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Damping in microscale modified couple stress thermoelastic circular Kirchhoff plate resonators

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Abstract

The vibrations of circular plate in modified couple stress thermoelastic medium using Kirchhoff- Love plate theory has been presented. The basic equations of motion and heat conduction equation for Lord Shulman (L-S, 1967) theory are written with the help of Kirchhoff-Love plate theory. The thermoelastic damping of micro beam resonators is studied by applying normal mode analysis method. The solutions for the free vibrations of plates under clamped, simply supported and free boundary conditions are obtained. The analytical expressions for thermoelastic damping of vibration and frequency shift are obtained for generalized couple stress thermoelastic and coupled thermoelastic plates. Numerical results with the help of MATLAB programming software in case of silicon material has been presented. A computer algorithm has been constructed to obtain the numerical results. The thermoelastic damping and frequency shift with varying values of length and thickness for clamped, simply supported and free boundary conditions in the absence and presence of couple stress are presented graphically. Comparisons are made with different modes of vibrations and also with and without couple stress parameter. Some special cases are obtained in the present problem. The thermoelastic damping have many applications in (sensors) resonators in detecting Infrared (IR) and imaging in addition to chemical and biological agent sensing and their temperature dependence in design/construction of precision thermometers and communications.

Keywords: Modified couple stress theory; Kirchhoff-Love plate theory; Normal mode analysis; Thermoelastic damping; Frequency shift

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1. Introduction

The classical couple stress elasticity has been observed to describe the Cosserat size-dependent effects [Mindlin (1962, 1963), Toupin (1962)], which is based on Cosserat continuum theory. However, as it involves four material constants for isotropic elastic materials where two of them are separate material length scale parameters, it is a very difficult task to experimentally determine the micro-structural material length scale parameter. Yang et al. (2002) developed couple stress based strain gradient theory for elasticity. The concept of representative volume element was introduced and the force and couple applied to a single material particle were defined. They developed a new set of equilibrium relations for a system of material particles to account for the rotations of these material particles and then results were generalized to the couple stress theory of continuum. By the introduction of a higher order equilibrium condition, the arbitrary nature of couples in the classical couple stress theory was resolved without the use of rigid vector attachment condition, as was used in the micropolar theory Eringen (1966).

When an elastic solid is put in motion, it is taken out of equilibrium, having an excess of kinetic and potential energy. The conjugation of strain field to a temperature field provides an energy dissipation mechanism that allows the system to relax back to equilibrium. This process of energy dissipation is called thermoelastic damping. The existence of thermoelastic damping process by considering the transverse vibration of homogeneous and isotropic thin beam was constructed by Zener (1937, 1938). Thermoelastic damping originates from thermal currents generated due to the contraction/expansion of elastic media. The bending of the reed causes dilations of opposite signs to exist on the upper and lower halves. While, one side is compressed and heated, the other side is stretched and cooled. Consequently, a transverse temperature gradient is produced due to the presence of finite thermal expansion. The temperature gradient generates local heat currents, which cause increase of the entropy of the reed and lead to energy dissipation. Circular plates are common elements in many sensors and resonators Bao and Jiang (1998). A micro-resonator based high sensor array to be used as Infrared sensors was presented by Vig et al. (1996). These sensors can be used for infrared detection, imaging, chemical and biological agent sensing, while they are not suitable for precision frequency control applications due to their extremely high sensitivity of mass loading.

Thermoelastic damping in single-crystal silicon and silicon nitride micro-resonators at room temperature was observed by Yasumura et al. (1999). Srikar and Senturia (2002) investigated thermoelastic damping in fine-grained polysilicon flexural beam resonators. Nayfeh and Younis (2004) developed analytical expressions for the quality factors of microplate of general shapes due to thermoelastic damping. Thermoelastic damping of the n-plane vibration of thin silicon rings of rectangular cross-section was presented by Wong et al. (2006). A new model of Kirchhoff plate in modified couple stress theory was constructed by Tsiatas (2009). In this model, the static analysis of isotropic micro-plates with arbitrary shape containing only one internal material length scale parameter which can capture the size effect. Sun and Tohmyoh (2009) studied the thermoelastic damping effect on the vibration of circular plate and gives the derivation of thermoelastic equations for a circular plate under out-of-plane flexural vibration. The basic equations of coupled thermoelastic theory are constructed by Sun and Saka (2010) for out of plane vibration of a circular plate resonators.

Sharma and Sharma (2011) studied the damping in micro-scale generalized thermoelastic circular plate resonators under clamped plate and simply-supported plate. Fang et al. (2013) investigated the problem of thermoelastic damping in the axisymmetric vibration of circular microplate resonators with two dimensional heat conduction. A new model of Kirchhoff plate in modified couple stress theory was developed by Shaat et al. (2014), which studies the effects of surface energy and microstructure on the plate rigidity and deflection. Forced vibration analysis of a microplate on the basis of modified couple stress theory and Kirchhoff plate theory was presented by Simsek et al. (2015). Darijani and Shahdadi (2015) developed a new non-classical shear deformation plate model in modified couple stress theory including two unknown functions. Gao and Zhang (2016) constructed a non-classical Kirchhoff plate model by applying modified couple stress theory, surface elasticity theory and two–parameter elastic foundation. Reddy et al. (2016) discussed the problem of functionally graded circular plates with modified couple stress theory by using finite element method. Chen and Wang (2016) investigated a model for composite laminated Reddy plate of new modified couple-stress theory and global local theory.

This paper deals with the study of vibrations of circular plate resonators in modified couple stress thermoelastic medium using Kirchhoff-Love plate theory. The theory of generalized (non-classical) thermoelasticity given by Lord and Shulman (1967) has been applied to solve the problem. The effects of thermoelastic damping and frequency shift on vibrations of circular plate resonators are presented analytically and shown graphically with varying values of length and thickness for clamped, simply-supported and free plates in the absence and presence of couple stress. Some special cases are also given.

2. Basic equations

Following Yang et al. (2002) and Rao (2007), the constitutive equation, the equations of motion and the equation of heat conduction for Lord-Shulman (L-S, 1967) theory in a modified couple stress thermoelastic model in the absence of body forces and body couples are

Constitutive relations

$$t_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} - \frac{1}{2} e_{kij} m_{lk,l} - \beta T \delta_{ij}, \tag{1}$$

$$m_{ij} = 2\alpha\chi_{ij},\ \chi_{ij} = \frac{1}{2}\left(\omega_{i,j} + \omega_{j,i}\right),\ \omega_i = \frac{1}{2} e_{ipq} u_{q,p}, \tag{2}$$

Equation of motion

$$\left(\lambda + \mu + \frac{\alpha}{4}\Delta\right)\nabla\left(\nabla.\boldsymbol{u}\right) + \left(\mu - \frac{\alpha}{4}\Delta\right)\nabla^2\boldsymbol{u} - \beta\nabla T = \rho\ddot{\boldsymbol{u}}, \tag{3}$$

Equation of heat conduction

$$K\Delta T-\left(\frac{\partial}{\partial t}+\tau_0\frac{\partial^2}{\partial t^2}\right)\left(\rho c_e T+T_0\beta\left(\nabla.\boldsymbol{u}\right)\right)=0, \tag{4}$$

where t_{ij} are the components of stress tensor, λ and μ are Lame' constants, δ_{ij} is Kronecker's delta, e_{ij} are the components of strain tensor, e_{ijk} is alternate tensor, m_{ij} are the components of couple-stress, $\beta=(3\lambda+2\mu)\alpha_t$. Here α_t are the coefficients of linear thermal expansion respectively, T is the temperature change, α is the couple stress parameter, χ_{ij} is symmetric curvature, ω_i is the rotational vector. $\boldsymbol{u}$ is the displacement vector, ρ is the density, Δ is the Laplacian operator, ∇ is del operator. K is the coefficient of the thermal conductivity, c_e is the specific heat at constant strain, T_0 is the reference temperature assumed to be such that $T/T_0<<1$. Here τ_0 is the thermal relaxation time.

3. Formulation of the problem

We consider a thin couple stress thermoelastic circular plate with uniform thickness h and radius b. The origin of the Cartesian coordinate system (r,θ,z) is taken at the centre of the plate. To simply the problem, we apply the fundamental Kirchhoff-Love plate hypothesis for the coupling plate vibrations following Huang (2005) for the present case:

(i) Normal stress t_{zz} can be neglected relative to the principal stresses, i.e., $t_{zz}=0$.
(ii) The rectilinear element normal to the middle surface before deformation remains perpendicular to the strained surface after deformation and their elongation can be neglected, i.e. $e_{rz}=e_{\theta z}=0$.
(iii) For small deflection vibration, the deformation along the middle surface can be neglected, i.e., $e_{zz}=0$.

In equilibrium conditions, the plate is unstrained, unstressed and continues at uniform environmental temperature T_0 everywhere. We define the displacement components $u(r,\theta,z,t)$, $v(r,\theta,z,t)$, $w(r,\theta,z,t)$ and temperature $T(r,\theta,z,t)$. The displacement components are given by

$$u(r,\theta,z,t)=-z\frac{\partial w(r,\theta,t)}{\partial r},\ v(r,\theta,z,t)=-\frac{z}{r}\frac{\partial w(r,\theta,t)}{\partial\theta},\ w(r,\theta,z,t)=w(r,\theta,t), \tag{5}$$

The strain components are taken as

$$\varepsilon_r=\frac{\partial u}{\partial r}-z\frac{\partial^2 w}{\partial r^2}, \tag{6}$$

$$\varepsilon_\theta = \frac{u}{r} + \frac{\partial v}{r\partial\theta} = -z\left(\frac{\partial w}{r\partial r} + \frac{\partial^2 w}{r^2\partial\theta^2}\right), \tag{7}$$

$$\gamma_{r\theta} = 2\varepsilon_{r\theta} = \frac{\partial v}{\partial r} + \frac{1}{r}\left(\frac{\partial u}{\partial\theta} - v\right) = -2z\frac{\partial}{\partial r}\left(\frac{\partial w}{r\partial\theta}\right). \tag{8}$$

and stress components are established by the constitutive relation as:

$$t_r = \frac{E}{1-\nu^2}\left[\varepsilon_r + \nu\varepsilon_\theta - (1+\nu)\alpha_T T\right], \tag{9}$$

$$t_\theta = \frac{E}{1-\nu^2}\left[\varepsilon_\theta + \nu\varepsilon_r - (1+\nu)\alpha_T T\right], \tag{10}$$

$$t_{r\theta} = \mu\gamma_{r\theta}, \tag{11}$$

The bending and torsion moments are defined as:

$$M_r = \int_{-h/2}^{h/2} t_r z dz + \int_{-h/2}^{h/2} m_{\theta r} dz, \tag{12}$$

$$M_\theta = \int_{-h/2}^{h/2} t_\theta z dz - \int_{-h/2}^{h/2} m_{r\theta} dz, \tag{13}$$

$$M_{r\theta} = \int_{-h/2}^{h/2} t_{r\theta} z dz + \frac{1}{2}\int_{-h/2}^{h/2} \left(m_\theta - m_r\right) dz, \tag{14}$$

$$M_r = -D\left[\frac{\partial^2 w}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)\right] - \frac{E\alpha_T M_T}{(1-\nu)\beta d} + \alpha h\left[\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2} - \frac{\partial^2 w}{\partial r^2}\right], \tag{15}$$

$$M_\theta = -D\left[\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2} + \nu\left(\frac{\partial^2 w}{\partial r^2}\right)\right] - \frac{E\alpha_T M_T}{(1-\nu)\beta d} - \alpha h\left[\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2} - \frac{\partial^2 w}{\partial r^2}\right], \tag{16}$$

$$M_{r\theta} = -\frac{\mu h^3}{6}\frac{\partial}{\partial r}\left(\frac{\partial w}{r\partial\theta}\right) + 2\alpha h\left(\frac{\partial w}{r^2\partial\theta} - \frac{\partial^2 w}{r\partial\theta\partial r}\right), \tag{17}$$

The equations for shear force resultants are

$$Q_r = \frac{1}{r}\left[\frac{\partial(rM_r)}{\partial r} - M_\theta + \frac{\partial M_{r\theta}}{\partial\theta}\right],\ Q_\theta = \frac{1}{r}\left[\frac{\partial(rM_{r\theta})}{\partial r} + \frac{\partial M_\theta}{\partial\theta} + M_{r\theta}\right]. \tag{18}$$

The equation of motion (force equilibrium z in the direction) is given by

$$\frac{\partial Q_r}{\partial r}+\frac{\partial Q_\theta}{r\partial\theta}+\frac{Q_r}{r}-\rho h\frac{\partial^2 w}{\partial t^2}=0. \tag{19}$$

The equation of motion for micro plate with symmetry about $y-$ axes is taken as

$$\left(D+\alpha h\right)\nabla^4 w+\frac{E\alpha_T}{\left(1-\nu\right)\beta d}\nabla^2 M_T+\rho h\frac{\partial^2 w}{\partial t^2}=0, \tag{20}$$

where $D=Eh^3/12\left(1-\nu^2\right)$ is the flexural rigidity of the plate, $E=\mu(3\lambda+2\mu)/(\lambda+\mu)$ is Young's modulus, $\nu=\lambda/2(\lambda+\mu)$ is the Poisson ratio and $q_0\left(x,t\right)$ represents the load acting along the thickness direction.

The equation of heat conduction for L-S theory is given by

$$K\nabla^2 T+K\frac{\partial^2 T}{\partial z^2}-\left(\frac{\partial}{\partial t}+\tau_0\frac{\partial^2}{\partial t^2}\right)\left(\rho c_e T-T_0\beta z\frac{\partial^2 w}{\partial x^2}\right)=0, \tag{21}$$

For convenience, we define the non-dimensional parameters as:

$$\left(x^{'},z^{'},u^{'},w^{'}\right)=\frac{\left(x,z,u,w\right)}{L},\ \tau_0^{'}=\frac{\tau_0 v}{L},\ t^{'}=\frac{tv}{L},\ T^{'}=\frac{T}{T_0},M_T^{'}=\frac{M_T}{d\beta T_0 h^2},\ v^2=\frac{E}{\rho}. \tag{22}$$

Using non-dimensional quantities defined by equation (22) on equations (20) and (21), we get

$$\left(1+\frac{\alpha h}{D}\right)\nabla^4 w+\frac{E\alpha_T T_0 h^2 L}{\left(1-\nu\right)D}\nabla^2 M_T+\frac{\rho h v^2 L^2}{D}\frac{\partial^2 w}{\partial t^2}=0, \tag{23}$$

$$\nabla^2 T+\frac{\partial^2 T}{\partial z^2}-\left(\frac{\partial}{\partial t}+\tau_0\frac{\partial^2}{\partial t^2}\right)\left(\frac{\rho c_e v L}{K}T-\frac{\beta v L}{K}z\nabla^2 w\right)=0, \tag{24}$$

4. Problem solution

Assuming the time harmonic vibrations as:

$$w\left(r,\theta,t\right)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}W_{mn}\left(r\right)e^{i\left(\omega_{mn}t+n\theta\right)},\ T\left(r,\theta,z,t\right)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\Theta_{mn}\left(r,z\right)e^{i\left(\omega_{mn}t+n\theta\right)}. \tag{25}$$

Here, ω_{mn} is the frequency of the plate and $W_{mn}\left(r\right)e^{i\omega_{mn}t}$ the mode shape of the displacement. Here $m=0,1,2,3,...$ denote the number of nodal diameters and $n=1,2,3,...$ the number of nodal circles. In general, the frequency ω_{mn} is complex. The real part $\mathrm{Re}(\omega_{mn})$ gives us new eigen-frequency of the plate in the presence of thermoelastic coupling, thermal relaxation time and mechanical relaxation times. The magnitude of imaginary part $\left|\mathrm{Im}\, g(\omega_{mn})\right|$ provides us the attenuation of vibrations.

Substituting equation (25) in equations (23) and (24), we obtain,

$$\left(1+\frac{\alpha h}{D}\right)\nabla^{*4}W_{mn}+\frac{E\alpha_T T_0 h^2 L}{(1-\nu)D}\nabla^{*2}M_{T_1}-\frac{\rho h v^2 L^2\omega_{mn}^2}{D}W_{mn}=0, \qquad (26)$$

and

$$\nabla^{*2}\Theta_{mn}+\frac{\partial^2\Theta_{mn}}{\partial z^2}-i\omega_{mn}\left(1+i\tau_0\omega_{mn}\right)\left(\frac{\rho c_e v L}{K}\Theta_{mn}-\frac{\beta L v}{K}z\nabla^{*2}W_{mn}\right)=0,. \qquad (27)$$

5. Thermal field on the thickness direction

We assume that the thermal gradient of the beam is very small as compared to that along its thickness direction,

$$\left(i.e.\ \left|\frac{\partial\Theta_{mn}}{\partial x}\right|<<\left|\frac{\partial\Theta_{mn}}{\partial z}\right|\right),$$

as in Sharma and Kaur (2014). Therefore, the equation (27) take the form,

$$\frac{\partial^2\Theta_{mn}}{\partial z^2}+p^2\Theta_{mn}=-\frac{z\beta L v\delta_t^1}{K}\nabla^{*2}W_{mn}, \qquad (28)$$

where

$$p^2=-\frac{\rho c_e L v\delta_t^1}{K},\ \delta_t^1=i\omega_{mn}\left(1+i\tau_0\omega_{mn}\right),\ M_{T_1}=\beta d\int_{-h/2}^{h/2}\Theta_{mn}(r,z)z\,dz. \qquad (29)$$

Let us assume that there is no heat flow across the upper and lower surfaces of the beam. Then,

$$\frac{\partial\Theta_{mn}}{\partial z}=0 \quad \text{at} \quad z=\pm h/2. \qquad (30)$$

With the use of this condition, the general solution of equation (28), can be written as

$$\Theta_{mn}(r,z) = -\frac{\beta L\nu\delta_t^1}{Kp^2}\left[z - \frac{sin(pz)}{pcos\left(\frac{p}{2}\right)}\right]\nabla^{*2}W_{mn}, \tag{31}$$

Substituting equation (31) in equation (29), the thermal moment is given by

$$M_{T_1} = -\frac{\beta^2 dL\nu h^3\delta_t^1}{12Kp^2}\left(1+f(p)\right)\nabla^{*2}W_{mn}, \tag{32}$$

where

$$f(p) = \frac{24}{(ph)^3}\left(\frac{ph}{2} - tan\frac{ph}{2}\right). \tag{33}$$

Using equation (32) in equation (26) , yields:

$$D_\omega\nabla^{*4}W_{mn} - a_1\omega_{mn}^2 W_{mn} = 0, \tag{34}$$

$$\nabla^{*4}W_{mn} - p^4 W_{mn} = 0, \tag{35}$$

where

$$\varepsilon_1 = -\frac{E\alpha_T T_0 h^5 L^2\beta^2 d\nu\delta_t^1}{12Kp^2 D(1-\nu)},\ D_\omega = \left[1+\varepsilon_1\left(1+f(p)\right)+\frac{\alpha h}{2D}\right],\ a_1 = \frac{\rho h\nu^2 L^2}{D},\ p^4 = \frac{a_1\omega_{mn}^2}{D_\omega}. \tag{36}$$

For axisymmetric vibration of circular plate, the solution of equation (35) is

$$W_{mn}(r) = A_1 J_m(pr) + A_2 Y_m(pr) + A_3 I_m(pr) + A_4 K_m(pr), \tag{37}$$

The coefficients through A_1 and A_4 allowed values of p are determined by the boundary conditions. In case of limitation of $W_{mn}(r)$ at the plate center, we get $A_2 = A_4 = 0$, then

$$W_{mn}(r) = A_1 J_m(pr) + A_3 I_m(pr), \tag{38}$$

Here, J_m and I_m represents the Bessel and modified Bessel functions of order n and of first kind.

6. Boundary Conditions

Let us consider following set of boundary conditions:

(i) Clamped plate

$$W_{mn}\big|_{r=a}=0,\ \left.{dW_{mn}}\middle/{dr}\right|_{r=a}=0. \tag{39}$$

(ii) Simply supported plate

$$W_{mn}\big|_{r=a}=0,\ \left[\nabla^2 W_{mn}+(1+\nu)\alpha_T M_{T_1}\right]\Big|_{r=a}=0. \tag{40}$$

Equation (42), with the aid of (34) reduces to

$$W_{mn}\big|_{r=a}=0,\ \nabla^2 W_{mn}\big|_{r=a}=0. \tag{41}$$

(iii) Free plate

$$\left.\frac{\partial^2 W_{mn}}{\partial r^2}+\nu\left(\frac{1}{r}\frac{\partial W_{mn}}{\partial r}+\frac{1}{r^2}\frac{\partial^2 W_{mn}}{\partial\theta^2}\right)\right|_{r=a}=0, \tag{42}$$

$$\left.\frac{\partial}{\partial r}\left(\nabla^2 W_{mn}\right)+\frac{(1-\nu)}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial^2 W_{mn}}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial W_{mn}}{\partial\theta}\right)\right|_{r=a}=0. \tag{43}$$

7. Characteristic Equations

Substituting equation (38) in the boundary conditions (39), (41) and (42) and (43), we get

(I) Clamped plate

$$A_1 J_m(pa)+A_3 I_m(pa)=0, \tag{44}$$

$$A_1 p\left[-J_{m+1}(pa)+\left(\frac{m}{pa}\right)J_m(pa)\right]+A_3 p\left[I_{m+1}(pa)+\left(\frac{m}{pa}\right)I_m(pa)\right]=0, \tag{45}$$

For nontrivial solutions of equations (44) and (45), we obtain

$$\begin{vmatrix} J_m(pa) & I_m(pa) \\ -J_{m+1}(pa)+\left(\frac{m}{pa}\right)J_m(pa) & I_{m+1}(pa)+\left(\frac{m}{pa}\right)I_m(pa) \end{vmatrix}=0. \tag{46}$$

(II) Simply supported plate

$$\begin{vmatrix} J_m(pa) & I_m(pa) \\ -J_m(pa) & I_m(pa) \end{vmatrix}=0. \tag{47}$$

(III) Free plate

$$\begin{vmatrix} (pa)^2 J_m(pa)+(1-\nu)\left\{(pa)J_m'(pa)-m^2 J_m(pa)\right\} & (pa)^2 I_m'(pa)+(1-\nu)m^2\left\{(pa)J_m'(pa)-J_m(pa)\right\} \\ (pa)^2 I_m(pa)-(1-\nu)\left\{(pa)I_m'(pa)-m^2 I_m(pa)\right\} & (pa)^2 I_m'(pa)-(1-\nu)\left\{(pa)I_m'(pa)-I_m(pa)\right\} \end{vmatrix}=0, \tag{48}$$

The solution of characteristic equations (46), (47) and (48) as $pa=\sqrt{q_{mn}}$, where the values of $q_{mn}\left(m=0,1,2,3;n=1,2,3\right)$ are described in Table 1 for clamped, simply supported and free plates.

8. Frequency shift and Damping

Now the vibration frequency of the plate in the presence of thermoelastic coupling and thermal relaxation time is given by

$$q^4 = {a_1\omega_{mn}^2}/{D_\omega}, \tag{49}$$

$$\omega_{mn} = q_{mn}^2\sqrt{{D_\omega}/{a_1}} = \omega_0\left[1+{\alpha h}/{D}+\varepsilon_1\left(1+f\left(p\right)\right)\right]^{\frac{1}{2}}. \tag{50}$$

where

$$\omega_0 = \frac{q_{mn}^2}{\sqrt{a_1}}. \tag{51}$$

and following Sharma (2011), we can replace $f(p)$ with $f(\omega_0)$ and expand equation (50) upto first order, we obtain

$$\omega_{mn} = \omega_0\left[1-\frac{\alpha h}{4D}-\left(\frac{i\omega E\alpha_T T_0 h^2 L^2\beta\nu\delta_t^1}{24Kp^2D(1-\nu)}\right)\left(1+f\left(p\right)\right)\right]. \tag{52}$$

The thermal gradients in the plane of cross-section along the thickness direction of the plate are much larger than those along its length and hence $\frac{\partial^2\Theta_{mn}}{\partial x^2}\cong 0$ so that

$$p^2+{i\omega_{mn}\rho c_e L\nu\delta_t^2}/{K}=0, \tag{53}$$

This implies that

$$p=p_0 e^{\frac{-i\alpha_1}{2}},\ p_0=\sqrt{{\rho c_e\omega_{mn}L\nu s^*}/{K}},\ s^*=\sqrt{1+\tau_0\omega_{mn}^2},\ \alpha_1=tan^{-1}\left(-{1}/{\tau_0\omega_{mn}}\right). \tag{54}$$

Replacing ω_{mn} with ω_0 in equations (55) and (56), we obtain

$$p=\sqrt{2}p_0^*\left(\cos\frac{\Phi}{2}-i\sin\frac{\Phi}{2}\right), \tag{55}$$

and

$$p_0^*=\sqrt{\rho c_e\omega_0 L\nu s_0^* / 2K},\ s_0^*=\sqrt{1+\tau_0\omega_0^2},\ \Phi=tan^{-1}\left(-1/\tau_0\omega_0\right). \tag{56}$$

The frequency ω_{mn} is complex in nature and hence we take

$$\omega_{mn}=\omega_{mn}^R+i\omega_{mn}^I,\ \omega_{mn}^R=\mathrm{Re}\left(\omega_{mn}\right),\ \omega_{mn}^I=\mathrm{Img}\left(\omega_{mn}\right), \tag{57}$$

and

$$\omega_{mn}^R=\omega_0\left[1+\alpha h/4D+\frac{\varepsilon_1}{2}\left(1+f\left(R\right)\right)\right],\ \omega_{mn}^I=\omega_0\varepsilon_1 f\left(I\right). \tag{58}$$

where

$$f\left(R\right)=\frac{6\cos\Phi}{\left(p_0^*h\right)^2}-\frac{6\sqrt{2}\cos\frac{3\Phi}{2}}{\left(p_0^*h\right)^3}\left[\frac{\sin\eta_1+\tan\frac{3\Phi}{2}\sinh\eta_1\eta_2}{\cos\eta_1+\cosh\eta_1\eta_2}\right], \tag{59}$$

and

$$f\left(I\right)=\frac{6\sin\Phi}{\left(p_0^*h\right)^2}-\frac{6\sqrt{2}\cos\frac{3\Phi}{2}}{\left(p_0^*h\right)^3}\left[\frac{\tan\frac{3\Phi}{2}\sin\eta_1-\sinh\eta_1\eta_2}{\cos\eta_1+\cosh\eta_1\eta_2}\right], \tag{60}$$

where $\eta_1=\sqrt{2}p_0^*h\cos\left(\frac{\Phi}{2}\right),\ \eta_2=\tan\left(\frac{\Phi}{2}\right)$ and ω_0 is taken from equation (51).

The frequency shift and damping in a thermoelastic plate are taken from Sharma (2011)

$$\omega_s=\left|\frac{\omega_{mn}^R-\omega_0}{\omega_0}\right|. \tag{61}$$

and

$$Q^{-1}=2\left|\frac{\omega_{mn}^I}{\omega_{mn}^R}\right|. \tag{62}$$

9. Particular cases

(i). Coupled thermoelastic (CT) plate

In the absence of thermal relaxation time $(\tau_0=0)$, we obtain

$$p = p_0(1-i),\ p_0^* = \sqrt{\frac{\rho c_e \omega_0 L \nu}{2K}},\ s_0^* = 1,\ \Phi = \frac{\pi}{2},\ \eta_2 = 1,\ \eta_1 = p_0^* h.$$

Accordingly, equations (61) and (62) became

$$f(R) = \frac{6}{(p_0^* h)^3}\left[\frac{\sin\eta_1 - \sinh\eta_1}{\cos\eta_1 + \cosh\eta_1}\right], \tag{63}$$

and

$$f(I) = \frac{6}{(p_0^* h)^2} - \frac{6}{(p_0^* h)^3}\left[\frac{\sin\eta_1 + \sinh\eta_1}{\cos\eta_1 + \cosh\eta_1}\right]. \tag{64}$$

(ii). If couple stress parameter $(\alpha = 0)$, equation (60) reduces to

$$\omega_{mn}^R = \omega_0\left[1 - \left(\frac{i\omega_0 E\alpha_T T_0 h^2 L^2 \beta\nu\delta_t^1}{24Kp^2 D(1-\nu)}\right)(1 + f(R))\right],\quad \omega_{mn}^I = \left(\frac{i\omega_0^2 E\alpha_T T_0 h^2 L^2 \beta\nu\delta_t^1}{24Kp^2 D(1-\nu)}\right) f(I). \tag{65}$$

where

$$f(R) = \frac{6\cos\Phi}{(p_0^* h)^2} - \frac{6\sqrt{2}\cos\frac{3\Phi}{2}}{(p_0^* h)^3}\left[\frac{\sin\eta_1 + \tan\frac{3\Phi}{2}\sinh\eta_1\eta_2}{\cos\eta_1 + \cosh\eta_1\eta_2}\right],\ f(I) = \frac{6\sin\Phi}{(p_0^* h)^2} - \frac{6\sqrt{2}\cos\frac{3\Phi}{2}}{(p_0^* h)^3}\left[\frac{\tan\frac{3\Phi}{2}\sin\eta_1 - \sinh\eta_1\eta_2}{\cos\eta_1 + \cosh\eta_1\eta_2}\right].$$

(66)

10. Numerical results and discussion

For the purpose of numerical computations, we have prepared mathematical model with silicon material. The physical data of the problem are taken from Sun and Saka (2010). The values of thermoelastic damping and frequency shift of vibration modes (1,1) and (2,1) with thickness h and length L in a clamped plate, simply-supported plate and free plate have been computed in the absence and presence of couple stress. The numerical computations have been carried out with the help of MATLAB software. The computer simulated results have been presented graphically in figs. 1-12.

Table 1. Following [Sharma and Sharma (2011), Rao (2007)], the values of $q_{mn}\ (m = 0,1,2,3; n = 1,2,3)$ for clamped, simply-supported and free plates

n	Clamped plate			Simply supported plate			Free plate		
	q_{1n}	q_{2n}	q_{3n}	q_{1n}	q_{2n}	q_{3n}	q_{1n}	q_{2n}	q_{3n}
0	9.8596	39,4384	88.7364	5.5460	30.1950	74.5632	4.21	5.253	12.23
1	22.1841	61.6225	120.7801	15.4056	49.9142	104.1420	20.52	35.25	52.91
2	39.4384	88.7364	157.7536	30.1950	74.5632	138.6506	59.86	83.9	111.3
3	61.6225	120.7801	199.6569	49.9142	104.1420	178.0890	119.0	154.0	192.1

Physical data for Silicon material:

Quantity	Silicon material	Unit
E	165.9	Kg m^{-1} s^{-2}
ν	0.22	
ρ	1.74×10^{3}	Kg m^{-3}
T_0	293	K
c_e	713	J Kg^{-1} K^{-1}
K	156	W m^{-1} K^{-1}
α_t	2.59	K^{-1}
α	25	N
ω	10	Sec^{-1}
τ_0	0.002	Sec
d	1	m
H	.01	m
L	20	m

Damping

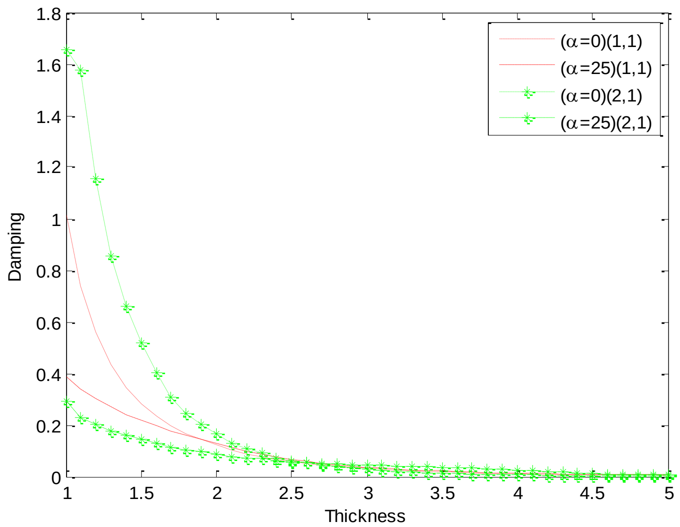

Figure 1. Thermoelastic damping Q^{-1} of vibration modes (1,1) and (2,1) with thickness h in a clamped plate.

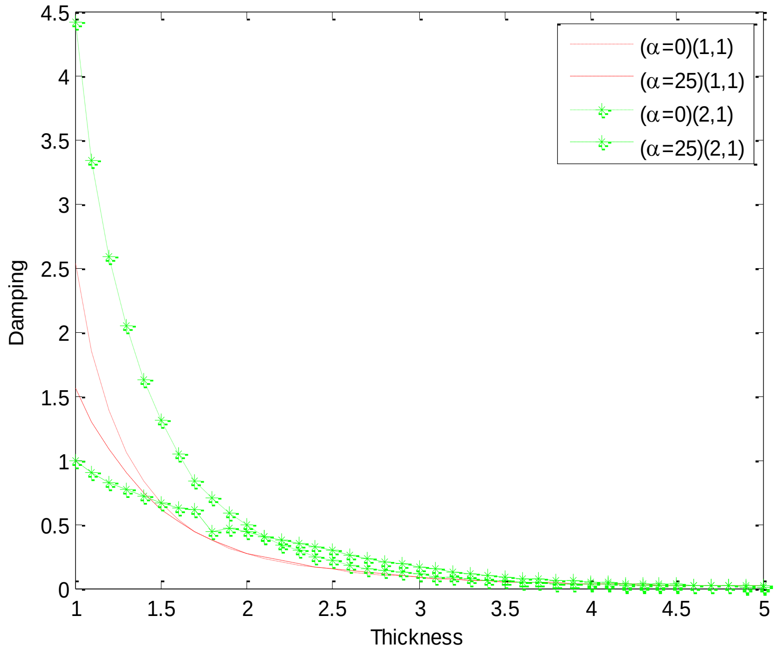

Figure 2. Thermoelastic damping Q^{-1} of vibration modes (1,1) and (2,1) with thickness h in a simply supported plate.

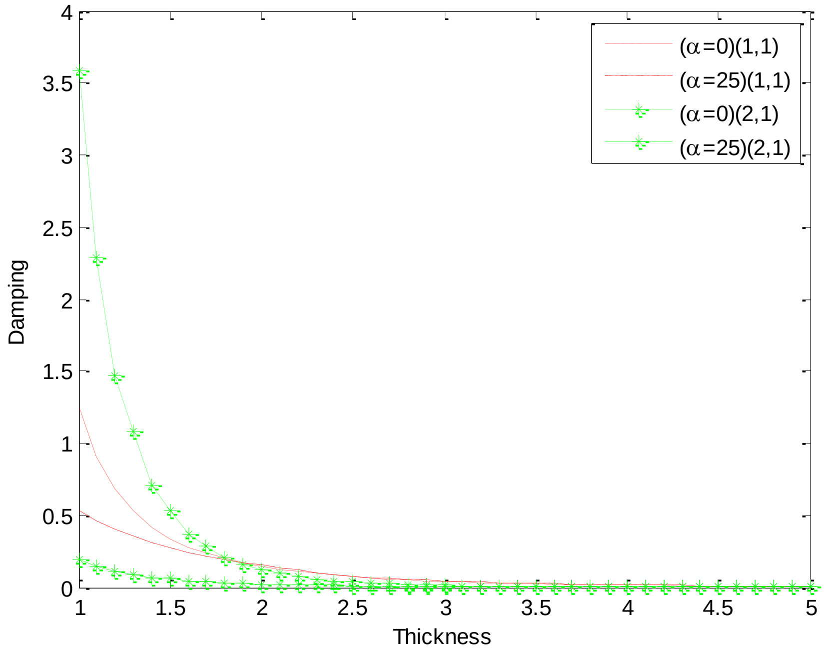

Figure 3. Thermoelastic damping Q^{-1} of vibration modes (1,1) and (2,1) with thickness h in a free plate.

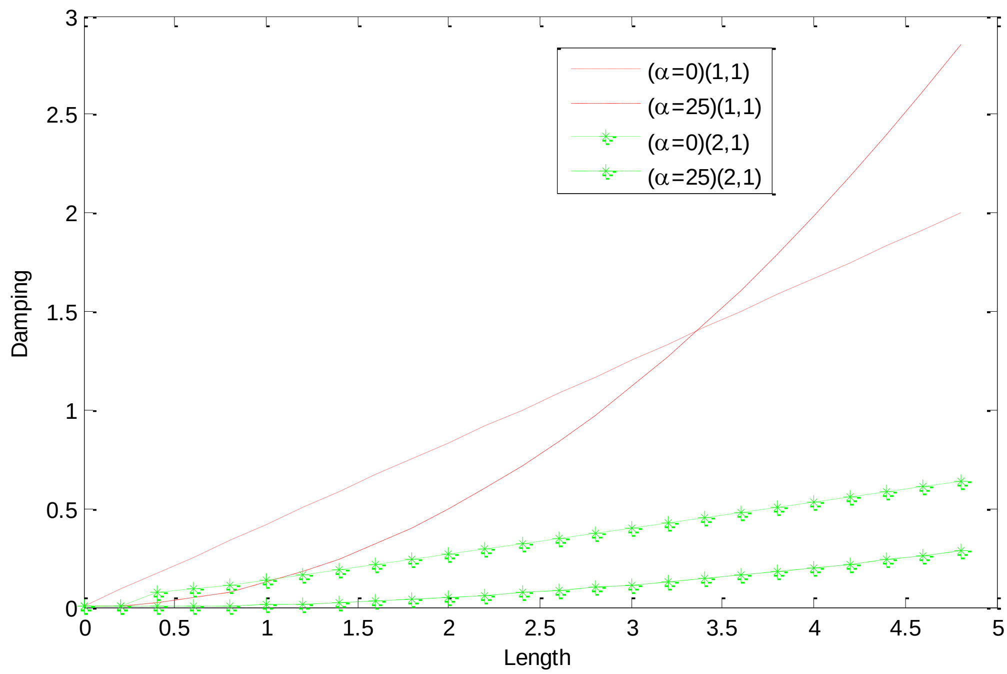

Figure 4. Thermoelastic damping Q^{-1} of vibration mode (1,1) and (2,1) with length L in a clamped plate.

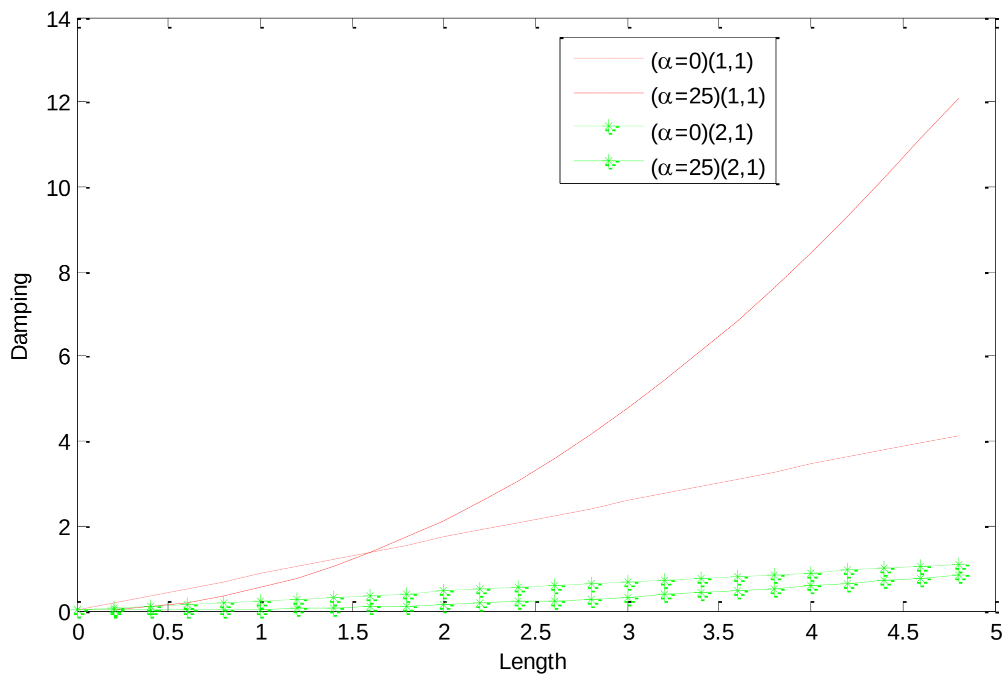

Figure 5. Thermoelastic damping Q^{-1} of vibration mode (1,1) and (2,1) with length L in a simply supported plate.

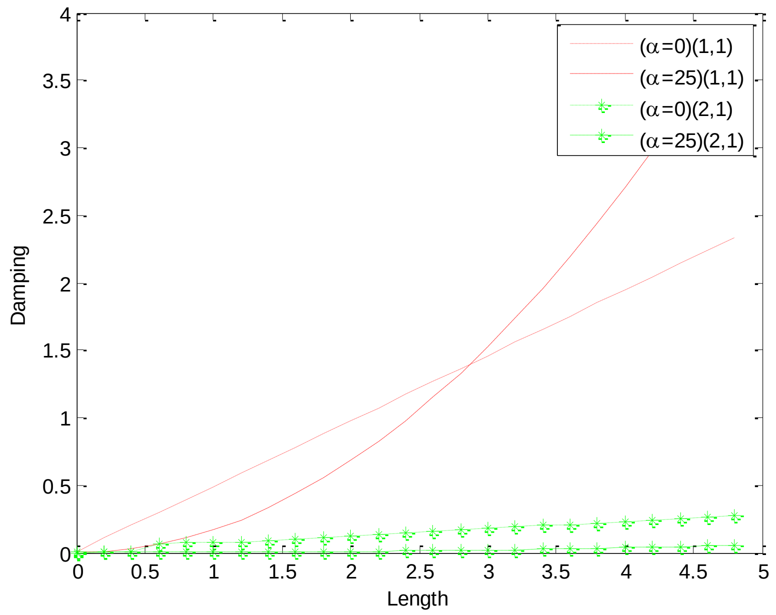

Figure 6. Thermoelastic damping Q^{-1} of vibration mode (1,1) and (2,1) with length L in a free plate.

Frequency shift

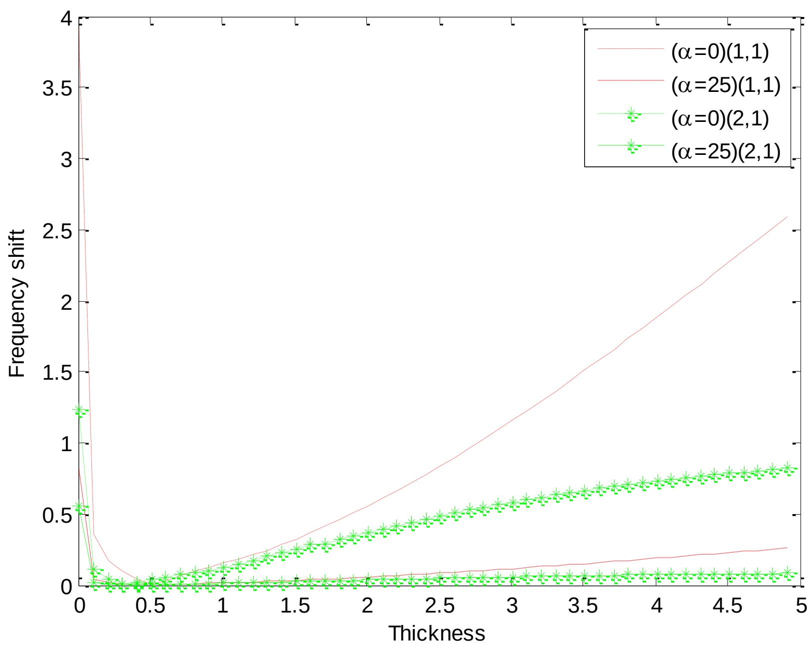

Figure 7. Frequency shift ω_s of vibration modes (1,1) and (2,1) with thickness h in a clamped plate.

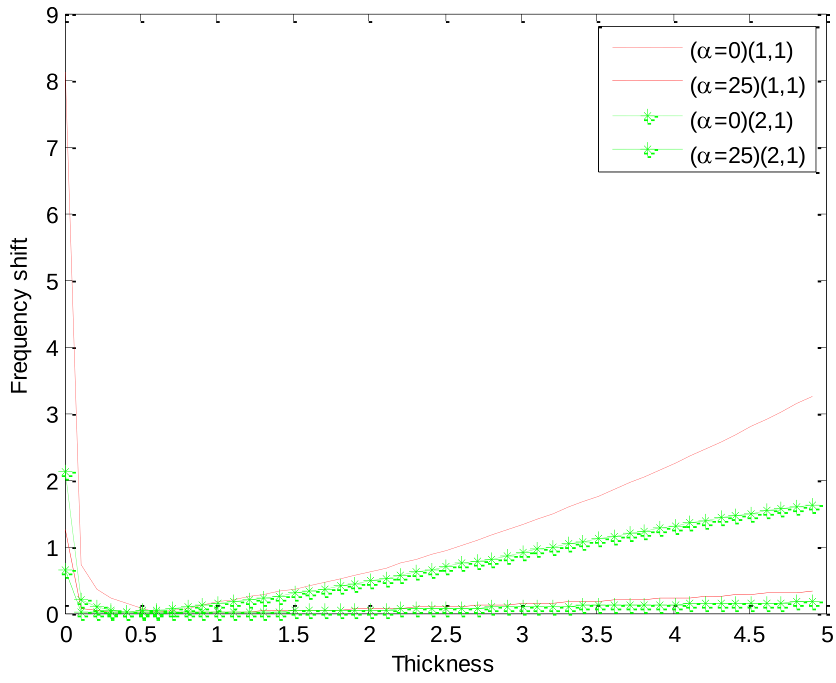

Figure 8. Frequency shift ω_s of vibration modes (1,1) and (2,1) with thickness h in a simply supported plate.

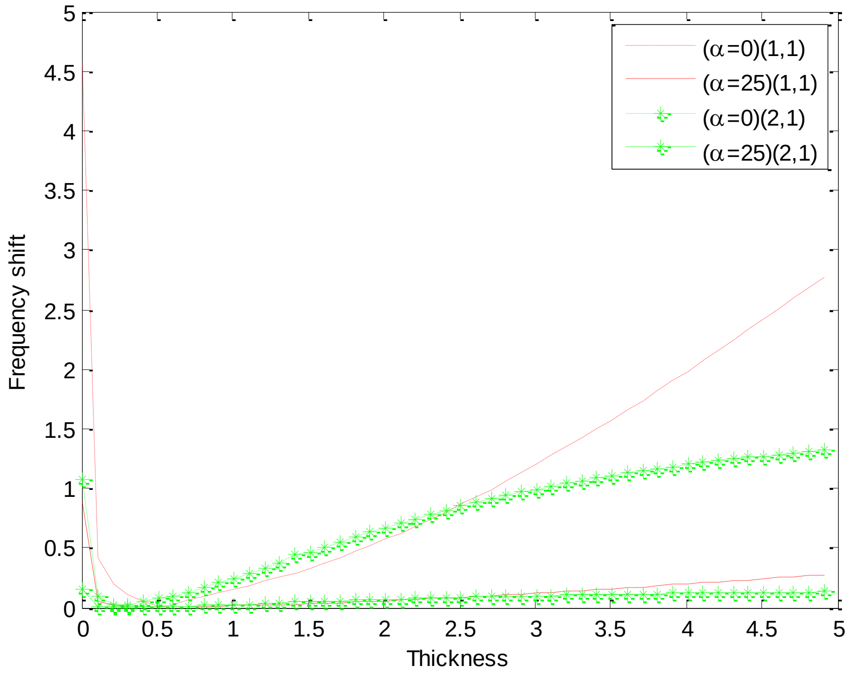

Figure 9. Frequency shift ω_s of vibration modes (1,1) and (2,1) with thickness h in a free plate.

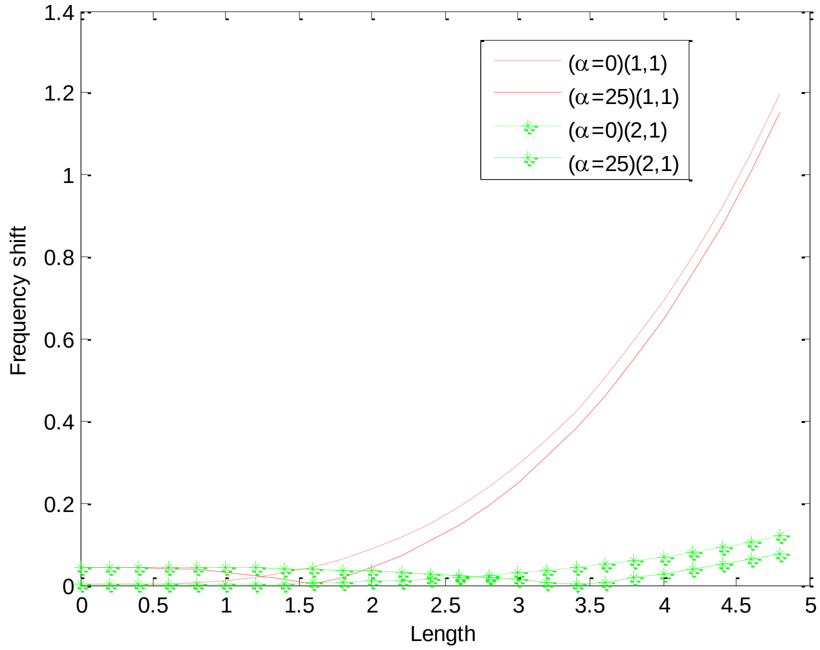

Figure 10. Frequency shift ω_s of vibration mode (1,1) and (2,1) with length L in a clamped plate.

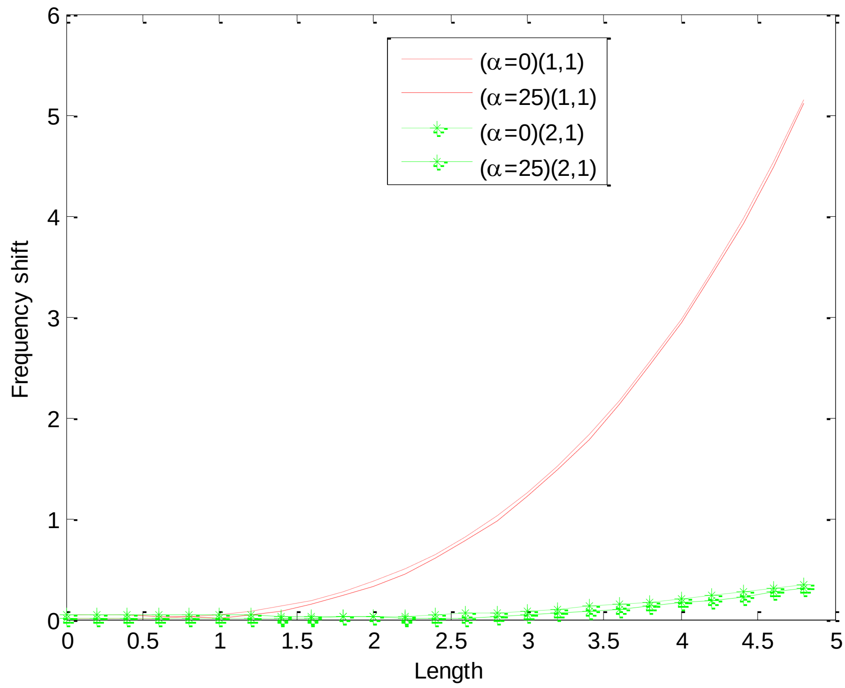

Figure 11. Frequency shift ω_s of vibration modes (1,1) and (2,1) with length L in a simply supported plate.

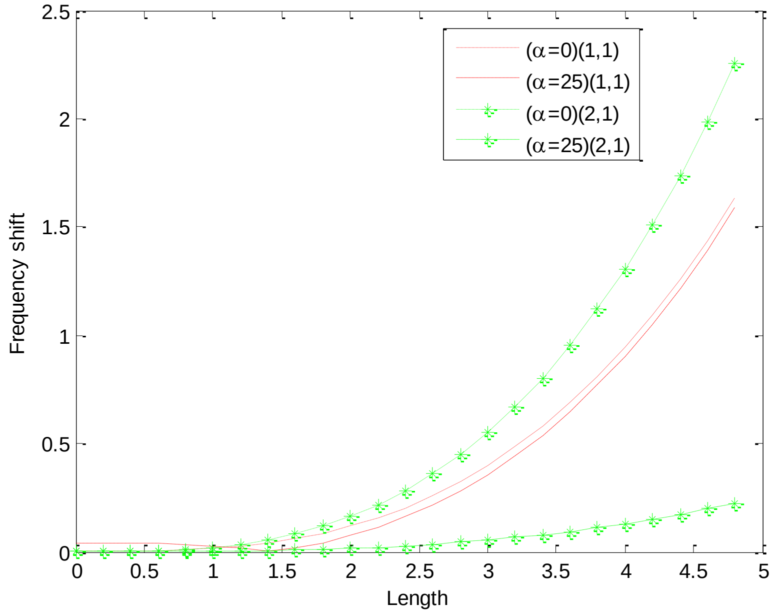

Figure 12. Frequency shift ω_s of vibration modes (1,1) and (2,1) with length L in a free plate.

Damping

Figure 1 shows the thermoelastic damping with thickness h in a clamped plate of vibration modes (1,1) and (2,1). It is observed that the damping factor decreases monotonically in the considered region of thickness for all cases. The values of damping factor is greater in the absence of couple stress and smaller for presence of couple stress. Figure 2 depicts the thermoelastic damping with thickness h in a simply supported plate of vibration modes (1,1) and (2,1). It is noticed that the behaviour and variation of damping factor are same but difference between their values. The values of damping factor of vibration mode (2,1) is observed to have more value than that of vibration mode (1,1) for absence and presence of couple stress. Figure 3 represents the thermoelastic damping with thickness h in a free plate of vibration modes (1,1) and (2,1). The value of damping factor is larger for small values of thickness and remains stationary for higher value of thickness in the assumed range for all cases of couple stress.

Figure 4 depicts the thermoelastic damping with length L in a clamped plate of vibration modes (1,1) and (2,1). The value of thermoelastic damping increases with increasing value of length. It is observed that the value of damping factor is greater in case of vibration mode (1,1) in comparison to vibration mode (2,1) for $\alpha = 0, 25$. Figure 5 represents the thermoelastic damping with length L in a simply supported plate of vibration modes (1,1) and (2,1). The values of damping factor increases smoothly with increase in length for all cases of couple stress and also different vibration modes in the considered range. The damping factor of vibration mode (1,1) have greater value than that of vibration mode (2,1) in the presence of couple stress, whereas its value of vibration mode (1,1) is more in comparison with vibration mode (2,1) for absence of couple stress. Figure 6 shows the thermoelastic damping with length L in a free plate of vibration modes (1,1) and (2,1). It is observed that the value of damping factor increases with increasing values of length (L). Moreover, damping factor have more value in the vibration mode (1,1) in comparison with vibration mode (2,1) for absence and presence of couple stress.

Frequency shift

Figure 7 represents the frequency shift of vibration mode (1,1) and (2,1) with thickness h in a clamped plate. It is clear that the frequency shift of vibration modes (1,1) and (2,1) first decreases for small values of thickness and then increases for higher value of thickness for the considered range. The value of frequency shift of vibration mode (1,1) has greater value of vibration mode (2,1) for $\alpha = 0, 25$. Figure 8 represents the frequency shift of vibration mode (1,1) and (2,1) with thickness h in a simply supported plate. The behaviour and variation are almost same but difference between their values. It is evident from the figure that the value of frequency shift of vibration mode (1,1) has more value in absence of couple stress than that of presence of couple stress. Also, frequency shift of vibration mode (2,1) is observed greater value for $\alpha = 0$ in comparison with $\alpha = 25$. Figure 9 represents the frequency shift of vibration mode (1,1) and (2,1) with thickness h in a free plate. The value of frequency shift initially decreases and then increases smoothly in the assumed range of thickness. It is noticed that the values of frequency shift of vibration modes (1,1) and (2,1) have greater value for absence of couple stress in comparison with presence of couple stress.

Figure 10 represents the frequency shift of vibration mode (1,1) and (2,1) with length L in a clamped plate. It is observed from the figure that the value of frequency shift increases with increasing value of length. The value of frequency shift is observed to have greater value in case of vibration mode (1,1) than that of vibration mode (2,1) for all cases of couple stress. Figure 11 represents the frequency shift of vibration mode (1,1) and (2,1) with length L for a simply supported plate. The values of frequency shift increases monotonically with increase in the value of length for all cases of couple stress and vibration modes (1,1) and (2,1). It is clear from the figure that value of frequency shift has more for absence of couple stress in comparison with presence of couple stress for vibration mode (1,1). Similarly, frequency shift of vibration mode (2,1) has less value for presence of couple stress than that of absence of couple stress. Figure 12 represents the frequency shift of vibration mode (1,1) and (2,1) with length L for a free plate. It is noticed that the behaviour and variation are same for vibration modes (1,1) and (2,1). The values of frequency shift for vibration mode (2,1) have greater value in case of vibration mode (1,1) in the absence of couple stress, whereas opposite behaviour is observed for presence of couple stress.

11. Conclusions

This paper devoted to the study of vibrations of circular plate in modified couple stress thermoelastic medium in the context of Kirchhoff-Love plate theory and Lord-Shulman thermoelasticity theory. The mathematical expressions for thermoelastic damping of vibration and frequency shift are obtained in the absence and presence of couple stress for (1,1) and (2,1) vibration modes of the plate. Damping factor and frequency shift with varying values of length and thickness are shown graphically to show the effect of couple stress for vibration modes (1,1) and (2,1) with clamped plate, simply supported plate and free plate. It is concluded from the figures that the damping factor decreases with increase in the value of thickness, whereas its value increases with increasing value of thickness for both cases of couple stress and both vibration modes (1,1) and (2,1). The value of frequency shift initially decreases and then increase in the value of thickness but the value of frequency shift increases with increase in the value of length in the considered range. It is also concluded that the thermoelastic damping factor and frequency shift of vibration modes (1,1) and (2,1) with increase in the values of thickness attains larger value in the absence of couple stress than that of presence of couple stress for clamped, simply-supported and free plates. The thermoelastic damping factor and frequency shift for absence and presence of couple stress for vibration mode (1,1) have more prominent value in comparison with vibration mode (2,1) for clamped, simply-supported and free plates. The results of this problem may be useful for Infra-Red (IR) detections and imaging in addition to chemical and biological agent sensing.

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