Applications and Applied Mathematics: An International Journal (AAM)

Volume 13 | Issue 1 Article 13

6-2018

Distance Product of Graphs

H. S. Mehta
Sardar Patel University

U. P. Acharya
A. D. Patel Institute of Technology

Mehta and Acharya: Distance Product of Graphs
Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 13, Issue 1 (June 2018), pp. 190–198

Distance Product of Graphs

[1]**H. S. Mehta and** [2]**U. P. Acharya**

[1]Department of Mathematics
Sardar Patel University
Vallabh Vidyanagar-388120, India
hs_mehta@spuvvn.edu

[2]A. D. Patel Institute of Technology
New V. V. Nagar -388121, India
maths.urvashi@adit.ac.in

Received: August 23, 2017; Accepted: March 30, 2018

Abstract

In graph theory, different types of product of two graphs have been studied, e.g. Cartesian product, Tensor product, Strong product, etc. Later on, Cartesian product and Tensor product have been generalized by $2-$Cartesian product and $2-$Tensor product. In this paper, we give one more generalize form, distance product of two graphs. Mainly we discuss the connectedness, bipartiteness and Eulerian property in this product.

Keywords: Connected graph; Cartesian product; Tensor product; $12-$product of graphs; Bipartite

MSC 2010 No.: 05C69, 05C70, 05C76

1. Introduction

The Cartesian product and Tensor product of graphs are well-studied by Hammack et al. (2011) and Sampathkumar (1975). Later on $2-$Cartesian product and $2-$Tensor product are defined and discussed by Acharya et al. (2014, 2015, 2017). These product are defined using the concept of vertices at distance two. The graph with this concept, e.g. the square graph $G^{[2]}$ and the derived graph G', are also studied in detail. In fact this concept is useful in studying energy of the derived graph which has wide application in chemical graph theory by Ayyaswamy et al. (2010), Hande et al. (2013) and Jog et al. (2012). But the unfortunately $2-$Cartesian product and $2-$Tensor product does not preserve connectedness. So, using two and usual cartesian product and tensor product we define the distance product of graphs in this paper. Mainly, we prove that the new product will give connected.

Let $G = (V(G), E(G))$ be a finite and simple graph with the vertex set $V(G)$ and the edge set $E(G)$. A graph G is connected, if there is a path between every pair of vertices. If G is a connected graph, then $d_G(u, u')$ is the length of the shortest path between u and u' in G. The diameter of G, denoted by $D(G)$, is defined as $\max\,\{d_G(u, u') : u, u^{'} \in V(G)\}$. The null graph is a graph with empty edge set.

Throughout this paper, we fix G and H to be finite, connected and simple graphs. For the basic terminology, concepts and results of graph theory, we refer to Balakrishnan et al. (2012) and Godsil et al. (2011).

2. Distance Product $G \otimes_{12} H$

In this section we define distance product of two graphs and discuss the connectedness of this new product.

Definition 2.1.

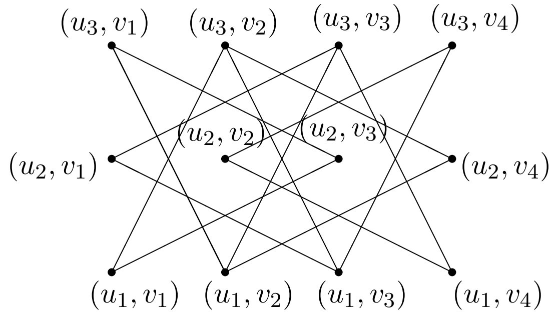
Let $G = (U,\ E_1)$ and $H = (V,\ E_2)$ be two connected graphs. The distance product or $12-$product of G and H, denoted by $G \otimes_{12} H$, is the graph with the vertex set $U \times V$ and two vertices (u, v) and (u', v') in $U \times V$ are adjacent in $G \otimes_{12} H$ if

(i) $d_G(u, u') = 1$ and $d_H(v, v') = 2$, or
(ii) $d_G(u, u') = 2$ and $d_H(v, v') = 1$.

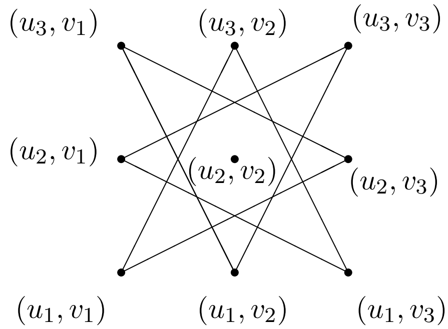
Note that $G \otimes_{12} H$ is isomorphic to $H \otimes_{12} G$. Also the graph $G \otimes_{12} H$ is a null graph if $D(G) < 2$ and $D(H) < 2$.

Example 2.2.

(i) Let $G = P_3$ with $u_1 \to u_2 \to u_3$ and $H = P_4$ with $v_1 \to v_2 \to v_3 \to v_4$. Then $P_3 \otimes_{12} P_4$ is as follows:



(ii) Let $G = P_3 = H$ with $u_1 \to u_2 \to u_3$ and $v_1 \to v_2 \to v_3$ in G and H respectively. Then the graph $P_3 \otimes_{12} P_3$ is not connected and it is as follows:



To obtain the connectedness of $G \otimes_{12} H$, we fix G and H both are connected graphs with $N^2(w) \neq \phi$ for every $w \in V(G) \cup V(H)$, where $N^2(u) = \{v \in V(G) : d_G(u, v) = 2\}$.

Theorem 2.3.

Let G and H be two graphs with $N^2(w) \neq \phi$, for every $w \in V(G) \cup V(H)$. Then $G \otimes_{12} H$ is connected if and only if G and H are connected.

Proof:

Let G and H be two graphs with vertex sets U and V respectively. Then $V(G \otimes_{12} H) = U \times V$. First, assume that G and H are connected.

Let (u, v) and (u', v') be in $U \times V$. Also, $u = u_0 \to u_1 \to \ldots \to u_m = u'$ is a path in G and $v = v_0 \to v_1 \to \ldots \to v_n = v'$ is a path in H.

As $N^2(u) \neq \phi$ in G, $\exists\, x' \in N^2(u)$ such that $d_G(u, x') = 2$. Let $u \to x \to x'$ be a path between u and x' in G. Similarly $\exists\, y' \in N^2\ (v_1)$ such that $d_H(v_1, y') = 2$. Then $P_1 : (u, v) \to (x', v_1) \to (x, y') \to (u, v_1)$ is a path between (u, v) and (u, v_1) in $G \otimes_{12} H$. By continuing the same process, there is a path between (u, v) and (u, v') in $G \otimes_{12} H$. Similarly, there is a path between (u, v') and (u', v') in $G \otimes_{12} H$. Using these paths, we get a path between (u, v) and (u', v') in $G \otimes_{12} H$. Hence, $G \otimes_{12} H$ is a connected graph.

Conversely, suppose that $G \otimes_{12} H$ is connected. Let u and u' be in $V(G)$ with $u \neq u'$.

Let $P : (u, v) = (u_0, v_0) \to (u_1, v_1) \to \ldots \to (u_n, v_n) = (u', v')$ be a path between (u, v) and (u', v') in $G \otimes_{12} H$ for $v, v' \in V(H)$. Now, from path P, we have $(u_0, v_0) \to (u_1, v_1)$ which gives that $d_G(u_0, u_1) = 1$ or 2 in G. Similarly, for any two adjacent vertices $(u_i, v_i) \to (u_{i+1}, v_{i+1})$ on path P gives that $d_G(u_i, u_{i+1}) = 1$ or 2 in G. So, there is a walk between u and u' in G. Hence, there is a path between u and u' in G. Therefore G is a connected graph. By similar arguments, H is also a connected graph. ∎

Next, we consider the case in which only one of the graph has property $N^2(u) \neq \phi$, for every u.

Theorem 2.4.

Let G be a non-complete graph and H be a graph with $N^2(b) \neq \phi;\ \forall\, b \in V(H)$. Then $G \otimes_{12} H$ is connected if and only if G and H are connected.

Proof:

Assume that G and H are connected graphs. We continue the notations of Theorem 2.3. Let (u, v) and (u', v') be in $U \times V$. If $N^2(u) \neq \phi$ in G, then we get the path between (u, v) and (u', v') as in Theorem 2.3.

Suppose $N^2(u) = \phi$. Then $d_G(u, u') = 1$ for every $u' \in V(G)$. In particular $d_G(u, u_1) = 1$. As G is non-complete, $\exists\, x_1,\ x_2 \in N(u)$ such that $d_G(x_1, x_2) = 2$. Also, as $N^2(v) \neq \phi$, $\exists\ y' \in V(H)$ such that $d_H(v, y') = 2$ with a path $v \to y \to y'$ in H. Now, we show that there is a path from (u, v) to (u_1, v) and from (u, v) to (u, v_1) in $G \otimes_{12} H$.

If $u_1 = x_1$ in G, then $(u, v) \to (x_1, y') \to (x_2, y) \to (x_1, v) = (u_1, v)$ is a path between (u, v) and (u_1, v). Suppose $x_1 \neq u_1 \neq x_2$ in G. If $d_G(u_1, x_1) = 1$, then $(u, v) \to (x_1, y') \to (u_1, v)$ and if $d_G(u_1, x_1) = 2$, then $(u, v) \to (x_2, y') \to (x_1, y) \to (u_1, v)$ are the paths between (u, v) and (u_1, v) in $G \otimes_{12} H$. Thus in all cases there is a path from (u, v) to (u_1, v) in $G \otimes_{12} H$.

Next, we show that there exists a path between (u, v) and (u, v_1) in $G \otimes_{12} H$. As $v_1 \in V(H)$, $\exists\ y'' \in V(H)$ such that $d(v_1, y'') = 2$ with (say) $v_1 \to t' \to y''$ in H. Then $(u, v) \to (x_1, y') \to (x_2, y) \to (x_1, v) \to (x_2, v_1) \to (x_1, t') \to (x_2, y'') \to (u, v_1)$ is a path between (u, v) and (u, v_1) in $G \otimes_{12} H$.

Similarly there is a path between (u, v) and (u', v') in $G \otimes_{12} H$. Hence $G \otimes_{12} H$ is a connected graph.

By similar arguments as given in Theorem 2.3, we can show that if $G \otimes_{12} H$ is a connected graph, then G and H are connected. ■

The next result shows that if we drop the condition $N^2(w) \neq \phi$, for every w from both the graphs, then $G \otimes_{12} H$ may not be connected.

Proposition 2.5.

Let $G = K_{1,n}$ and $H = K_{1,m}$ with $m, n \geq 3$. Then $G \otimes_{12} H$ has two components.

Proof:

Let $G = K_{1,n}$ with $V(G) = \{u_0\} \cup U$ and $H = K_{1,m}$ with $V(H) = \{v_0\} \cup V$ with $\mid U \mid \geq 3$ and $\mid V \mid \geq 3$.

It is clear that the vertex (u_0, v_0) in $V(G \otimes_{12} H)$ cannot be adjacent with any other vertex say (u', v') in $G \otimes_{12} H$ as $d_G(u_0, u') = 1 = d_H(v_0, v')$ for $u' \neq u_0$ and $v' \neq v_0$.

Let (u, v) and (u', v') be two distinct vertices other than (u_0, v_0) in $V(G \otimes_{12} H)$.

Case (i): If $u = u_0$ and $v \neq v_0$, then (u, v) adjacent to (u', v') as $d_G(u, u') = 1$ and $d_H(v, v') = 2$, $v' \neq v_0$. If $v' = v_0$ or $u' = u_0$, then for some $u'' \in V(G)$ and $v'' \in V(H)$, $(u_0, v) \to (u'', v'') \to$

(u', v_0) or (u_0, v') is a path between (u, v) and (u', v') in $G \otimes_{12} H$.

Case (ii): If $u \neq u_0$ and $v \neq v_0$, then for some $u'' \neq u_0 \in V(G)$, we get a path $P' : (u, v) \to (u'', v_0) \to (u', v')$ in $G \otimes_{12} H$. Also, for some $v'' \neq v_0 \in V(H)$, we get a path $P'' : (u, v) \to (u_0, v'') \to (u', v')$ in $G \otimes_{12} H$.

Thus in all cases (u, v) is connected with (u', v') in $G \otimes_{12} H$. So, except (u_0, v_0) other vertices will give connected component. ■

Next, we prove that Theorem 2.4 is not true if $G = K_n$. In fact, we prove that the number of components of $K_n \otimes_{12} H$ depends on the bipartiteness of H.

Proposition 2.6.

Let H be a connected bipartite graph with $N^2(v) \neq \phi$, $\forall v \in V(H)$. Then the graph $K_n \otimes_{12} H$ has two components.

Proof:

Let K_n be a compete graph with vertex set U and H be a connected bipartite graph with partite sets V_1 and V_2. Then,

$$V(K_n \otimes_{12} H) = [U \times V_1] \cup [U \times V_2]$$

Let $(u, v) \in U \times V_1$ and $(u', v') \in U \times V_2$. Then (u, v) can not be adjacent with (u', v') as $d(u, u') \neq 2$ in K_n and $d_H(v, v') \neq 2$. So, $U \times V_1$ and $U \times V_2$ will give two disconnected subgraphs in $K_n \otimes_{12} H$.

Let (u, v) and (u', v') be in $U \times V_1$. As H is a connected graph, there is a path $P : v = v_0 \to v_1 \to v_2 \to \cdots \to v_m = v'$ between v and v' of even length. If $m = 4k + 2$, then $P' : (u, v) = (u, v_0) \to (u', v_2) \to (u, v_4) \to \ldots (u, v_{4k}) \to (u', v_{4k+2})$ is a path between (u, v) and (u', v') in $K_n \otimes_{12} H$.

Suppose $m = 4k$. Since $n \geq 3$, $\exists u'' \in U$ with $u \neq u' \neq u''$.
$P'' : (u, v) = (u, v_0) \to (u'', v_2) \to (u', v_4) \to \ldots (u, v_{4k-2}) \to (u', v_{4k})$ is a path between (u, v) and (u', v') in $K_n \otimes_{12} H$.

Since $N^2(v) \neq \phi$, $\exists b \in V(H)$ such that $d_H(v, b) = 2$. So, in case of $v = v'$, $(u, v) \to (u'', b) \to (u', v)$ is a path between (u, v) and (u', v') in $K_n \otimes_{12} H$.

Similarly $U \times V_2$ also gives a connected subgraph in $K_n \otimes_{12} H$. Thus, the graph $K_n \otimes_{12} H$ has two components. ■

To prove the result for non-bipartite graph, we shall use the following result.

Proposition 2.7.

Let G be a non-bipartite connected graph with $N^2(u) \neq \phi$, for every $u \in V(G)$. Assume that G contains C_{2l+1}, $l > 1$. Then between every pair of vertices, there exists a walk of length $4k$ as well as $4k + 2$; $(k \in \mathbb{N} \cup \{0\})$ form in G.

Proposition 2.8.

Let H be a non-bipartite connected graph containing C_{2l+1} $(l > 1)$ and $N^2(v) \neq \phi$, for every $v \in V(H)$. Then $K_n \otimes_{12} H$ is a connected graph.

Proof:

Let K_n and H be two graphs with vertex sets U and V respectively. Then $V(K_n \otimes_{12} H) = U \times V$. Let (u, v) and (u', v') be in $U \times V$. We continue the notation of Proposition 2.6. If $l(P)$ is even, then as in Proposition 2.6,we get the path in $K_n \otimes_{12} H$.

If $l(P)$ is odd, then by Proposition 2.7, there is a walk $W : v = w_0 \to w_1 \to \ldots \to w_p = v'$ of even length between v and v' in H with $d_H(w_i, w_{i+2}) = 2$. Then using W, as earlier we get the path in $K_n \otimes_{12} H$. So, the graph $K_n \otimes_{12} H$ is a connected graph. ■

Finally, for $G = K_2$, the following examples show that the number of components of $G \otimes_{12} H$ is not related with bipartiteness, e.g.,

$$K_2 \otimes_{12} C_{4k} = \bigcup_{\substack{\circ \\ i=1}}^{4} (C_{2k})^{(i)}, \quad K_2 \otimes_{12} C_{4k+2} = \bigcup_{\substack{\circ \\ i=1}}^{2} (C_{4k+2})^{(i)} \quad \text{and}$$
$$K_2 \otimes_{12} C_{4k+1} = C_{2(4k+1)}$$

Remark.

(i) Suppose G and H are two connected graphs with $D(G) \geq 2$ and $D(H) \geq 3$ respectively. Then the graph $G \otimes_{12} H$ is a connected graph. In particular $K_{1,n} \otimes_{12} H$ is connected with $D(H) \geq 3$.

(ii) It is known that the usual tensor product $G \otimes H$ is disconnected if G and H both are bipartite graphs. But as we have proved in Theorem 2.3 and Theorem 2.4, in general the connectedness of $G \otimes_{12} H$ is independent of bipartiteness of G and H.

3. Bipartiteness and Eulerian Property of $G \otimes_{12} H$

In this section, we discuss bipartiteness and Eulerian property of distance product graph $G \otimes_{12} H$. We fix G and H to be connected graphs with $N^2(w) \neq \phi$ for every $w \in V(G) \cup V(H)$.

It is known that if G and H both are bipartite graphs, then $G \otimes H$ is bipartite, but the graphs $G \times H$, $G \times_2 H$ and $G \otimes_2 H$ may not be bipartite graphs Acharya et al. (2015, 2017).

Proposition 3.1.

Let G and H be connected graphs. The graph $G \otimes_{12} H$ is bipartite if and only if G and H both are bipartite graphs.

Proof:

Let G and H be two bipartite graphs with partite sets $U_1,\ U_2$ and $V_1,\ V_2$ respectively. Then

$V(G \otimes_{12} H) = \{[U_1 \times V_1] \cup [U_2 \times V_2]\} \cup \{[U_1 \times V_2] \cup [U_2 \times V_1]\} = W_1 \cup W_2$, where $W_1 = [U_1 \times V_1] \cup [U_2 \times V_2]$ and $W_2 = [U_1 \times V_2] \cup [U_2 \times V_1]$.

Let (u, v) and (u', v') be in W_1. If both (u, v) and (u', v') are in the same $U_i \times V_i$ $(i = 1$ or $2)$, then (u, v) can not be adjacent to (u', v') in $G \otimes_{12} H$, as $d_G(u, u')$ and $d_H(v, v')$ both are even integers. Also if $(u, v) \in U_1 \times V_1$ and $(u', v') \in U_2 \times V_2$, then (u, v) can not be adjacent to (u', v'), as $d_G(u, u')$ and $d_H(v, v')$ are odd integers. So, W_1 is an independent set in $G \otimes_{12} H$. Similarly W_2 is also an independent set in $G \otimes_{12} H$.

Conversely, assume that G is a connected non-bipartite graph. So, G contains an odd cycle, say $C : u = u_0 \to u_1 \to \ldots \to u_{2n} \to u_{2n+1} = u_0$. Let v and v' be in $V(H)$ with $d_H(v, v') = 2$ and path $v \to b \to v'$.

Suppose $n = 2m$. Then $(u, v) = (u_0, v) \to (u_2, b) \to \ldots \to (u_{4m}, v') \to (u_{4m+1}, v) = (u, v)$ is an odd cycle of length $2m + 1 = n + 1$ in $G \otimes_{12} H$.

Also, if $n = 2m + 1$, then $(u, v) = (u_0, v) \to (u_2, b) \to (u_4, v) \to \ldots \to (u_{4m}, v) \to (u_{4m-1}, v') \to (u_{4m+1}, b) \to (u_{4m+3}, v) = (u, v)$ is an odd cycle of length $2m + 3 = n + 2$ in $G \otimes_{12} H$. Thus, in each case $G \otimes_{12} H$ contains an odd cycle. So, $G \otimes_{12} H$ is not a bipartite graph. ■

Next, we discuss degree of the vertex (u, v) in $G \otimes_{12} H$. We define $deg_2(u) = |N^2(u)|$, for $u \in V(G)$.

Proposition 3.2.

For $(u, v) \in G \otimes_{12} H$, $deg(u, v) = deg(u)\ deg_2(v)\ +\ deg_2(u) deg(v)$.

Proof:

Assume that $deg(u) = k$ with $N(u) = \{x_1, \ldots, x_k\}$ and $deg_2(u) = m$ with $N^2(u) = \{w_1, \ldots, w_m\}$ in G. Also $deg(v) = n$ with $N(v) = \{z_1, \ldots, z_n\}$ and $deg_2(v) = t$ with $N^2(v) = \{y_1, \ldots, y_t\}$ in H. The vertex (u, v) in $G \otimes_{12} H$ is adjacent to the following vertices: $\{(x_i, y_j); 1 \leq i \leq k$ with $1 \leq j \leq t\}$ and $\{(w_i, z_j); 1 \leq i \leq m$ with $1 \leq j \leq n\}$ in $G \otimes_{12} H$. So, $deg(u, v) = kt + mn$. Thus
$deg(u, v) = |N_1(u)||N^2(v)| + |N^2(u)||N_1(v)| = deg(u)\ deg_2(v)\ +\ deg_2(u) deg(v)$. ■

Proposition 3.3.

If G and H both are connected, Eulerian graphs, then $G \otimes_{12} H$ is an Eulerian graph.

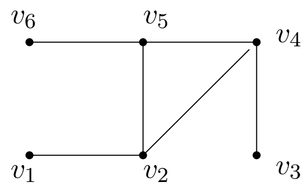
Proof:

Let G and H be connected Eulerian graphs. Let $(u, v) \in V(G \otimes_{12} H)$ with $u \in V(G)$ and $v \in V(H)$. Then $deg(u) = 2k$ in G and $deg(v) = 2t$ in H. So, by Proposition 3.2, $deg(u, v) = 2k\ deg_2(v) + 2t\ deg_2(u)$, an even number. Also, $G \otimes_{12} H$ is connected and so $G \otimes_{12} H$ is an Eulerian graph. ■

Remark.

(i) The converse of Proposition 3.3 is not true. For example, if $G = P_n$ and $H = C_m$, then it can be checked that $P_n \otimes_{12} C_m$ is Eulerian.

(ii) If G or H is not Eulerian graph, then Proposition 3.3 is not true, e.g., if $G = K_{2,4}$ and H is as follows, then $G \otimes_{12} H$ is not Eulerian.



4. Conclusion

We have defined new product of graphs, distance product $G \otimes_{12} H$ using the concept of distance between two vertices. We have proved that in most of non-trivial graphs (except star graph) distance product $G \otimes_{12} H$ is connected if and only if G and H are connected. We have also obtained number of components for $K_n \otimes_{12} H$. For bipartiteness we proved the strong result but in Eulerian property we get one way result.

Acknowledgement:

The authors are thankful to both the referees and the Editor-in-Chief Professor Aliakbar Montazer Haghighi for their valuable suggestions and comments towards the improvement of this paper.

REFERENCES

Acharya, U.P. and Mehta, H.S. (2014). 2-Cartesian Product of Special Graphs, Int. J. of Math. and Soft Comp., Vol. 4, No. 1, pp. 139–144.

Acharya, U.P. and Mehta, H.S. (2014). 2-tensor Product of special Graphs, Int. J. of Math. and Scietific Comp., Vol. 4, No. 1, pp. 21–24.

Acharya, U.P. and Mehta, H.S. (2015). Generalized Cartesian product of Graphs, Int. J. of Math. and Scietific Comp., Vol. 5, No. 1, pp. 4–7.

Acharya, U.P. and Mehta, H.S. (2017). Connectedness and Distance in $G \otimes_2 H$, IOSR Journal of Mathematics, Vol. 13, Issue 1, Ver. IV, pp. 1–7.

Ayyaswamy, S.K., Balachandran, S. and Gutman, I. (2010). On Second Stage Spectrum and energy of a graph, Kragujevac J. Math., Vol. 34, pp. 139–146.

Balakrishnan, R. and Ranganathan, K. (2012) . *A Textbook of Graph Theory*, 2nd edition, Universitext, Springer, New York.

Godsil, C. and Riyle, G.F. (2001) . *Algebraic Graph Theory*, Springer-Verlag, New York.

Hammack, Richard, Imrich, Wilfried and Klavzar,Sandi. (2011) . *Hand book of product Graphs*, CRC Press, Taylor & Francis Group, 2nd edition, New York.

Hande, S.P., Jog, S.R., Ramane, H.S., Hampiholi, P.R., Gutman, I. and Durg, B.S. (2013) . Derived Graphs of Subdivision Graphs, Kragujevac J. of Math., Vol. 37, No. 2, pp. 319–323.

Jog, S.R., Hande, S.P., Gutman, I. and Bozkurt, S. Burcu. (2012) . Derived Graphs of Some Graphs, Kragujevac J. Math., Vol. 36, No. 2, pp. 309–314.

Sampathkumar, E. (1975) . On Tensor Product of Graphs, J. Austral. Math. Soc., Vol. 20 (Series A), pp. 268–273.