Applications and Applied Mathematics: An International Journal (AAM)

Volume 13 | Issue 1 Article 1

6-2018

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Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 13, Issue 1 (June 2018), pp. 1–18

Incomplete Generalized (p, q, r)-Tribonacci Polynomials

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Received: January 17, 2018; Accepted: April 19, 2018

Abstract

In this paper, we consider an extension of the tribonacci polynomial, which we will refer to as the generalized (p, q, r)-tribonacci polynomial, denoted by $T_{n,m}(x)$. We find an explicit formula for $T_{n,m}(x)$, which we use to introduce the incomplete generalized (p, q, r)-tribonacci polynomials and derive several properties. An explicit formula for the generating function of the incomplete generalized polynomials is determined and a combinatorial interpretation is provided yielding further identities.

Keywords: Tribonacci polynomials; Incomplete tribonacci numbers; Polynomial generalization; Generating function; Combinatorial identity; Fibonacci polynomials; Riordan array

MSC 2010 No.: 11B39, 05A15

1. Introduction

The tribonacci numbers are defined by the recurrence relation

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}, \qquad n \geq 3, \tag{1}$$

with initial conditions $T_0 = 0$ and $T_1 = T_2 = 1$. Alladi and Hoggatt (1977) defined the tribonacci triangle (see Table 1) to obtain several properties of tribonacci numbers.

Table 1. Tribonacci triangle.

n/i	0	1	2	3	4	5	6	7	$\cdots$
0	1								
1	1	1							
2	1	3	1						
3	1	5	5	1					
4	1	7	13	7	1				
5	1	9	25	25	9	1			
6	1	11	41	63	41	11	1		
7	1	13	61	129	129	61	13	1	
$\vdots$					$\vdots$				

The sum of the elements along a rising diagonal of this triangle is given by the tribonacci number, that is,

$$T_{n+1} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} B\left(n-i, i\right), \tag{2}$$

where $B\left(n, i\right)$ is the n-th row and i-th column entry of the tribonacci triangle. In Barry (2006), an explicit representation of the tribonacci numbers is given by

$$T_n = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{j=0}^{i} \binom{i}{j} \binom{n-1-i-j}{i}. \tag{3}$$

Several generalizations of the tribonacci numbers have been considered. The tribonacci p-numbers where $p \geq 1$ were studied by Kuhapatanakul (2012) and are defined by the recurrence

$$T_p(n+2) = T_p(n+1) + T_p(n) + T_p(n-p), \qquad n \geq 0, \tag{4}$$

with initial conditions $T_p\left(1\right) = 1$ and $T_p\left(i\right) = 0$ for $-p \leq i \leq 0$. The tribonacci numbers are obtained by taking $p = 1$. Similar to the tribonacci triangle, Kuhapatanakul (2012) constructed the tribonacci p-triangle whose sum of elements along a rising diagonal is the tribonacci p-number.

A different generalization was considered by Hoggatt and Bicknell (1973) who defined the tribonacci polynomials $T_n\left(x\right)$ by the recurrence relation

$$T_n\left(x\right) = x^2 T_{n-1}\left(x\right) + x T_{n-2}\left(x\right) + T_{n-3}\left(x\right), \qquad n \geq 3, \tag{5}$$

with initial conditions $T_0\left(x\right) = 0$, $T_1\left(x\right) = 1$ and $T_2\left(x\right) = x^2$. Ramírez and Sirvent (2014) found an explicit formula for the tribonacci polynomials by constructing the tribonacci polynomial triangle. Then they introduced the incomplete tribonacci polynomials as

$$T_n^{(s)}(x) = \sum_{i=0}^{s} \sum_{j=0}^{i} \binom{i}{j} \binom{n-1-j-i}{i} x^{2n-2-3(i+j)}, \qquad 0 \leq s \leq \left\lfloor \frac{n-1}{2} \right\rfloor, \tag{6}$$

and established several properties of these polynomials. In Shattuck (2015), a combinatorial interpretation for $T_n^{(s)}(x)$ was provided in obtaining an explicit formula for the generating function of $T_n^{(s)}(x)$, which had been requested earlier in Ramírez and Sirvent (2014). Ramírez and Sirvent (2015) later generalized the tribonacci triangle to the k-bonacci by using a family of Riordan arrays. For other generalizations of the tribonacci and incomplete tribonacci polynomials, see Djordjevic and Djordjevic (2016). Finally, we refer the reader to such works as Belbachir and Belkhir (2014), Djordjevic (2004), Filipponi (1996), Pinter and Srivastava (1999), Tan (2018), and Tan and Ekin (2015) for related results involving incomplete Fibonacci (and Lucas) numbers and polynomials.

Here, we first define the generalized (p, q, r)-tribonacci polynomials, which extend both the tribonacci p-numbers and the classical tribonacci polynomials. (See Wang and Wang (2017), where generalized (p, q)-Fibonacci polynomials are defined analogously.) We then construct the generalized tribonacci polynomial triangle and find a formula for the entries of this triangle. Using the explicit formula for the generalized (p, q, r)-tribonacci polynomials, we introduce an incomplete version of these polynomials and establish several properties. A formula for the generating function of the incomplete generalized polynomials is determined and a combinatorial interpretation is provided which leads to some further identities.

2. Generalized (p, q, r)-tribonacci polynomials

Let $p(x)$, $q(x)$ and $r(x)$ be non-zero polynomials with real coefficients, where at times we will suppress the x argument.

Definition 2.1.

Let $m \geq 3$ be a fixed positive integer. The generalized (p, q, r)-tribonacci polynomials $T_{n,m}(x)$ are defined by the recurrence relation

$$T_{n,m}(x) = p(x)\, T_{n-1,m}(x) + q(x)\, T_{n-2,m}(x) + r(x)\, T_{n-m,m}(x), \qquad n \geq m, \tag{7}$$

with initial conditions $T_{0,m}(x) = 0$ and $T_{i,m}(x) = F_{p,q,i}(x)$ for $i = 1, \ldots, m-1$, where $F_{p,q,n}(x)$ is the (p, q)-Fibonacci polynomial (Lee and Asci (2012)) defined by

$$F_{p,q,n}(x) = p(x)\, F_{p,q,n-1}(x) + q(x)\, F_{p,q,n-2}(x), \qquad n \geq 2, \tag{8}$$

with $F_{p,q,0}(x) = 0$ and $F_{p,q,1}(x) = 1$.

The $m = 3, 4, 5$ cases of $T_{n,m}(x)$ for the first few n are given in the following tables.

We note the following special cases of $T_{n,m}(x)$:

(1) When $m = 3$ in (7), then we get the generalized tribonacci polynomials from Ramìrez and Sirvent (2015), the $p = x^2, q = x, r = 1$ case of which corresponds to the original tribonacci polynomials from Hoggatt and Bicknell (1973).
(2) The $p = q = r = 1$ case of (7) corresponds to the tribonacci p-numbers from Kuhapatanakul (2012) where $p + 2$ is equivalent to our m. Note that the $m = 4$ and $m = 5$ cases correspond to sequences A060945 and A079971 in Sloane (2010), respectively.

Table 2. The terms of $T_{n,3}(x)$

n	$T_{n,3}(x)$
0	0
1	1
2	p
3	p^2+q
4	$p^3+2pq+r$
5	$p^4+3p^2q+q^2+2pr$
6	$p^5+4p^3q+3pq^2+3p^2r+2qr$
7	$p^6+5p^4q+6p^2q^2+q^3+4p^3r+6pqr+r^2$
$\vdots$	$\vdots$

Table 3. The terms of $T_{n,4}(x)$

n	$T_{n,4}(x)$
0	0
1	1
2	p
3	p^2+q
4	p^3+2pq
5	$p^4+3p^2q+q^2+r$
6	$p^5+4p^3q+3pq^2+2pr$
7	$p^6+5p^4q+6p^2q^2+q^3+3p^2r+2qr$
8	$p^7+6p^5q+10p^3q^2+4pq^3+4p^3r+6pqr$
$\vdots$	$\vdots$

Now we define the generalized (p,q,r)-tribonacci polynomial triangle. Let the entry in the n-th row and i-th column of this array be denoted by $B_m(n,i)(x)$.

We define the $B_m(n,i)(x)$ as follows. For integers i and n with $1 \leq i < n$, let

$$B_m(n,i)(x) = pB_m(n-1,i)(x) + qB_m(n-1,i-1)(x) + rB_m(n-m+1,i-1)(x), \quad (9)$$

where $B_m(n,0)(x) = p^n$, $B_m(n,n)(x) = q^n$ and $B_m(n,i)(x) = 0$ for $i > n$.

The generalized (p,q,r)-tribonacci polynomial triangles for $m = 3,4,5$ are given in the tables below.

Note that if $p(x) = q(x) = r(x) = 1$, then we get the tribonacci p-triangle (Kuhapatanakul (2012)), while if $m = 3$ with $p(x) = x^2$, $q(x) = x$ and $r(x) = 1$, we get the tribonacci polynomial triangle (Ramírez and Sirvent (2014)).

Table 4. The terms of$T_{n,5}(x)$

n	$T_{n,5}(x)$
0	0
1	1
2	p
3	p^2+q
4	p^3+2pq
5	$p^4+3p^2q+q^2$
6	$p^5+4p^3q+3pq^2+r$
7	$p^6+5p^4q+6p^2q^2+q^3+2pr$
8	$p^7+6p^5q+10p^3q^2+4pq^3+3p^2r+2qr$
$\vdots$	$\vdots$

Table 5. The generalized (p,q,r)-tribonacci polynomial triangle.

n/i	0	1	2	3	4	$\cdots$
0	$B_m(0,0)(x)$					
1	$B_m(1,0)(x)$	$B_m(1,1)(x)$				
2	$B_m(2,0)(x)$	$B_m(2,1)(x)$	$B_m(2,2)(x)$			
3	$B_m(3,0)(x)$	$B_m(3,1)(x)$	$B_m(3,2)(x)$	$B_m(3,3)(x)$		
4	$B_m(4,0)(x)$	$B_m(4,1)(x)$	$B_m(4,2)(x)$	$B_m(4,3)(x)$	$B_m(4,4)(x)$	
$\vdots$			$\vdots$		$\vdots$	

Table 6. The case of $m=3$.

n/i	0	1	2	3	4	$\cdots$
0	1					
1	p	q				
2	p^2	$2pq+r$	q^2			
3	p^3	$3p^2q+2pr$	$3pq^2+2qr$	q^3		
4	p^4	$4p^3q+3p^2r$	$6p^2q^2+6pqr+r^2$	$4pq^3+3q^2r$	q^4	
$\vdots$			$\vdots$		$\vdots$	

By induction on n and the definition of $B_m(n,i)(x)$, we have

$$B_m(n,i)(x)=\sum_{j=0}^{i}\binom{i}{j}\binom{n-(m-2)j}{i}p^{n-i-(m-2)j}q^{i-j}r^j, \tag{10}$$

where the second binomial coefficient is taken to be zero if $n<(m-2)j$.

Table 7. The case of $m = 4$.

n/i	0	1	2	3	4	5	$\cdots$
0	1						
1	p	q					
2	p^2	$2pq$	q^2				
3	p^3	$3p^2q+r$	$3pq^2$	q^3			
4	p^4	$4p^3q+2pr$	$6p^2q^2+2qr$	$4pq^3$	q^4		
5	p^5	$5p^4q+3p^2r$	$10p^3q^2+6pqr$	$10p^2q^3+3q^2r$	$5pq^4$	q^5	
$\vdots$			$\vdots$		$\vdots$		

Table 8. The case of $m = 5$.

n/i	0	1	2	3	4	5	$\cdots$
0	1						
1	p	q					
2	p^2	$2pq$	q^2				
3	p^3	$3p^2q$	$3pq^2$	q^3			
4	p^4	$4p^3q+r$	$6p^2q^2$	$4pq^3$	q^4		
5	p^5	$5p^4q+2pr$	$10p^3q^2+2qr$	$10p^2q^3$	$5pq^4$	q^5	
$\vdots$			$\vdots$		$\vdots$		

Upon computing the generating function formulas of both quantities and comparing (which works out to $\frac{1}{1-px-qx^2-rx^m}$ in each case), we see that rising diagonal sums of the generalized (p,q,r)-tribonacci polynomial triangle are given by generalized (p,q,r)-tribonacci polynomials, i.e.,

$$T_{n,m}(x) = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} B_m(n-1-i,i)(x). \tag{11}$$

Thus, the explicit formula for $T_{n,m}(x)$ is given by

$$T_{n,m}(x) = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{j=0}^{i} \binom{i}{j} \binom{n-1-i-(m-2)j}{i} p^{n-1-2i-(m-2)j} q^{i-j} r^j, \qquad n \geq 1. \tag{12}$$

Remark.

Note that the (p,q,r)-tribonacci polynomial triangle corresponds to the *Riordan matrix* given by

$$[B_m(n,i)]_{n,i\geq 0} = \left(\frac{1}{1-px}, \frac{qx+rx^{m-1}}{1-px} \right),$$

where we refer the reader to Shapiro et al. (1991) for the definition. Therefore, by the summation property of Riordan matrices, the generating function for the row sum equals

$$\frac{1}{1-px}\cdot\frac{1}{1-\frac{qx+rx^{m-1}}{1-px}}=\frac{1}{1-(p+q)x-rx^{m-1}},$$

the $m=3$ and $m=4$ cases of which when $p=q=r=1$ correspond to the Pell number sequence and to sequence A008998 in Sloane (2010), respectively.

3. Incomplete generalized (p,q,r)-tribonacci polynomials

Let $m\geq 3$. We define the incomplete generalized (p,q,r)-tribonacci polynomials using the explicit formula for $T_{n,m}(x)$ as follows.

Definition 3.1.

Let $T_{n,m}^{(s)}(x)$ be defined by

$$\begin{aligned}T_{n,m}^{(s)}(x)&=\sum_{i=0}^{s}B_m\left(n-1-i,i\right)(x)\\&=\sum_{i=0}^{s}\sum_{j=0}^{i}\binom{i}{j}\binom{n-1-i-(m-2)\,j}{i}p^{n-1-2i-(m-2)j}q^{i-j}r^{j},\end{aligned}\tag{13}$$

where $n\geq 1$ and $0\leq s\leq\left\lfloor\frac{n-1}{2}\right\rfloor$. We will refer to $T_{n,m}^{(s)}(x)$ as the *incomplete generalized* (p,q,r)-*tribonacci polynomial*.

We have the following special cases of $T_{n,m}^{(s)}(x)$:

(1) $T_{n,m}^{(0)}\left(x\right)=p^{n-1},\quad n\geq 1,$

(2) $T_{n,m}^{(1)}\left(x\right)=p^{n-1}+\left(n-2\right)p^{n-3}q+\left(n-m\right)p^{n-m-1}r,\quad n\geq m,$

(3) $T_{n,m}^{\left(\left\lfloor\frac{n-1}{2}\right\rfloor\right)}(x)=T_{n,m}\left(x\right),\quad n\geq 1,$

(4) $T_{n,m}^{\left(\left\lfloor\frac{n-3}{2}\right\rfloor\right)}(x)=\begin{cases}T_{n,m}\left(x\right)-\frac{n}{2}pq^{\frac{n-2}{2}}-\delta_{m,3}\left(\frac{n-2}{2}\right)q^{\frac{n-4}{2}}r, & \text{if } n \text{ is even;}\\ T_{n,m}\left(x\right)-q^{\frac{n-1}{2}}, & \text{if } n \text{ is odd.}\end{cases}$

The terms of $T_{n,m}^{(s)}(x)$ for $1\leq n\leq 6$ are given in the following table.

The first few terms of the $T_{n,m}^{(s)}(x)$ for $m=3$ and $m=4$ are given in the tables below.

Proposition 3.2.

The incomplete generalized (p,q,r)-tribonacci polynomials $T_{n,m}^{(s)}(x)$ satisfy the non-linear recurrence

$$T_{n,m}^{(s+1)}\left(x\right)=p\left(x\right)T_{n-1,m}^{(s+1)}\left(x\right)+q\left(x\right)T_{n-2,m}^{(s)}\left(x\right)+r\left(x\right)T_{n-m,m}^{(s)}\left(x\right)\tag{14}$$

Table 9. The polynomials $T_{n,m}^{(s)}(x)$.

n/s	0	1	2
1	$B_m(0,0)$		
2	$B_m(1,0)$		
3	$B_m(2,0)$	$B_m(2,0)+B_m(1,1)$	
4	$B_m(3,0)$	$B_m(3,0)+B_m(2,1)$	
5	$B_m(4,0)$	$B_m(4,0)+B_m(3,1)$	$B_m(4,0)+B_m(3,1)+B_m(2,2)$
6	$B_m(5,0)$	$B_m(5,0)+B_m(4,1)$	$B_m(5,0)+B_m(4,1)+B_m(3,2)$

Table 10. The polynomials $T_{n,3}^{(s)}(x)$.

n/s	0	1	2
1	1		
2	p		
3	p^2	p^2+q	
4	p^3	$p^3+2pq+r$	
5	p^4	p^4+3p^2q+2pr	$p^4+3p^2q+2pr+q^2$
6	p^5	$p^5+4p^3q+3p^2r$	$p^5+4p^3q+3p^2r+3pq^2+2qr$

Table 11. The polynomials $T_{n,4}^{(s)}(x)$.

n/s	0	1	2
1	1		
2	p		
3	p^2	p^2+q	
4	p^3	p^3+2pq	
5	p^4	p^4+3p^2q+r	$p^4+3p^2q+r+q^2$
6	p^5	p^5+4p^3q+2pr	$p^5+4p^3q+2pr+3pq^2$

and the non-homogeneous linear recurrence

$$\begin{aligned} T_{n,m}^{(s)}(x) = {} & p(x)\,T_{n-1,m}^{(s)}(x) + q(x)\,T_{n-2,m}^{(s)}(x) + r(x)\,T_{n-m,m}^{(s)}(x) \\ & -q(x)\,B_m(n-3-s,s)(x) - r(x)\,B_m(n-m-1-s,s)(x), \end{aligned} \tag{15}$$

where $n \geq m+1$ and $0 \leq s \leq \left\lfloor \frac{n-1}{2} \right\rfloor$.

Proof:

In showing (14), one may assume $s \leq \lfloor \frac{n-3}{2} \rfloor$, for it is clear if $s = \lfloor \frac{n-1}{2} \rfloor$. By (13) and (9), we have

$$
\begin{aligned}
& pT^{(s+1)}_{n-1,m} + qT^{(s)}_{n-2,m} + rT^{(s)}_{n-m,m} \\
&= p\sum_{i=0}^{s+1} B_m(n-2-i,i) + q\sum_{i=0}^{s} B_m(n-3-i,i) + r\sum_{i=0}^{s} B_m(n-m-1-i,i) \\
&= p\sum_{i=0}^{s+1} B_m(n-2-i,i) + q\sum_{i=1}^{s+1} B_m(n-2-i,i-1) \\
&\quad + r\sum_{i=1}^{s+1} B_m(n-m-i,i-1) \\
&= \sum_{i=0}^{s+1} \left(pB_m(n-2-i,i) + qB_m(n-2-i,i-1) + rB_m(n-m-i,i-1)\right) \\
&\quad - qB_m(n-2,-1) - rB_m(n-m,-1) \\
&= \sum_{i=0}^{s+1} B_m(n-1-i,i)(x) = T^{(s+1)}_{n,m}(x),
\end{aligned}
$$

which gives (14). To show (15), first note that by (13) and (14), we have

$$
\begin{aligned}
\sum_{i=0}^{s} B_m(n-1-i,i) &= p\sum_{i=0}^{s} B_m(n-2-i,i) + q\sum_{i=0}^{s-1} B_m(n-3-i,i) \\
&\quad + r\sum_{i=0}^{s-1} B_m(n-m-1-i,i).
\end{aligned}
$$

The preceding equation may be rewritten as

$$
\begin{aligned}
\sum_{i=0}^{s} B_m(n-1-i,i) &= p\sum_{i=0}^{s} B_m(n-2-i,i) + q\sum_{i=0}^{s} B_m(n-3-i,i) \\
&\quad + r\sum_{i=0}^{s} B_m(n-m-1-i,i) \\
&\quad - qB_m(n-3-s,s) - rB_m(n-m-1-s,s),
\end{aligned}
$$

which implies

$$
\begin{aligned}
T^{(s)}_{n,m}(x) &= pT^{(s)}_{n-1,m}(x) + qT^{(s)}_{n-2,m}(x) + rT^{(s)}_{n-m,m}(x) \\
&\quad - qB_m(n-3-s,s) - rB_m(n-m-1-s,s).
\end{aligned}
$$

■

Corollary 3.3.

If $s \geq 0$, then

$$T_{n+1,m}^{(s)}(x) = \sum_{i=0}^{s} \left(pq^i T_{n-2i,m}^{(s-i)}(x) + rq^i T_{n-2i-m+1,m}^{(s-i-1)}(x) \right), \qquad n \geq 2s+1. \tag{16}$$

Proof:

Induct on s, the $s=0$ case clear since $T_{n+1,m}^{(s)}(x) = p^n$ for $n \geq 0$. If $s \geq 0$ and $n \geq 2s+3$, then by (14), we have

$$\begin{aligned}
T_{n+1,m}^{(s+1)}(x) &= pT_{n,m}^{(s+1)}(x) + rT_{n-m+1,m}^{(s)}(x) + qT_{n-1,m}^{(s)}(x) \\
&= pT_{n,m}^{(s+1)}(x) + rT_{n-m+1,m}^{(s)}(x) + q\sum_{i=0}^{s} \left(pq^i T_{n-2i-2,m}^{(s-i)}(x) + rq^i T_{n-2i-m-1,m}^{(s-i-1)}(x) \right) \\
&= pT_{n,m}^{(s+1)}(x) + rT_{n-m+1,m}^{(s)}(x) + \sum_{i=1}^{s+1} \left(pq^i T_{n-2i,m}^{(s-i+1)}(x) + rq^i T_{n-2i-m+1,m}^{(s-i)}(x) \right) \\
&= \sum_{i=0}^{s+1} \left(pq^i T_{n-2i,m}^{(s-i+1)}(x) + rq^i T_{n-2i-m+1,m}^{(s-i)}(x) \right),
\end{aligned}$$

which completes the induction. ■

Theorem 3.4.

Let $a, b \geq 0$. Then the incomplete generalized (p, q, r)-tribonacci polynomials $T_{n,m}^{(s)}(x)$ satisfy

$$\sum_{s=0}^{a} \binom{s}{b} T_{n,m}^{(s)}(x) = \binom{a+1}{b+1} T_{n,m}^{(a)}(x) - \sum_{i=0}^{a}\sum_{j=0}^{i} \binom{i}{b+1}\binom{i}{j}\binom{n-1-i-(m-2)j}{i} p^{n-1-2i-(m-2)j} q^{i-j} r^j. \tag{17}$$

Proof:

From the definition $T_{n,m}^{(s)}(x) = \sum_{i=0}^{s} B_m(n-1-i,i)(x)$, we have

$$\begin{aligned}
\sum_{s=0}^{a} \binom{s}{b} T_{n,m}^{(s)}(x) &= \binom{0}{b} T_{n,m}^{(0)}(x) + \binom{1}{b} T_{n,m}^{(1)}(x) + \binom{2}{b} T_{n,m}^{(2)}(x) + \cdots + \binom{a}{b} T_{n,m}^{(a)}(x) \\
&= \binom{0}{b} B_m(n-1,0)(x) + \left(\binom{1}{b} B_m(n-1,0)(x) + \binom{1}{b} B_m(n-2,1)(x) \right) \\
&+ \left(\binom{2}{b} B_m(n-1,0)(x) + \binom{2}{b} B_m(n-2,1)(x) + \binom{2}{b} B_m(n-3,2)(x) \right) \\
&\vdots
\end{aligned}$$

$$+\left(\binom{a}{b}B_m\left(n-1,0\right)\left(x\right)+\binom{a}{b}B_m\left(n-2,1\right)\left(x\right)+\cdots+\binom{a}{b}B_m\left(n-1-a,a\right)\left(x\right)\right)$$

$$=\left[\binom{a+1}{b+1}-\binom{0}{b+1}\right]B_m\left(n-1,0\right)\left(x\right)+\left[\binom{a+1}{b+1}-\binom{1}{b+1}\right]B_m\left(n-2,1\right)\left(x\right)$$

$$+\cdots+\left[\binom{a+1}{b+1}-\binom{a}{b+1}\right]B_m\left(n-1-a,a\right)\left(x\right),$$

where we have used Identity 5.9 from Graham et al. (1994) in the last equality. Thus, we obtain

$$\sum_{s=0}^{a}\binom{s}{b}T_{n,m}^{(s)}\left(x\right)=\sum_{i=0}^{a}\left[\binom{a+1}{b+1}-\binom{i}{b+1}\right]B_m\left(n-1-i,i\right)\left(x\right)$$

$$=\sum_{i=0}^{a}\binom{a+1}{b+1}B_m\left(n-1-i,i\right)\left(x\right)-\sum_{i=0}^{a}\binom{i}{b+1}B_m\left(n-1-i,i\right)\left(x\right)$$

$$=\binom{a+1}{b+1}T_{n,m}^{(a)}\left(x\right)-\sum_{i=0}^{a}\binom{i}{b+1}B_m\left(n-1-i,i\right)\left(x\right),$$

which implies (17). ■

Remark.

Formula (17) generalizes Proposition 5 in Ramírez and Sirvent (2014), reducing to it when $m=3$, $a=\lfloor\frac{n-1}{2}\rfloor$, $b=0$, $p=x^2$, $q=x$ and $r=1$. Formula (14) reduces to Proposition 2 in Ramírez and Sirvent (2014), upon taking the same values for m, p, q and r.

4. Generating function formula for $T_{n,m}^{(s)}(x)$

In order to determine a generating function formula for $T_{n,m}^{(s)}(x)$, we will proceed in a combinatorial manner. Given $m\geq 3$, let $\mathcal{T}_n=\mathcal{T}_{n,m}$ denote the set of tilings of length n consisting of squares, dominos and m-inos, where an m-ino is a $1\times m$ rectangular piece. When $m=3$, members of $\mathcal{T}_n$ correspond to the tribonacci tilings (see, e.g., Benjamin and Quinn (2003)). In what follows, a *longer piece* will refer to either a domino or an m-ino. Define the weight of $\lambda\in\mathcal{T}_n$ by $\text{wght}(\lambda)=p^{u_1(\lambda)}q^{u_2(\lambda)}r^{u_3(\lambda)}$, where u_1, u_2 and u_3 record the number of squares, dominos and m-inos, respectively, of λ.

Given $s\geq 0$, let $\ell_{n,m}=\ell_{n,m}^{(s)}(p,q,r)$ denote the sum of the weights of all members of $\mathcal{T}_{n+2s}$ containing exactly $s+1$ longer pieces and ending in a longer piece. There is the following generating function formula for $\ell_{n,m}$.

Lemma 4.1.

If $s \geq 0$, then

$$\sum_{n\geq 0} \ell_{n,m} z^n = z^2 \left(\frac{q + rz^{m-2}}{1 - pz} \right)^{s+1}. \tag{18}$$

Proof:

To find an expression for $\ell_{n,m}$, let $\mathcal{S}$ denote the subset of $\mathcal{T}_{n+2s}$ enumerated by $\ell_{n,m}$. We first count $\pi \in \mathcal{S}$ ending in a domino. Suppose π contains j m-inos and $s-j+1$ dominos for some $0 \leq j \leq s$, and hence $n + 2s - mj - 2(s - j + 1) = n - (m-2)j - 2$ squares. Upon selecting the positions for the squares and the dominos, it is seen that there are $\binom{s}{j}\binom{n+s-(m-2)j-2}{s}$ possible tilings, each of weight $p^{n-(m-2)j-2} q^{s-j+1} r^j$. Considering all possible j gives a total weight of

$$\sum_{j=0}^{s} \binom{s}{j} \binom{n+s-(m-2)j-2}{s} p^{n-(m-2)j-2} q^{s-j+1} r^j$$

for all members of $\mathcal{S}$ ending in a domino. Upon finding a similar expression for the weight of all members of $\mathcal{S}$ ending in an m-ino, we get

$$\ell_{n,m} = \sum_{j=0}^{s} \binom{s}{j} \binom{n+s-(m-2)j-2}{s} p^{n-(m-2)j-2} q^{s-j+1} r^j$$

$$+ \sum_{j=0}^{s} \binom{s}{j} \binom{n+s-(m-2)j-m}{s} p^{n-(m-2)j-m} q^{s-j} r^{j+1}, \qquad n \geq 0, \tag{19}$$

where binomial coefficients with negative upper indices are taken here to be zero. Multiplying both sides of (19) by z^n, summing over $n \geq 0$ and interchanging summation implies

$$\sum_{n\geq 0} \ell_{n,m} z^n = \frac{qz^2(q + rz^{m-2})^s}{(1-pz)^{s+1}} + \frac{rz^m(q + rz^{m-2})^s}{(1-pz)^{s+1}},$$

which gives (18). ■

For $m \geq 3$, note that $T_{n,m} = T_{n,m}(x)$ is also given recursively by

$$T_{n,m} = pT_{n-1,m} + qT_{n-2,m} + rT_{n-m,m}, \qquad n \geq 2,$$

with $T_{1,m} = 1$ and $T_{0,m} = T_{-1,m} = \cdots = T_{-(m-2),m} = 0$. Define the generating function $R_m^{(s)}(z)$ by

$$R_m^{(s)}(z) = \sum_{n\geq 2s+1} T_{n,m}^{(s)}(x) z^n, \qquad s \geq 0.$$

Note that by the definitions, $T_{n,m}(x)$ is the sum of the weights of all members of $\mathcal{T}_{n-1}$, while $T_{n,m}^{(s)}(x)$ is the restriction of $T_{n,m}(x)$ to those members of $\mathcal{T}_{n-1}$ containing at most s longer pieces, where $0 \leq s \leq \lfloor (n-1)/2 \rfloor$.

We have the following generating function formula for $R_m^{(s)}(z)$.

Theorem 4.2.

If $s \geq 0$, then

$$\frac{R_m^{(s)}(z)}{z^{2s+1}} = \frac{T_{2s+1,m} + (qz + rz^{m-1})T_{2s,m} + r\sum_{j=1}^{m-2} T_{2s-j,m}z^{m-j-1} - z^2\left(\frac{q+rz^{m-2}}{1-pz}\right)^{s+1}}{1 - pz - qz^2 - rz^m}. \tag{20}$$

Proof:

First note that the product $\ell_{i,m}T_{n-2s-i,m}$ gives the total weight of all members of $\mathcal{T}_{n-1}$ containing at least $s+1$ longer pieces where the $(s+1)$-st longer piece ends at position $i+2s$. Considering all possible i gives the weight of all members of $\mathcal{T}_{n-1}$ containing strictly more than s longer pieces. Thus, by subtraction, we have

$$T_{n,m}^{(s)}(x) = T_{n,m} - \sum_{i=0}^{n-2s-1} \ell_{i,m}T_{n-2s-i,m}, \qquad n \geq 2s+1. \tag{21}$$

To find an appropriate recurrence for $T_n = T_{n,m}$, consider whether or not there is a piece covering the boundary between positions $2s$ and $2s+1$ within a member of $\mathcal{T}_{n-1}$, and if so, whether or not that piece is a domino or an m-ino. Note that in the latter case, the leftmost position covered by the m-ino would be $2s-j$ for some $0 \leq j \leq m-2$. This leads to the following recurrence for $m \leq 2s+1$:

$$T_n = T_{n-2s}T_{2s+1} + qT_{n-2s-1}T_{2s} + r\sum_{j=0}^{m-2} T_{2s-j}T_{n-m-2s+j+1}, \qquad n \geq 2s+1. \tag{22}$$

By the values of T_n when $n < 0$, formula (22) is seen also to hold for $m > 2s+1$. Multiplying both sides of (22) by z^n, and summing over $n \geq 2s+1$, gives

$$\begin{aligned}\sum_{n\geq 2s+1} T_n z^n &= T_{2s+1}\sum_{n\geq 2s+1} T_{n-2s}z^n + qT_{2s}\sum_{n\geq 2s+1} T_{n-2s-1}z^n \\ &\quad + r\sum_{j=0}^{m-2} T_{2s-j}\sum_{n\geq 2s+1} T_{n-m-2s+j+1}z^n \\ &= \left(T_{2s+1}z^{2s} + (qz^{2s+1} + rz^{m+2s-1})T_{2s} + rz^{m-1}\sum_{j=1}^{m-2} T_{2s-j}z^{2s-j}\right)T_m(z),\end{aligned}$$

where $T_m(z) = \sum_{n\geq 0} T_{n,m}z^n = \frac{z}{1-pz-qz^2-rz^m}$. Let $L_m^{(s)}(z)$ denote the generating function in (18). Multiplying both sides of (21) by z^n, and summing over $n \geq 2s+1$, then implies

$$\begin{aligned}R_m^{(s)}(z) &= \sum_{n\geq 2s+1} T_{n,m}z^n - \sum_{n\geq 2s+1} z^n \sum_{i=0}^{n-2s-1} \ell_{i,m}T_{n-2s-i,m} \\ &= z^{2s}\left(T_{2s+1} + (qz + rz^{m-1})T_{2s} + r\sum_{j=1}^{m-2} T_{2s-j}z^{m-j-1}\right)T_m(z) - z^{2s}L_m^{(s)}(z)T_m(z),\end{aligned}$$

from which (20) follows from (18). ∎

Remark.

Letting $m = 3$ and $p = x^2$, $q = x$, $r = 1$ in formula (20) gives Theorem 0.9 in Shattuck (2015), while letting $m = 3$ and $p = q = r = 1$ gives Theorem 8 from Ramírez and Sirvent (2014).

5. Further identities for $T_{n,m}^{(s)}(x)$

Using the combinatorial interpretation for $T_{n,m}^{(s)}(x)$ discussed above, it is possible to find additional identities.

Proposition 5.1.

If $n \geq 2$ and $s \geq 0$, then

$$pT_{n+1,m}^{(s)}(x) + rT_{n-m+2,m}^{(s-1)}(x) = p^{n+1} + q\sum_{i=1}^{n-1} p^i T_{n-i,m}^{(s-1)}(x) + r\sum_{i=0}^{n-m+1} p^i T_{n-m-i+2,m}^{(s-1)}(x). \tag{23}$$

Proof:

Let $\mathcal{T}_n^{(s)}$ denote the subset of $\mathcal{T}_n$ whose members contain at most s longer pieces. Note first that the left-hand side of (23) is seen to enumerate all members of $\mathcal{T}_{n+1}^{(s)}$ ending in either a square or an m-ino. On the other hand, if $\lambda \in \mathcal{T}_{n+1}^{(s)}$ ends in i squares preceded by a domino where $1 \leq i \leq n-1$, then there are $p^i q T_{n-i,m}^{(s-1)}$ possibilities since the remaining tiles can contain at most $s-1$ longer pieces among them. Considering all i gives all members of $\mathcal{T}_{n+1}^{(s)}$ ending in a non-empty sequence of squares preceded by a domino. Note that this corresponds to the first sum on the right side of (23). Similarly, the second sum counts all members of $\mathcal{T}_{n+1}^{(s)}$ ending in a (possibly empty) sequence of squares preceded by an m-ino. Finally, the p^{n+1} term accounts for the all-squares tiling and combining the previous cases gives (23). ■

Taking $s = \lfloor n/2 \rfloor$ in (23), and replacing n by $n+2$, and p, q, r by c_1, c_2, c_3, respectively, gives

$$c_1T_{n+3,m} + c_3T_{n-m+4,m} = c_1^{n+3} + c_2\sum_{i=1}^{n+1} c_1^i T_{n-i+2,m} + c_3\sum_{i=0}^{n-m+3} c_1^i T_{n-m-i+4,m}$$

$$= c_1^{n+3} + c_2\sum_{i=0}^{n} c_1^{n-i+1} T_{i+1,m} + c_3\sum_{i=m-3}^{n} c_1^{n-i} T_{i-m+4,m}$$

$$= c_1^n(c_1^3 + c_1c_2 + T_{4-m,m}c_3) + \sum_{i=1}^{n} c_1^{n-i}\left(c_1c_2T_{i+1,m} + c_3T_{i-m+4,m}\right) - c_3\sum_{i=0}^{m-4} c_1^{n-i} T_{i-m+4,m}.$$

When $m = 3$ in the last equality, one gets

$$c_1T_{n+3,3} + c_3T_{n+1,3} = c_1^n(c_1^3 + c_1c_2 + c_3) + (c_1c_2 + c_3)\sum_{i=1}^{n} c_1^{n-i} T_{i+1,3},$$

which is equivalent to the case of Identity 77 from Benjamin and Quinn (2003) where $p_i = c_i$ for $i = 1, 2, 3$. Considering members of $\mathcal{T}_{n+1}^{(s)}$ ending in a domino or an m-ino leads in a similar manner as above to the following formulas for $n \geq 2s$ and $n \geq ms$, respectively:

$$qT_{n,m}^{(s-1)}(x) + rT_{n-m+2,m}^{(s-1)}(x) = p\sum_{i=1}^{s} q^i T_{n-2i+1,m}^{(s-i)}(x) + r\sum_{i=0}^{s-1} q^i T_{n-m-2i+2,m}^{(s-i-1)}(x) \tag{24}$$

and

$$qT_{n,m}^{(s-1)}(x) + rT_{n-m+2,m}^{(s-1)}(x) = p\sum_{i=1}^{s} r^i T_{n-mi+1,m}^{(s-i)}(x) + q\sum_{i=0}^{s-1} r^i T_{n-mi,m}^{(s-i-1)}(x). \tag{25}$$

When $m = 3$, we were unable to find in the literature the formulas (24) or (25), in particular in the complete case when $s = \lfloor (n+1)/2 \rfloor$.

Considering the number i of dominos and the number j of m-inos among the final s pieces of a member of $\mathcal{T}_{n-1}^{(s)}$ gives

$$T_{n,m}^{(s)}(x) = \sum_{i=0}^{s}\sum_{j=0}^{s-i} p^{s-i-j} q^i r^j \binom{s}{i, j, s-i-j} T_{n-s-i-(m-1)j}^{(s-i-j)}(x), \qquad n \geq ms+1, \tag{26}$$

which generalizes Identity 0.7 in Shattuck (2015).

Let

$$F_n^{(s)}(x) = \sum_{i=0}^{s} p^{n-1-2i} q^i \binom{n-1-i}{i}, \qquad n \geq 1,$$

denote a polynomial analogue of the incomplete Fibonacci numbers.

Proposition 5.2.

If $n \geq m$, then

$$T_{n+1,m}^{(s)}(x) = F_{n+1}^{(s)}(x) + r\sum_{i=0}^{n-m}\sum_{j=0}^{\alpha} p^{i-2j} q^j \binom{i-j}{j} T_{n-i-m+1,m}^{(s-j-1)}(x), \tag{27}$$

where $\alpha = \min\{s-1, \lfloor i/2 \rfloor\}$, and

$$T_{n+1,m}^{(s)}(x) = \sum_{i=0}^{n} \sum_{\substack{j=\beta \\ mj \equiv i \text{ (mod 2)}}}^{\gamma} q^{\frac{i-mj}{2}} r^j \binom{j + \frac{i-mj}{2}}{j} \left(pT_{n-i,m}^{((2s-i+(m-2)j)/2)}(x) + \delta_{n,i} \right), \tag{28}$$

where $\beta = \max\{0, \lceil \frac{i-2s}{m-2} \rceil\}$ and $\gamma = \min\{s, \lfloor i/m \rfloor\}$.

Proof:

To show (27), consider the leftmost position $i+1$ covered by the first m-ino piece (if it exists) and the number j of dominos to the left of the first m-ino within $\lambda \in \mathcal{T}_n^{(s)}$. There are $p^{i-2j} q^j \binom{i-j}{j}$ possibilities for the pieces to the left of the first m-ino which itself contributes a factor of r towards the weight. There are then $T_{n-m-i+1,m}^{(s-j-1)}(x)$ ways in which to tile the remaining $n-m-i$ positions of λ. Note that $0 \leq j \leq \alpha$ in order for λ to exist. Summing over all possible i and j gives the weight of all members of $\mathcal{T}_n^{(s)}$ containing at least one m-ino. From its definition, it is seen that there are $F_{n+1}^{(s)}(x)$ possible tilings which do not contain an m-ino, from which formula (27) follows.

For (28), consider now the position $i+1$ of the first square (if it exists) of $\lambda \in \mathcal{T}_n^{(s)}$, where $0 \leq i \leq n-1$, and the number j of m-inos to the left of the first square. Observe that $j \leq s$ and $mj \leq i$, whence $j \leq \gamma$. Also, we must have $mj \equiv i \pmod 2$, in which case there are $\frac{i-mj}{2}$ dominos to the left of the first square. Since $j + \frac{i-mj}{2} \leq s$, we have $j \geq \beta$. The square in the $(i+1)$-st position contributes p towards the weight and the tiles beyond this position constitute a member of $\mathcal{T}_{n-i-1,m}^{(u)}$, where $u = \frac{2s-i+(m-2)j}{2}$ (note that $j \geq \beta$ and $mj \equiv i \pmod 2$ implies u is a non-negative integer). Considering all possible i and j then gives the first part of the sum on the right-hand side of (28). On the other hand, if λ contains no squares, then it is seen that there are

$$\sum_{\substack{j=\beta \\ mj\equiv n \pmod 2}}^{\gamma} q^{\frac{n-mj}{2}} r^j \binom{j+\frac{n-mj}{2}}{j}$$

possibilities. Combining this case with the previous completes the proof of (28). ■

The identities in the preceding proposition seem to be new also in the case $m=3$. For example, taking $p=q=r=1$ and $m=3$ in (27) and (28) gives

$$T_{n+1}^{(s)} = F_{n+1}^{(s)} + \sum_{i=0}^{n-3}\sum_{j=0}^{\alpha} \binom{i-j}{j} T_{n-i-2}^{(s-j-1)}, \qquad n \geq 2,$$

and

$$T_{n+1}^{(s)} = \sum_{i=0}^{n} \sum_{\substack{j=\beta \\ j\equiv i \pmod 2}}^{\gamma} \binom{\frac{i-j}{2}}{j} \left(T_{n-i}^{((2s-i+j)/2)} + \delta_{n,i}\right), \qquad n \geq 2,$$

where $T_n^{(s)}$ denotes the incomplete tribonacci number (see, e.g., Ramírez and Sirvent (2014)).

6. Conclusion

In this paper, we have considered a class of generalized tribonacci polynomials $T_{n,m}(x)$ which reduces to the usual tribonacci polynomials when $m=3$ with $p(x)=x^2$, $q(x)=x$ and $r(x)=1$. We derive an explicit formula for $T_{n,m}(x)$ as a doubly-indexed sum which allows one to define and ascertain several properties of the incomplete generalized tribonacci polynomials $T_{n,m}^{(s)}(x)$. A combinatorial approach is employed to determine a formula for the generating function of $T_{n,m}^{(s)}(x)$ for a fixed m and s, which extends earlier formulas from Ramírez and Sirvent (2014) and Shattuck (2015). The combinatorial interpretation for $T_{n,m}^{(s)}(x)$ is then used to prove further identities, several of which appear to be new also in the case $m=3$ (see, for example, formulas (24)-(28) above). Perhaps it would be interesting to investigate incomplete versions of the generalized tribonacci polynomials where the terms corresponding to the complete case do not satisfy a linear recurrence. For example, one could replace $q(x)$ with say q^{n-2} or $r(x)$ with r^{n-m} in the defining recurrence (7) and consider properties of the incomplete tribonacci polynomials that result. Finally, one might try to find a combinatorial interpretation for the unsigned inverse matrix $[B_m(n,i)]_{n,i\geq 0}^{-1}$ (see remark above for notation) when $p = q = r = 1$, the $m = 3$ case of which has been studied by Yang et al. (2013) and Ramírez and Sirvent (2018).

Acknowledgement:

We wish to thank the referee for carefully reading our manuscript and for pointing out to us the connection to Riordan arrays.

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