Applications and Applied Mathematics: An International Journal (AAM)

Volume 12 | Issue 2 Article 23

12-2017

Certain integrals associated with the generalized Bessel-Maitland function

D. L. Suthar
Wollo University

Hafte Amsalu
Wollo University

Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 12, Issue 2 (December 2017), pp. 1002 - 1016

Certain integrals associated with the generalized Bessel-Maitland function

[1]D.L. Suthar and [2]Hafte Amsalu

Department of Mathematics
Wollo University, Dessie
P.O. Box: 1145
South Wollo, Amhara Region, Ethiopia
Email- [1]dlsuthar@gmail.com, [2]hafamsalu@gmail.com

Received: February 14, 2017; Accepted: September 2, 2017

Abstract

The aim of this paper is to establish two general finite integral formulas involving the generalized Bessel-Maitland functions $J_{\nu,\,q}^{\mu,\gamma}(z)$. The result given in terms of generalized (Wright's) hypergeometric functions ${}_p\psi_q$ and generalized hypergeometric functions ${}_pF_q$. These results are obtained with the help of finite integral due to Lavoie and Trottier. Some interesting special cases involving Bessel-Maitland function, Struve's functions, Bessel functions, generalized Bessel functions, Wright function, generalized Mittag-Leffler functions are deduced.

Keywords: Lavoie-Trottier integral formula; Gamma function; Hypergeometric functions; Bessel function; Generalized Bessel-Maitland function; Generalized Wright Hypergeometric functions

MSC 2010 No.: 26A33, 33B15, 33C10, 33C20

1. Introduction and Preliminaries

In Applied sciences, many important functions are defined via improper integrals or series (or finite products). These important functions are generally known as special functions. In special functions, one of the most important functions (Bessel function) is widely used in physics and engineering; therefore, they are of interest to physicists and engineers as well as mathematicians. In recent years, a remarkably large number of integral formulas involving a variety of special functions have been developed by many authors: Brychkov (2008), Choi and Agarwal (2013), Choi et al. (2014), Agarwal et al. (2014), Manaria et al. (2014), Khan and Kashmin (2016), Parmar and Purohit (2016), Suthar and Haile (2016) and Nisar et al. (2016, 2017). We aim at presenting two generalized integral formulas involving the Bessel-

Maitland function, which are expressed in terms of the generalized (Wright's) hypergeometric and generalized hypergeometric functions.

For our purpose, we begin by recalling some known functions and earlier works. The Bessel-Maitland function $J_{\nu}^{\mu}(z)$ is defined through a series representation by Marichev (1983) as follows:

$$J_{\nu}^{\mu}(z) = \sum_{m=0}^{\infty} \frac{(-z)^m}{\Gamma(\nu + \mu m + 1)\, m!} \text{, where } \mu > 0,\ z \in \mathbb{C}. \tag{1.1}$$

The generalized Bessel function of the form $J_{\nu,\sigma}^{\mu}(z)$ is defined by Jain and Agarwal (2015) as follows:

$$J_{\nu,\sigma}^{\mu}(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{z}{2}\right)^{\nu+2\sigma+2m}}{\Gamma(\nu + \sigma + \mu m + 1)(\sigma + m + 1)}, \tag{1.2}$$

where

$$z \in \mathbb{C} \setminus (-\infty, 0];\ \mu > 0,\ \nu,\ \sigma \in \mathbb{C}.$$

Further, generalization of the generalized Bessel-Maitland function $J_{\nu,q}^{\mu,\gamma}(z)$ defined by Pathak (1966) is as follows:

$$J_{\nu,q}^{\mu,\gamma}(z) = \sum_{m=0}^{\infty} \frac{(\gamma)_{qm}(-z)^m}{\Gamma(\nu + \mu m + 1)\, m!}, \tag{1.3}$$

where

$$\mu, \nu,\ \gamma \in \mathbb{C},\ \Re(\mu) \geq 0,\ \Re(\nu) \geq -1,\ \Re(\gamma) \geq 0 \text{ and } q \in (0, 1) \cup \mathbb{N}$$

and

$$(\gamma)_0 = 1, \quad (\gamma)_{qm} = \frac{\Gamma(\gamma + qm)}{\Gamma(\gamma)},$$

is known as generalized Pochhammer symbol defined by Mittag-Leffler (1903).

From the generalization of the generalized Bessel-Maitland function (1.3), it is possible to find some special cases by giving particular values to the parameters μ, ν, γ and q.

1) If $q = 1$, $\gamma = 1$ and ν is replaced by $\nu + \sigma$ and z is replaced by $\frac{z^2}{4}$ in (1.3), then we obtain

$$J_{\nu+\sigma,1}^{\mu,1}\left(\frac{z^2}{4}\right) = \Gamma(\sigma + m + 1)\left(\frac{z}{2}\right)^{-\nu-2\sigma} J_{\nu,\sigma}^{\mu}(z), \tag{1.4}$$

where $J_{v,\,\sigma}^{\mu}(z)$ denotes Bessel-Maitland function defined by Agarwal et al. (2014).

2) If we replace μ by 1 and σ by $\frac{1}{2}$ in (1.4), we obtain

$$J_{v+\frac{1}{2},\,1}^{1,\,1}\left(\frac{z^2}{4}\right)=\Gamma\left(m+\frac{3}{2}\right)\left(\frac{z}{2}\right)^{-v-1}H_v(z), \tag{1.5}$$

where $H_v(z)$ denotes Struve's function defined by Erdẻlyi et al. (1954).

$$H_v(z)=\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{z}{2}\right)^{v+2m+1}}{\Gamma\left(m+\frac{3}{2}\right)\Gamma\left(v+m+\frac{3}{2}\right)}. \tag{1.6}$$

3) If $q=0$, then (1.3) reduces to

$$J_{v,\,0}^{\mu,\,\gamma}(z)=J_v^{\mu}(z), \tag{1.7}$$

where $J_v^{\mu}(z)$ is generalized Bessel function defined by Agarwal (2015).

4) If $q=0$, $\mu=1$ and z is replaced by $\frac{z^2}{4}$ then (1.3) reduces to

$$J_{v,\,0}^{1,\,\gamma}\left(\frac{z^2}{4}\right)=\left(\frac{z}{2}\right)^{-v}J_v(z), \tag{1.8}$$

where $J_v(z)$ is called Bessel's function of the first kind and of order v, where v is any non-negative constant.

5) If $q=0$ and v is replaced by $v-1$ and z is replaced by $-z$, then (1.3) reduces to

$$J_{v-1,\,0}^{\mu,\,\gamma}(-z)=\phi(\mu,v;z), \tag{1.9}$$

where $\phi(\mu,v;z)$ is known as Wright function, defined by Choi et al. (2014).

6) If v is replaced by $v-1$ and z is replaced by $-z$, then (1.3) reduces to

$$J_{v-1,\,q}^{\mu,\,\gamma}(-z)=E_{\mu,\,v}^{\gamma,\,q}(z), \tag{1.10}$$

where $E_{\mu,\,v}^{\gamma,\,q}(z)$ is generalized Mittag -Leffler function, and was given by Shukla and Prajapati (2007).

7) If $q=1$, v is replaced by $v-1$ and z is replaced by $-z$, then (1.3) reduces to

$$J_{v-1,\,1}^{\mu,\,\gamma}(-z)=E_{\mu,\,v}^{\gamma}(z), \tag{1.11}$$

was introduced by Prabhakar (1971).

8) If $q=1, \gamma=1,\ v$ is replaced by $v-1$ and z is replaced by $-z$, (1.3) reduces to

$$J_{v-1,\,1}^{\mu,\,1}(-z)=E_{\mu,v}(z), \tag{1.12}$$

where $\mu\in\mathbb{C},\ \Re(\mu)>0,\ \Re(v)>0$, and was studied by Wiman (1905).

9) If $q=1, \gamma=1,\ v=0$ and z is replaced by $-z$, (1.3) reduces to

$$J_{0,\,1}^{\mu,\,1}(-z)=E_{\mu}(z). \tag{1.13}$$

where $\mu\in\mathbb{C},\Re(\mu)>0$, and was introduced by Mittag-Leffler (1903).

Further, another representation of the generalized Bessel–Maitland function $J_{v,\,q}^{k,\,\gamma}(z)$ defined by Singh et al. (2014): if $\mu=k\in\mathbb{N}$ and $q\in\mathbb{N}$, then (1.3) reduces to

$$J_{v,\,q}^{k,\,\gamma}(z)=\frac{1}{\Gamma(v+1)^{q}}\,{}_{q}F_{k}\left[\begin{matrix}\Delta(q;\gamma); \\ \Delta(k;\ v+1);\end{matrix}\ -\frac{q^{q}}{k^{k}}z\right], \tag{1.14}$$

where ${}_{q}F_{k}(.)$ is the generalized hypergeometric function and the symbols $\Delta(q;\gamma)$ is a $q-$tuple $\frac{\gamma}{q},\frac{\gamma+1}{q},\ \frac{\gamma+2}{q},\ldots,\frac{\gamma+q-1}{q}$ and $\Delta(k;v+1)$ is a $k-$tuple $\frac{v+1}{k},\frac{v+2}{k},\ \frac{v+3}{k},\ldots,\frac{v+k}{k}$.
The generalization of the generalized hypergeometric series ${}_{p}F_{q}$ is due to Fox (1928) and Wright (1935, 1940((a), (b))) who studied the asymptotic expansion of the generalized Wright hypergeometric function defined by (see, also (1985)).

The generalized Wright hypergeometric function ${}_{p}\psi_{q}(z)$ (see, for detail, Srivastava and Karlsson (1985)), for $z\in\mathbb{C}$ complex, $a_i,\ b_j\in\mathbb{C}$ and $\alpha_i,\ \beta_j\in\mathbb{R}$, where $(\alpha_i,\ \beta_j\neq 0;$ $i=1,2,\ldots,p;\ j=1,\ 2,\ldots,q)$, is defined as below:

$${}_{p}\psi_{q}(z)={}_{p}\psi_{q}\left[\begin{matrix}(a_i,\ \alpha_i)_{1,\ p} \\ (b_j,\ \beta_j)_{1,\ q}\end{matrix}\Big|z\right]=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i k)\,z^{k}}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j k)\ k!}, \tag{1.15}$$

Introduced by Wright (1935), the generalized Wright function and proved several theorems on the asymptotic expansion of ${}_{p}\psi_{q}(z)$ for all values of the argument z, under the condition:

$$\sum_{j=1}^{q}\beta_j-\sum_{i=1}^{p}\alpha_i>-1. \tag{1.16}$$

It is noted that the generalized (Wright) hypergeometric function ${}_p\psi_q$ in (1.15) whose asymptotic expansion was investigated by Fox (1928) and Wright is an interesting further generalization of the generalized hypergeometric series as follows:

$$ {}_p\Psi_q\left[\begin{matrix}(\alpha_1, 1), \ldots, (\alpha_p, 1) \\ (\beta_1, 1), \ldots, (\beta_q, 1)\end{matrix}\; z\right] = \frac{\prod_{j=1}^{p}\Gamma(\alpha_j)}{\prod_{j=1}^{q}\Gamma(\beta_j)}\, {}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p; \\ \beta_1,\ldots,\beta_q;\end{matrix}\; z\right], \tag{1.17}$$

where ${}_pF_q$ is the generalized hypergeometric series defined by (see : (2012), Section 1.5)

$$ {}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p; \\ \beta_1,\ldots,\beta_q;\end{matrix}\; z\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!} = {}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z), \tag{1.18}$$

For our present investigation, we also need to recall the following Lavoie-Trottier integral formula (1969):

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}dx = \left(\frac{2}{3}\right)^{2\alpha}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{1.19}$$

provided $\Re(\alpha)>0,\ \Re(\beta)>0.$

2. Main Results

In this section, we established two generalized integral formulas, which are expressed in terms of generalized (Wright) hypergeometric functions, by inserting the generalized Bessel-Maitland function (1.3) with suitable argument in to the integrand of (1.19).

Theorem 2.1.

The following integral formula holds true for $\alpha, \beta, \mu, v, \gamma \in \mathbb{C}$ with $\mathrm{Re}(v) \geq -1,\ \mathrm{Re}(\gamma) > 0,\ \mathrm{Re}(\alpha) > 0,\ \mathrm{Re}(\beta) > 0$ and x > 0, we have

$$\int_0^\infty x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} J_{v,q}^{\mu,\gamma}\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)dx$$

$$= \left(\frac{2}{3}\right)^{2\alpha}\frac{\Gamma(\alpha)}{\Gamma(\gamma)}\, {}_2\Psi_2\left[\begin{matrix}(\gamma,\ q),\ (\beta,\ 1) & ; \\ (v+1,\ \mu),\ (\beta+\alpha,\ 1); & -y\end{matrix}\right]. \tag{2.1}$$

Proof:

Now applying (1.3) to the integrand of (2.1) and then interchanging the order of integration and summation, which is verified by uniform convergence of the involved series under the given conditions in Theorem 2.1, we get

$$\int_0^\infty x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} J_{v,\,q}^{\mu,\,\gamma}\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)dx$$

$$=\sum_{m=0}^{\infty}\frac{(\gamma)_{qm}(-1)^m y^m}{\Gamma(v+\mu m+1)\, m!}\int_0^\infty x^{\alpha-1}(1-x)^{2(\beta+m)-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta+m-1}dx,$$

By considering the condition given in Theorem 2.1, since $\mathrm{Re}(v)\geq -1,\quad \mathrm{Re}(\alpha)>0\quad \mathrm{Re}(\beta)>0$, $q\in(0,1)\cup\mathbb{N}$ and applying (1.19),

$$=\sum_{m=0}^{\infty}\frac{(\gamma)_{qm}(-1)^m y^m}{\Gamma(v+\mu m+1)m!}\left(\frac{2}{3}\right)^{2\alpha}\frac{\Gamma(\alpha)\,\Gamma(\beta+m)}{\Gamma(\alpha+\beta+m)},$$

$$=\left(\frac{2}{3}\right)^{2\alpha}\frac{\Gamma(\alpha)}{\Gamma(\gamma)}\sum_{m=0}^{\infty}\frac{\Gamma(\gamma+q\,m)\Gamma(\beta+m)(-1)^m y^m}{\Gamma(v+\mu m+1)\,\Gamma(\alpha+\beta+m)\,m!},$$

which upon using the definition (1.17), we get the desired result (2.1).

Theorem 2.2.

The following integral formula holds true: for $\alpha,\beta,\mu,v,\gamma\in\mathbb{C}$ with $\mathrm{Re}(v)\geq -1,\ \mathrm{Re}(\gamma)>0,\ \mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0$ and x > 0, we have

$$\int_0^\infty x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} J_{v,\,q}^{\mu,\,\gamma}\left(yx\left(1-\frac{x}{3}\right)^2\right)dx$$

$$=\left(\frac{2}{3}\right)^{2\alpha}\frac{\Gamma(\beta)}{\Gamma(\gamma)}\,{}_2\Psi_2\left[\begin{matrix}(\gamma,\ q),\ (\alpha,\ 1) & ; \\ (v+1,\ \mu),\ (\alpha+\beta,\ 1); & \end{matrix}\ -\frac{4y}{9}\right]. \tag{2.2}$$

By similar manner as in proof of Theorem 2.1, we can also prove the integral formula (2.2).

Next, we consider other variations of Theorem 2.1 and Theorem 2.2. In fact, we establish some integral formulas for the generalized Bessel-Maitland function expressed in terms of the generalized hypergeometric function. To do this, we recall the well-known Legendre duplication formula (see, (2012)) as

$$(\lambda)_{2m}=2^{2m}\left(\frac{\lambda}{2}\right)_m\left(\frac{\lambda+1}{2}\right)_m,\quad m\in\mathbb{N}_0. \tag{2.3}$$

Corollary 2.3.

Assuming the condition of Theorem 2.1 is satisfied and replacing μ by k in the generalized Bessel-Maitland function $J_{v,\,q}^{\mu,\,\gamma}(z)$ in (2.1) and using (1.14), it can be shown that the following integral formula holds true:

$$\int_0^\infty x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} J_{v,\,q}^{k,\,\gamma}\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)dx$$

$$=\left(\frac{2}{3}\right)^{2\alpha}\frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(v+1)\;\Gamma(\alpha+\beta)}\;{}_{q+1}F_{k+1}\left[\begin{matrix}\Delta(q;\;\gamma),\;(\beta)\quad ; \\ \Delta(k;\;v+1),\;(\beta+\alpha);\end{matrix}\quad -\frac{yq^q}{k^k}\right], \tag{2.4}$$

where

$\Re(\alpha)>0,\;\;\Re(\beta)>0$, $\Delta(q;\gamma)$ is a $q-$tuple $\frac{\gamma}{q},\frac{\gamma+1}{q},\;\frac{\gamma+2}{q},\ldots,\frac{\gamma+q-1}{q}$,

and

$\Delta(k;v+1)$ is a $k-$tuple $\frac{v+1}{k},\frac{v+2}{k},\;\frac{v+3}{k},\ldots,\frac{v+k}{k}$.

Corollary 2.4.

Let the condition of Theorem 2.2 be satisfied and replacing μ by k in the generalized Bessel-Maitland function $J_{v,\,q}^{\mu,\,\gamma}(z)$ in (2.2) and using (1.14). Then the following integral formula holds true:

$$\int_0^\infty x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} J_{v,\,q}^{k,\,\gamma}\left(yx\left(1-\frac{x}{3}\right)^2\right)dx$$

$$=\left(\frac{2}{3}\right)^{2\alpha}\frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(v+1)\;\Gamma(\alpha+\beta)}\;{}_{q+1}F_{k+1}\left[\begin{matrix}\Delta(q;\;\gamma),\;(\alpha)\quad ; \\ \Delta(k;\;v+1),\;(\beta+\alpha);\end{matrix}\quad -\frac{4yq^q}{9k^k}\right]. \tag{2.5}$$

Proof:

By writing the right hand side of (2.1) in the original summation formula, after a little simplification, we find that, when the last resulting summation is expressed in the term of hypergeometric in the relation (1.14), this completes the proof of Corollary 2.3. Similarly, it is easy to see that a similar argument as in proof of Corollary 2.3 will established the integral formula (2.4). Therefore, we omit the details of the proof of the Corollary 2.4.

3. Special Cases

In this section, we represent certain cases of generalized form of Bessel-Maitland function (1.3).

On setting $\gamma=1,\;v=v+\sigma,\;q=1$ and z is replaced by $(z^2/4)$, in Theorem 2.1 and Theorem 2.2 and making use of the relation (1.4), then, the generalized Bessel-Maitland function will have the following relation with Bessel-Maitland function as follows:

Corollary 3.1.

Let the conditions of $\alpha, \beta, \nu, \sigma \in \mathbb{C}$, $\Re(\alpha) > 0$ and $\Re(\beta+\nu) > 0$ be satisfied, then the following integral formula holds true:

$$\int_0^\infty x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} J_{\nu,\sigma}^{\mu}\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)dx$$

$$=\left(\frac{2}{3}\right)^{2\alpha}\left(\frac{y}{2}\right)^{\nu+2\sigma}\Gamma(\alpha)\ {}_2\Psi_3\left[\begin{matrix}(\beta+\nu+2\sigma,\ 2),(1,\ 1) & ; \\ (\nu+\sigma+1,\ \mu),(1+\sigma,\ 1),(\alpha+\beta+\nu+2\sigma,\ 2); & \end{matrix}\ -\frac{y^2}{4}\right]. \tag{3.1}$$

Corollary 3.2.

Let the conditions of $\alpha, \beta, \nu, \sigma \in \mathbb{C}$, $\Re(\alpha+\nu) > 0$ and $\Re(\beta+\nu) > 0$ be satisfied, then the following integral formula holds true:

$$\int_0^\infty x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} J_{\nu,\sigma}^{\mu}\left(yx\left(1-\frac{x}{3}\right)^2\right)dx$$

$$=\left(\frac{2}{3}\right)^{2\alpha}\left(\frac{2y}{9}\right)^{\nu+2\sigma}\Gamma(\beta)\ {}_2\Psi_3\left[\begin{matrix}(\nu+\alpha+2\sigma,\ 2),(1,\ 1) & ; \\ (\nu+\sigma+1,\ \mu),\ (1+\sigma,\ 1),(\alpha+\beta+\nu+2\sigma,\ 2); & \end{matrix}\ -\frac{4y^2}{81}\right]. \tag{3.2}$$

On setting $\mu=1, \sigma=1/2$ in (3.1) and (3.2) and using the relation (1.5), then we get the integral formulas involving the Struve's function $H_\nu(z)$ as follows:

Corollary 3.3.

Let the conditions of $\alpha, \beta, \nu \in \mathbb{C}$ and $\Re(\beta+\nu) > 0$ be satisfied, then the following integral formula holds true:

$$\int_0^\infty x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} H_\nu\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)dx$$

$$=\left(\frac{2}{3}\right)^{2\alpha}\left(\frac{y}{2}\right)^{\nu+1}\Gamma(\alpha)\ {}_2\Psi_3\left[\begin{matrix}(\beta+\nu+1,\ 2),\ (1,\ 1); \\ (\alpha+\beta+\nu+1,\ 2),\ \left(\nu+\frac{3}{2},\ 1\right),\left(\frac{3}{2},\ 1\right); \end{matrix}\ -\frac{y^2}{4}\right]. \tag{3.3}$$

Corollary 3.4.

Let the conditions of $\alpha, \beta, \nu \in \mathbb{C}$ and $\Re(\alpha+\nu) > 0$ be satisfied, then the following integral formula holds true:

$$\int_0^\infty x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta+j-1} H_\nu\left(yx\left(1-\frac{x}{3}\right)^2\right)dx$$

$$=\left(\frac{2}{3}\right)^{2(\alpha+\nu)}\left(\frac{y}{2}\right)^{\nu+1}\Gamma(\beta)\,{}_2\Psi_3\left[\begin{matrix}(\alpha+\nu+1,\ 2),\ (1,\ 1)\,; \\ (\alpha+\nu+\beta+1,\ 2),\ \left(\nu+\frac{3}{2},\ 1\right),\ \left(\frac{3}{2},\ 1\right);\end{matrix}\ -\frac{4y^2}{81}\right]. \tag{3.4}$$

On setting $q=0$ in Theorem 2.1 and Theorem 2.2 and making use of the relation (1.7), then the generalized Bessel–Maitland function $J_{\nu,\,q}^{\mu,\,\gamma}(z)$ will have the following relation with Bessel-Maitland function $J_\nu^\mu(z)$ as follows:

Corollary 3.5.

Let the conditions of $\alpha,\ \beta,\mu,\ \nu\in\mathbb{C},\ \Re(\alpha)>0$ and $\Re(\beta)>0$ be satisfied, then the following integral formula holds true:

$$\int_0^\infty x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}J_\nu^\mu\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)dx$$

$$=\left(\frac{2}{3}\right)^{2\alpha}\Gamma(\alpha)\,{}_1\Psi_2\left[\begin{matrix}(\beta,\ 1)\ ; \\ (\nu+1,\ \mu),\ (\beta+\alpha,\ 1);\end{matrix}\ -y\right]. \tag{3.5}$$

Corollary 3.6.

Let the conditions of $\alpha,\ \beta,\mu,\ \nu\in\mathbb{C},\ \Re(\alpha)>0$ and $\Re(\beta)>0$ be satisfied, then the following integral formula holds true:

$$\int_0^\infty x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}J_\nu^\mu\left(xy\left(1-\frac{x}{3}\right)^2\right)dx$$

$$=\left(\frac{2}{3}\right)^{2\alpha}\Gamma(\beta)\,{}_1\Psi_2\left[\begin{matrix}(\alpha,1)\ ; \\ (\nu+1,\ \mu),(\beta+\alpha,\ 1);\end{matrix}\ -\frac{4y}{9}\right]. \tag{3.6}$$

Corollary 3.7.

Let the conditions of $\alpha,\ \beta,\ \nu\in\mathbb{C},\ \Re(\alpha)>0$ and $\Re(\beta)>0$ be satisfied, then the following integral formula holds true:

$$\int_0^\infty x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}J_\nu^k\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)dx$$

$$=\left(\frac{2}{3}\right)^{2\alpha}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\nu+1)\Gamma(\alpha+\beta)}\,{}_1F_{k+1}\left[\begin{matrix}(\beta)\ ; \\ \Delta(k,\nu+1),(\alpha+\beta);\end{matrix}\ -\frac{y}{k^k}\right].\ , \tag{3.7}$$

where k is positive integer.

Corollary 3.8.

Let the conditions of $\alpha, \beta, \nu \in \mathbb{C}$, $\Re(\alpha) > 0$ and $\Re(\beta) > 0$ be satisfied, then the following integral formula holds true:

$$\int_0^\infty x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} J_\nu^k\left(yx\left(1-\frac{x}{3}\right)^2\right)dx$$

$$=\left(\frac{2}{3}\right)^{2\alpha}\frac{\Gamma(\beta)\Gamma(\alpha)}{\Gamma(\nu+1)\Gamma(\alpha+\beta)}\;{}_1F_{k+1}\left[\begin{matrix}(\alpha) & ; \\ \Delta(k,\nu+1),\ (\alpha+\beta); & \end{matrix} -\frac{4y}{9k^k}\right]. \tag{3.8}$$

On setting $q=0$, $\mu=1$and z is replaced by $(z^2/4)$, in Theorem 2.1 and Theorem 2.2 and making use of the relation (1.8), we obtain the following integral formulas involving the ordinary Bessel function as follows:

Corollary 3.9.

Let the conditions of $\alpha, \beta, \nu, \gamma \in \mathbb{C}$, $\Re(\alpha) > 0$ and $\Re(\beta) > 0$ be satisfied, then the following integral formula holds true:

$$\int_0^\infty x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} J_\nu\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)dx$$

$$=\left(\frac{2}{3}\right)^{2\alpha}\left(\frac{y}{2}\right)^{\nu}\Gamma(\alpha)\;{}_1\Psi_2\left[\begin{matrix}(\beta+\nu,\ 2) & ; \\ (\nu+1,\ 1),\ (\beta+\alpha+\nu,\ 2); & \end{matrix} -\frac{y^2}{4}\right]. \tag{3.9}$$

Corollary 3.10.

Let the conditions of $\alpha, \beta, \nu, \gamma \in \mathbb{C}$, $\Re(\alpha) > 0$ and $\Re(\beta) > 0$ be satisfied, then the following integral formula holds true:

$$\int_0^\infty x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} J_\nu\left(yx\left(1-\frac{x}{3}\right)^2\right)dx$$

$$=\left(\frac{2}{3}\right)^{2(\alpha+\nu)}\left(\frac{y}{2}\right)^{\nu}\Gamma(\beta)\;{}_1\Psi_2\left[\begin{matrix}(\alpha+\nu,\ 2) & ; \\ (\nu+1,\ 1),\ (\beta+\alpha+\nu,\ 2); & \end{matrix} -\frac{4y^2}{81}\right]. \tag{3.10}$$

Remark

In (3.9) and (3.10), by making some suitable adjustments of the parameters, we arrive at the known result given by Agarwal et al. ((2014), p. 3, equations (2.1) and (2.3)).

On setting $q=0$ and ν is replaced by $\nu-1$ and z is replaced by $-z$, in Theorem 2.1 and Theorem 2.2 and making use of the relation (1.9), we obtain the following integral formulas involving the Weight function as follows:

Corollary 3.11.

Let the conditions of $\alpha, \beta, \mu, v, \gamma \in \mathbb{C}$, $\Re(\alpha) > 0$ and $\Re(\beta) > 0$ be satisfied, then the following integral formula holds true:

$$\int_0^\infty x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}\phi\left(\mu, v; y\left(1-\frac{x}{4}\right)(1-x)^2\right)dx$$

$$=\left(\frac{2}{3}\right)^{2\alpha}\Gamma(\alpha)\,{}_1\Psi_2\left[\begin{matrix}(\beta,\ 1) & ; \\ (v,\ \mu),\ (\beta+\alpha,\ 1); & \end{matrix}\ y\right]. \tag{3.11}$$

Corollary 3.12.

Let the conditions of $\alpha, \beta, \mu, v, \gamma \in \mathbb{C}$, $\Re(\alpha) > 0$ and $\Re(\beta) > 0$ be satisfied, then the following integral formula holds true*:*

$$\int_0^\infty x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}\phi\left(\mu, v; yx\left(1-\frac{x}{3}\right)^2\right)dx$$

$$=\left(\frac{2}{3}\right)^{2\alpha}\Gamma(\beta)\,{}_1\Psi_2\left[\begin{matrix}(\alpha,\ 1) & ; \\ (v,\ \mu),\ (\beta+\alpha,\ 1); & \end{matrix}\ \frac{4y}{9}\right]. \tag{3.12}$$

On setting v by $v-1$ and z is replaced by $-z$, in Theorem 2.1 and Theorem 2.2 and making use of the relation (1.10), we obtain the following integral formulas involving the generalized Mittag- Leffler function as follows:

Corollary 3.13.

Let the conditions of $\alpha, \beta, \mu, v, \gamma \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\mu) > 0$, $\Re(\gamma) > 0$, $\Re(v) \geq -1$ and $q \in (0, 1) \cup \mathbb{N}$ be satisfied, then the following integral formula holds true:

$$\int_0^\infty x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}E_{\mu,v}^{\gamma,q}\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)dx$$

$$=\left(\frac{2}{3}\right)^{2\alpha}\frac{\Gamma(\alpha)}{\Gamma(\gamma)}\,{}_2\Psi_2\left[\begin{matrix}(\gamma,\ q),\ (\beta,\ 1) & ; \\ (v,\ \mu),\ (\beta+\alpha,\ 1) & ; \end{matrix}\ y\right]. \tag{3.13}$$

Corollary 3.14.

Let the conditions of $\alpha, \beta, \mu, v, \gamma \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\mu) > 0$, $\Re(\gamma) > 0$, $\Re(v) \geq -1$ and $q \in (0, 1) \cup \mathbb{N}$ be satisfied, then the following integral formula holds true:

$$\int_0^\infty x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}E_{\mu,v}^{\gamma,q}\left(yx\left(1-\frac{x}{3}\right)^2\right)dx$$

$$=\left(\frac{2}{3}\right)^{2\alpha}\frac{\Gamma(\beta)}{\Gamma(\gamma)}\,{}_2\Psi_2\left[\begin{matrix}(\gamma,q),\ (\alpha,1) & ; & \frac{4y}{9}\\ (\nu,\mu),\ (\beta+\alpha,1); & & \end{matrix}\right]. \tag{3.14}$$

Remark

In (3.13) and (3.14), by making some suitable adjustments of the parameters, we arrive at the known result given by Manaria et al.((2016), p. 5, eq.(2.3) and (2.4)).

On setting $q=1$, ν by $\nu-1$and z is replaced by $-z$, in Theorem 2.1 and Theorem 2.2 and making use of the relation (1.11), we obtain the following integral formulas involving the generalized Mittag- Leffler function as follows:

Corollary 3.15.

Let the conditions of $\alpha, \beta, \mu, \nu, \gamma\in\mathbb{C}$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\mu)>0$, $\Re(\gamma)>0$ and $\Re(\nu)\geq -1$, be satisfied, then the following integral formula holds true:

$$\int_0^\infty x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}E^{\gamma}_{\mu,\nu}\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)dx$$

$$=\left(\frac{2}{3}\right)^{2\alpha+}\frac{\Gamma(\alpha)}{\Gamma(\gamma)}\,{}_2\Psi_2\left[\begin{matrix}(\gamma,1),(\beta,1) & ; & y\\ (\nu,\mu),(\beta+\alpha,1); & & \end{matrix}\right]. \tag{3.15}$$

Corollary 3.16.

Let the conditions of $\alpha, \beta, \mu, \nu, \gamma\in\mathbb{C}$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\mu)>0$, $\Re(\gamma)>0$ and $\Re(\nu)\geq -1$, be satisfied, then the following integral formula holds true:

$$\int_0^\infty x^{\alpha-1}(1-x)^{2\beta+-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}E^{\gamma}_{\mu,\nu}\left(yx\left(1-\frac{x}{3}\right)^2\right)dx$$

$$=\left(\frac{2}{3}\right)^{2\alpha}\frac{\Gamma(\beta)}{\Gamma(\gamma)}\,{}_2\Psi_2\left[\begin{matrix}(\gamma,1),\ (\alpha,1) & ; & \frac{4y}{9}\\ (\nu,\mu),\ (\beta+\alpha,1); & & \end{matrix}\right]. \tag{3.16}$$

On setting $\gamma=1$, ν by $\nu-1$and z is replaced by $-z$, in Theorem 2.1 and Theorem 2.2 and making use of the relation (1.12), we obtain the following integral formulas involving the generalized Mittag- Leffler function as follows:

Corollary 3.17.

Let the conditions of $\alpha, \beta, \mu, \nu\in\mathbb{C}$, $\Re(\alpha)>0$, $\Re(\beta)>0$ $\Re(\mu)>0$ and $\Re(\nu)\geq -1$, be satisfied, then the following integral formula holds true:

$$\int_0^\infty x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}E_{\mu,\nu}\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)dx$$

$$= \left(\frac{2}{3}\right)^{2\alpha} \Gamma(\alpha)\, {}_2\Psi_2\left[\begin{matrix} (\beta, 1), & (1, 1) & ; \\ (v, \mu), & (\beta+\alpha, 1) & ; \end{matrix}\; y\right]. \tag{3.17}$$

Corollary 3.18.

Let the conditions of $\alpha, \beta, \mu, v \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$ $\Re(\mu) > 0$ and $\Re(v) \geq -1$, be satisfied, then the following integral formula holds true:

$$\int_0^\infty x^{\alpha-1}(1-x)^{2\beta+-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} E_{\mu,\,v}\left(yx\left(1-\frac{x}{3}\right)^2\right)dx$$

$$= \left(\frac{2}{3}\right)^{2\alpha} \Gamma(\beta)\, {}_2\Psi_2\left[\begin{matrix} (\alpha, 1), & (1, 1) & ; \\ (v, \mu), & (\beta+\alpha, 1) & ; \end{matrix}\; \frac{4y}{9}\right]. \tag{3.18}$$

On setting $\gamma = 1$, $v = 0$, $q = 1$ and z is replaced by $-z$, in Theorem 2.1 and Theorem 2.2 and making use of the relation (1.13), we obtain the following integral formulas involving the Mittag- Leffler function as follows:

Corollary 3.19.

Let the conditions of $\alpha, \beta, \mu, v \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$ and $\Re(\mu) > 0$, be satisfied, then the following integral formula holds true:

$$\int_0^\infty x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} E_{\mu}\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)dx$$

$$= \left(\frac{2}{3}\right)^{2\alpha} \Gamma(\alpha)\, {}_2\Psi_2\left[\begin{matrix} (\beta, 1), & (1, 1) & ; \\ (1, \mu), & (\beta+\alpha, 1) & ; \end{matrix}\; y\right]. \tag{3.19}$$

Corollary 3.20.

Let the conditions of $\alpha, \beta, \mu, v \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$ and $\Re(\mu) > 0$, be satisfied, then the following integral formula holds true:

$$\int_0^\infty x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} E_{\mu}\left(yx\left(1-\frac{x}{3}\right)^2\right)dx$$

$$= \left(\frac{2}{3}\right)^{2\alpha} \Gamma(\beta)\, {}_2\Psi_2\left[\begin{matrix} (1, 1), & (\alpha, 1) & ; \\ (1, \mu), & (\beta+\alpha, 1) & ; \end{matrix}\; \frac{4y}{9}\right]. \tag{3.20}$$

4. Concluding remarks

In the present paper, we investigate new integrals involving the generalized Bessel-Maitland function $J_{v,q}^{\mu,\gamma}(z)$, in terms of generalized (Wright) hypergeometric functions and generalized hypergeometric function ${}_qF_k$. Certain special cases of integrals involving generalized Bessel Maitland function have been investigated by the authors in the literature with different arguments. Therefore the results presented in this paper are easily converted in terms of a similar type of new interesting integrals with different arguments after some suitable parametric replacements. In this sequel, one can obtain integral representation of more generalized special function, which has much application in physics and engineering science.

Acknowledgment:

The authors would like to thank the anonymous referees for their valuable comments and suggestions which have helped to improve the manuscript.

REFERENCES

Agarwal, P. (2015). Pathway fractional integral formulas involving Bessel function of the first kind, Adv. Stud. Contemp. Math., Vol. 25, No. 1, pp. 221-231.

Agarwal, P., Jain, S. Agarwal, S. and Nagpal, M. (2014). On a new class of integrals involving Bessel functions of the first kind, Commun. Numer. Anal., Art. ID 00216, pp. 1-7.

Agarwal, P., Jain, S., Chand, M., Dwivedi, S.K. and Kumar, S. (2014). Bessel functions associated with Saigo-Maeda fractional derivatives operators, J. Frac. Calc., Vol. 5, No. 2, pp. 102-112.

Brychkov, Y.A. (2008). *Handbook of Special Functions, Derivatives, Integrals, Series and Other Formulas*, CRC Press, Taylor and Francis Group, Boca Raton, London, and New York.

Choi, J., Agarwal, P., Mathur, S. and Purohit, S.D. (2014). Certain new integral formulas involving the generalized Bessel functions, Bull. Korean Math. Soc., Vol. 51, No. 4, pp. 995-1003.

Choi, J. and Agarwal, P. (2013). Certain unified integrals involving a product of Bessel functions of first kind, Honam Mathematical J., Vol. 35, No. 4, pp. 667-677.

Choi, J. and Agarwal, P. (2013). Certain unified integrals associated with Bessel functions, Bound. Value Probl., doi:10.1186/1687-2770-2013-95.

Erdėlyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F. (1954). T*able of integral transforms,* Vol. II, McGraw-Hill, New York.

Fox, C. (1928). The asymptotic expansion of generalized hypergeometric functions, Proc. London Math. Soc., Vol. 27, No. 2, pp. 389-400.

Jain, S. and Agarwal, P. (2015). A new class of integral relation involving general class of Polynomials and I- function, Walailak J. Sci. & Tech., Vol. 12, No. 11, pp.1009-1018.

Khan, N.U. and Kashmin, T. (2016). Some integrals for the generalized Bessel-Maitland functions, Electron. J. Math. Anal. Appl., Vol. 4, No.2, pp. 139-149.

Lavoie, J.L. and Trottier, G. (1969). On the sum of certain Appell's series, Ganita, Vol. 20, pp. 43-66.

Marichev, O.I. (1983). Handbook of integral transform and Higher transcendental functions, Ellis, Harwood, chichester (John Wiley and Sons); New York.

Manaria, N., Nisar, K.S. and Purohit, S.D. (2016). On a new class of integrals involving product of generalized Bessel function of the first kind and general class of polynomials, Acta Univ. Apulensis, Vol. 46, pp. 97-105.

Menaria, N., Baleanu, D. and Purohit, S.D. (2016). Integral formulas involving product of general class of polynomials and generalized Bessel function, Sohag J. Math., Vol. 3, No. 2, pp. 77-81.

Manaria, N., Purohit, S.D. and Parmar, R.K. (2016). On a new class of integrals involving generalized Mittag-Leffler function, Survey of mathematics and its applications, Vol. 11, pp. 1-9.

Mitag-Leffler, G.M. (1903). Sur la nouvelle function E_(x), C. R. Acad. Soc. Paris, Vol. 137, pp. 554-558.

Nisar, K.S., Parmar, R.K. and Abusufian, A.H. (2016). Certain new unified integrals with the generalized k-Bessel function, Far East J. Math. Sci., Vol. 100, pp. 1533-1544.

Nisar, K.S., Suthar, D.L., Purohit, S.D. and Aldhaifallah, M. (2017). Some unified integral associated with the generalized Struve function, Proc. Jangjeon Math. Soc., Vol. 20, No. 2, pp. 261-267.

Parmar, R.K. and Purohit S.D. (2016).On a new class of integrals involving generalized hypergeometric function, Int. Bull. Math. Res., Vol. 3, No. 2, pp. 24-27.

Pathak, R.S. (1966). Certain convergence theorems and asymptotic properties of a generalization of Lommel and Maitland transform, Proc. Nat. Acad. Sci. India Sect. A, Vol. 36, pp. 81-86.

Prabhaker, T.R. (1971). A singular integral equation with a generalized Mittag-Leffler function in the kernel, Yokohama Math. J., Vol. 19, pp. 7-15.

Shukla, A.K. and Prajapati, J.C. (2007). On a generalized Mittag-Leffler function and its properties, J. Math. Anal. Appl., Vol. 336, pp. 797-811.

Singh, M., Khan, M.A. and Khan, A.H. (2014). On some properties of a generalization of Bessel-Maitland function, Journal of Mathematics trends and Technology, Vol. 14, No. 1, pp. 46-54.

Srivastava, H.M. and Choi, J. (2012). *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier, Amsterdam.

Srivastava, H.M. and Karlsson, P.W. (1985). *Multiple Gaussian Hypergeometric Series,* Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane, and Toronto.

Suthar, D.L. and Habenom Haile (2016). Integrals involving generalized Bessel-Maitland function, J. Sci. Arts, Vol. 37, No. 4, pp.357-362.

Wiman, A. (1905). Uber den Fundamentalsatz in der Teorie der Funktionen, Acta Math., Vol. 29, pp. 191-201.

Wright, E.M. (1935). The asymptotic expansion of the generalized hypergeometric functions, J. London Math. Soc., Vol. 10, pp. 286-293.

Wright, E.M. (1940(a)). The asymptotic expansion of integral functions defined by Taylor series, Philos. Trans. Roy. Soc. London, A, Vol. 238, pp. 423-451.

Wright, E.M. (1940(b)). The asymptotic expansion of the generalized hypergeometric function II, Proc. Lond. Math. Soc., Vol. 46, pp. 389-408.