t)}^{\varphi_2(t)} \left|(\frac{100\cos v}{4+v^2} - \frac{200v\sin v}{(4+v^2)^2})dv\right| \\ &< \int_{-\infty}^{+\infty} \frac{60\,|u|}{(2+u^2)^2}du + \int_{-\infty}^{+\infty}\left(\frac{100}{4+v^2} + \frac{200\,|v|}{(4+v^2)^2}\right)dv \\ &= 50\pi,\end{aligned}$$

where

$$\omega_1(t) = \min\{x(0), x(t)\}, \qquad \omega_2(t) = \max\{x(0), x(t)\},$$

and

$$\varphi_1(t) = \min\{y(0), y(t)\}, \qquad \varphi_2(t) = \max\{y(0), y(t)\}.$$

By taking $\alpha = 2.5$ it follows easily that

$$0.06 = \frac{\Delta_p \Delta_f}{\delta_r} < \alpha < \delta_a \delta_q = 3,$$

and

$$\frac{1}{2}(\alpha + \delta_a\delta_q)\Delta_{a'}\Delta_q\Delta_{p^{-1}} - \delta_b(\alpha\Delta_p^{-1}\delta_r - \Delta_f) = -1.65 < 0.$$

We take $r\left(t\right) = \exp\left(-t^2\right)$, then, $0 \leq r(t) \leq \gamma$, $(\gamma > 0)$, and $r'\left(t\right) = -2t\exp\left(-t^2\right) \leq \beta_0$ for $0 < \beta_0 < 1$. Thus, all the conditions of Theorem 4.1 are satisfied.

5. Boundedness and the existence of periodic solutions

First, consider a system of delay differential equations

$$x' = F(t, x_t)\ ,\ x_t(\theta) = x(t+\theta)\ ,\ -r \le \theta \le 0\ ,\ t \ge 0, \tag{13}$$

where $F : \mathbb{R} \times C_H \longrightarrow \mathbb{R}^n$ is a continuous mapping and takes bounded set into bounded sets. The following lemma is a well-known result obtained by Burton (1985).

Lemma 5.1 (Burton (1985)).

Let $V(t,\phi) : \mathbb{R} \times C_H \longrightarrow \mathbb{R}$ be a continuous and local Lipschitz in ϕ. If
$(i)\ W(|x(t)|) \le V(t, x_t) \le W_1(|x(t)|) + W_2\left(\int_{t-r(t)}^{t} W_3(|x(s)|)\,ds\right)$,
$(ii)\ V'_{(13)} \le W_3(|x(s)|) + M$ for some $M > 0$, where $W(r), W_i\,(i = 1,2,3)$ are wedges,
then the solutions of (13) are uniformly bounded and uniformly ultimately bounded for bound B.

If (13) is a periodic system with period T, we have the following result.

Lemma 5.2 (Li Senlin and Wen Lizhi (1987)).

Suppose that, for $\alpha > 0$, there exists $L(\alpha) > 0$ such that $|f(t, x_t)| \le L(\alpha)$, for $t \in [-T, 0]$ and $\|x_t\| \le \alpha$, and suppose that the solutions of (13) are bounded and ultimately bounded for bound B, then, there exists a periodic solution of (13) of period T.

To study the boundedness and the existence of periodic solutions of (2), we would need to write (2) in the form

$$\begin{aligned}
X' &= P^{-1}(X)Y,\\
Y' &= Z,\\
Z' &= -A(t)\theta_2(t)Y - A(t)Q(X)P^{-1}(X)Z - B(t)R(X)P^{-1}(X)Y - C(t)F(X)\\
&\quad + C(t)\int_{\rho(t)}^{t} J_F(X(s))\,P^{-1}(X(s))\,Y(s)ds + H\Big(t, X, P^{-1}(X)Y, \theta_1(t)Y + P^{-1}(X)Z\Big).
\end{aligned} \tag{14}$$

Thus, our main theorem in this section is stated with respect to (14) as follows.

Theorem 5.3.

One assumes that all the assumptions of Theorem 4.1 and the assumption

$$\|H(t, X, Y, Z)\| \le h_1(t) + h_2(t)(\|X\| + \|Y\| + \|Z\|) \tag{15}$$

hold, where $h_1(t)$ and $h_2(t)$ are continuous functions and there exist $H_0, \epsilon > 0$ such that

$$h_1(t) \le H_0 \quad h_2(t) \le \epsilon.$$

Then all solutions of system (14) are uniformly bounded and uniformly ultimately bounded.

Proof:

Along any solution $(X(t), Y(t), Z(t))$ of (14), we have

$$\frac{d}{dt}U_{(14)} = \frac{d}{dt}U_{(7)} + < \beta X + (\alpha + \delta_a\delta_q)P^{-1}(X)Y + 2Z, H(t, X, P^{-1}(X)Y, X'') > .$$

From (12), we obtain

$$\frac{d}{dt}U_{(14)} \leq -\xi\Big(\|X\|^2 + \|Y\|^2 + \|Z\|^2 \Big) + \kappa_1(\|X\| + \|Y\| + \|Z\|)\|H(t, X, P^{-1}(X)Y, X'')\|,$$

where $\kappa_1 = \max\{\beta, (\alpha + \delta_a\delta_q)\delta_p^{-1}, 2\}$.

Choosing $\epsilon < 3^{-1}\kappa_1^{-1}\xi$, then, there exists $\kappa_2 = \xi - 3\kappa_1\epsilon > 0$.

In view of (15) we have

$$\frac{d}{dt}U_{(14)} \leq -\frac{\kappa_2}{2}(\|X\|^2 + \|Y\|^2 + \|Z\|^2) + \frac{3}{2}\kappa_1^2 H_0^2 \kappa_2^{-1}, \tag{16}$$

since

$$\frac{\kappa_2}{2}\left\{\Big(\|X\| - \kappa_1 H_0 \kappa_2^{-1}\Big)^2 + \Big(\|Y\| - \kappa_1 H_0 \kappa_2^{-1}\Big)^2 + \Big(\|Z\| - \kappa_1 H_0 \kappa_2^{-1}\Big)^2\right\} \geq 0,$$

for all X, Y and Z. From estimate (16), hypothesis (ii) of Lemma 5.1 is satisfied. Also from estimates (11) and by the fact that $W(t, \phi) \leq W_2(\|\phi\|) + W_3(\int_{\rho(t)}^{t} W_4(\phi(s))ds)$, is easily verified, then condition (i) of Lemma 5.1 follows. This completes the proof of the theorem. ■

The following theorem being a consequence of Theorem 5.3 and Lemma 5.2.

Theorem 5.4.

If hypotheses of Theorem 5.3 be satisfied and A, B, C, H are periodic functions of period T, then there exists a periodic solution of system (14) with the period T.

Proof:

It only remains to verify using the assumptions of Theorem 5.3 that the conditions of Lemma 5.2 follow easily. ■

6. Conclusion

Lyapunov's method has proved to be a popular and useful technique in the study of the stability and boundedness of solutions of higher order non-linear differential equations. In this paper we investigate the asymptotic stability of the zero solution and ultimate boundedness of solutions for certain third order non-linear non-autonomous vector differential equations with delay. Sufficient conditions were obtained for the existence of at least one periodic solution of the equation.

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