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A new hybrid method for solving nonlinear fractional differential equations

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Abstract

In this paper, numerical solution of initial and boundary value problems for nonlinear fractional differential equations is considered by pseudospectral method. In order to avoid solving systems of nonlinear equations resulting from the method, the residual function of the problem is constructed, as well as a suggested unconstrained optimization model solved by PSOGSA algorithm. Furthermore, the research inspects and discusses the spectral accuracy of Chebyshev polynomials in the approximation theory. The following scheme is tested for a number of prominent examples, and the obtained results demonstrate the accuracy and efficiency of the proposed method.

Keywords: Fractional differential equations; Pseudospectral method; Chebyshev polynomials; Modified Gravitational search algorithm

MSC 2010 No.: 34A08, 90C99

1. Introduction

Fractional differential equations (FDEs) arise in many areas of science, economics and engineering, such as biophysics, control theory, finance, bioengineering, electrodynamics of complex media, signal processing and viscoelastic materials. (for instance, see Magin (2004), Machando (2001), Sabatier et al. (2007), Kilbas et al. (2006), Koller (1984), Podlubny (1999) and Sheng et al. (2011))

The existence and uniqueness of solutions for FDEs has been discussed by many authors (for example, see Sabatier et al. (2007), Rehman et al. (2011), Diethelm (2010), Agarwal et al. (2010), Agarwal and Ahmad (2011) and Bayor and Torres (2016)). The algorithms for solving FDEs has attracted considerable attention, some of these methods are as follows:

Adomian decomposition method (Wang (2006), Momani et al. (2006)), variational iteration method (Inc (2008), Odibat and Momani (2006)), homotopy perturbation method (Gupta and Singh (2011), Odibat, and Momani (2008)), homotopy analysis method (Hashim et al. (2009), Odibat et al. (2010)), collocation method (Rawashdeh (2006)), perturbation Laplace method (Khan et al. (2012)) and fractional difference transform method (Erturk et al. (2008), Meerscharet and Tadjeran (2006)). Also the wavelet collocation method for the numerical solution of a class of FDEs is presented in (Heydari et al. (2012)). Doha et al. (2012) presented Jacobi operational matrix of fractional derivatives and used Jacobi collocation approximation for nonlinear FDEs. Diethelm et al. (2002) suggested predictor-corrector method for numerical solutions of FDEs. A numerical technique based Quasi Newton method and simplified reproducing kernel method is used to solve nonlinear FDEs (Jia et al. (2016)). Geng and Cui (2012) presented an algorithm based on reproducing kernel method for solving nonlocal fractional boundary value problems. Li et al. (2016) used finite difference methods with non-uniform meshes for solving nonlinear FDEs. Artificial neural networks is used to solve FDEs in (Pakdaman et al. (2017)). Saadatmandi and Dehghan (2010) presented a numerical algorithm based on Legendre polynomials to solve FDEs and also generalized Legendre operational matrix to fractional calculus.

In this paper, we consider the following class of FDEs:

$$D^{\alpha}u = f(x,u,D^{\beta_1}u,\dots,D^{\beta_k}u),\ x\in[a,b]. \tag{1}$$

with the initial conditions,

$$u^{(i)}(a) = d_i,\ i=0,\dots,n, \tag{2}$$

or the boundary conditions,

$$g_k(u(a),u'(a),\dots,u^{(n)}(a),u(b),u'(b),\dots,u^{(n)}(b))=0,\ \ k=0,\dots,n, \tag{3}$$

where $n<\alpha\le n+1,\ 0<\beta_1<\beta_2<\dots<\beta_k<\alpha$, D^{α} denotes the Caputo fractional derivative of order α , and we assume that f and g_k are given real nonlinear functions.

In this paper, pseudospectral method based on Chebyshev polynomials is used for solving the above FDEs. Also, by considering residual function of the FDE, an unconstrained optimization problem is introduced. We use modified gravitational search algorithm to find appropriate coefficients of the Chebyshev series. The efficiency of the proposed method is shown by some examples.
This paper is organized as follows: In section 2, we discuss about Chebyshev polynomials and their spectral accuracy in approximation theory. In section 3, we review fractional

derivatives. In section 4, we review a summary of modified gravitational search algorithm. Then, in section 5, hybrid pseudospectral method and PSOGSA for solving nonlinear FDEs is presented. In section 6, we present the results of numerical experiments, and finally in section 7, we will draw some conclusions based on the numerical analysis.

2. Orthogonal Chebyshev polynomials

It is well known that the eigenfunctions of certain singular Sturm-Lioville problems allow the approximation of $C^{\infty}[a,b]$ functions, where the truncation error approaches zero faster than any negative power of the number of basic functions used in the approximation, as that number (the order of truncation N) tends to infinity (Canuto et al. (1998)). This phenomenon is usually referred to as "spectral accuracy" (Gottlieb and Orzag (1979)). Throughout, we will use first kind orthogonal Chebyshev polynomials $\{T_k\}_{k=0}^{\infty}$, which are eigenfunctions of the singular Sturm-Liouville problem:

$$\frac{d}{dz}(\sqrt{1-z^2}\frac{d}{dz}T_n(z))+\frac{n^2}{\sqrt{1-z^2}}T_n(z)=0, \qquad -1<z<1, n=0,1,.... \tag{4}$$

The Chebyshev polynomials are orthogonal with respect to the L_2 inner product on the interval $[-1,1]$ by the weight function $w(z)=\frac{1}{\sqrt{1-z^2}}$, i.e.,

$$\int_{-1}^{1}T_m(z)T_n(z)w(z)dz=\frac{\pi\gamma_m}{2}\delta_{mn}, \tag{5}$$

where δ_{mn} denotes the Kronecker delta and

$$\gamma_m=\begin{cases}2, & m=0,\\ 1, & m\geq 1.\end{cases}$$
(6)

The Chebyshev expansion of a function is defined as:

$$u(z)=\sum_{k=0}^{\infty}a_kT_k(z). \tag{7}$$

The derivative of the above function expanded in Chebyshev polynomials can be represented by the following theorem:

Theorem 2.1. (Canuto et al. (1998)**).**

If $u(z)=\sum_{k=0}^{\infty}a_kT_k(z)$, then, the derivative of u can be represented by

$$u'(z)=\sum_{k=0}^{\infty}a_k^{(1)}T_k(z), \tag{8}$$

where

$$a_k^{(1)} = \frac{2}{\gamma_k} \sum_{\substack{p=k+1 \\ p+k \ odd}}^{\infty} p a_p. \tag{9}$$

Furthermore, we have an efficient way of differentiating a polynomial of degree N in Chebyshev space, i.e., since $a_k^{(1)} = 0$ for $k \geq N$, the non-zero coefficients are computed in decreasing order by the recurrence relation:

$$\gamma_k a_k^{(1)} = a_{k+2}^{(1)} + 2(k+1)a_{k+1}, \qquad 0 \leq k \leq N-1. \tag{10}$$

The generalization of this relation is (Canuto et al. (1998)):

$$\gamma_k a_k^{(q)} = a_{k+2}^{(q)} + 2(k+1)a_{k+1}^{(q-1)}, \qquad q = 2,3,.... \tag{11}$$

In the remaining part of this section, we present some theorems about the convergence of Chebyshev expansion.

Theorem 2.2. (Mason and Handscomb (2003)).

If $u(x)$ is continuous and either is of bounded variation or satisfies a Dini-Lipschitz condition on $[-1,1]$, then, its Chebyshev series expansion is uniformly convergent.

Theorem 2.3. (Boyd (2000)).

Let

$$u \in C[-1,1] \qquad \text{and} \qquad (P_N u)(x) = \sum_{n=0}^{N} a_n T_n(x),$$

where

$$a_k = \frac{2}{\pi \gamma_k} \int_{-1}^{1} \frac{T_k(x) u(x)}{\sqrt{1-x^2}}.$$

Then,

$$E_T(N) \equiv | u(x) - P_N u(x) | \leq \sum_{n=N+1}^{\infty} | a_n | \text{ for all } u(x), \text{ all } N \text{ and all } x \in [-1,1].$$

Theorem 2.4. (Mason and Handscomb (2003)).

If $u(x)$ has $m+1$ continuous derivatives on $x \in [-1,1]$ and

$$(P_N u)(x) = \sum_{n=0}^{N} a_n T_n(x),$$

where

$$a_n = \frac{2}{\pi\gamma_n}\int_{-1}^{1}\frac{T_n(x)u(x)}{\sqrt{1-x^2}},$$

then

$$|u(x) - P_N u(x)| = O(N^{-m}), \text{ for all } x \in [-1,1].$$

3. Fractional derivatives

In this section, we recall some essential facts of fractional calculus. (Podlubny (1999)). There are various definitions for fractional derivatives. However, three definitions of fractional derivatives are more applicable than others and are used in modelling the problems in different fields of sciences. These definitions are Grunwald-Letnikov, Riemann-Liouville and Caputo. Among these fractional derivatives, Riemann-Liouville and Caputo derivatives are the most popular fractional derivatives. Riemann-Liouville derivative has a lot of problems in modelling real-world phenomena, for example, the derivative of a constant function is not zero in Riemann-Liouville approach. But Caputo definition resolves problems of Riemann-Liuville definitions in modelling real-world phenomena, and so is more practical in science and engineering. The main advantage of the Caputo approach is that the initial conditions for FDEs is sufficient to prove the uniqueness of the solution, and so we use Caputo derivative in this paper.

Caputo derivative has the following properties:

$D^{\alpha} c = 0$, where c is a constant function and,

$$D^{\alpha} x^n = \begin{cases} 0, & n < \lceil \alpha \rceil,\ n \in Z^+, \\ \dfrac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{n-\alpha}, & n \geq \lceil \alpha \rceil,\ n \in Z^+, \end{cases} \tag{12}$$

where $\lceil \alpha \rceil$ is the ceiling function and denotes the smallest integer greater than or equal to α (Podlubny (1999)).

Definition 3.1.

The Caputo definition for the fractional-order derivatives is defined as

$$D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha+1-n}}dt, \qquad 0 \leq n-1 < \alpha \leq n, n \in N. \tag{13}$$

The Caputo's fractional differentiation is a linear operation, i.e.,

$$D^{\alpha}(\eta f(x) + \lambda g(x)) = \eta D^{\alpha} f(x) + \lambda D^{\alpha} g(x), \tag{14}$$

where η and λ are constants.

4. Modified gravitational search algorithm (PSOGSA)

In this section, we review gravitational search algorithm (GSA) and particle swarm optimization (PSO) in subsections (4.1) and (4.2) respectively, and a summary of hybrid population-based algorithm which combines PSO and GSA (PSOGSA) is presented in next subsection.

4.1. Gravitational Search Algorithm

Gravitational search algorithm (GSA) is a recent heuristic population-based method which has been introduced by Rashedi et al. (2009). This algorithm is based on the law of gravity (Newton (1729)).

GSA consists of a collection of agents that interact with each other through the gravity forces. The gravity forces cause a global movement, where each object moves toward other objects with heavier masses.

The position of m agents are initialized randomly. The position of i-th agent in D-dimensional searching space is defined by $X_i = (x_i^1, ..., x_i^d, ..., x_i^D)$ for $i = 1, ..., m$. The force from agent j on agent i is defined as (Rashedi et al (2009)):

$$F_{ij}^d(t) = G(t)\frac{M_{pi}(t)M_{aj}(t)}{R_{ij}(t)+\varepsilon}(x_j^d(t) - x_i^d(t)), \tag{15}$$

where M_{aj} and M_{pi} are active gravitational mass related to agent j, and the passive gravitational mass related to agent i respectively, ε is a small positive constant, and R_{ij} is the Euclidian distance between two agents i and j, and $G(t)$ is gravitational constant at time t, which is given by:

$$G(t) = G_0 \times e^{(-\alpha\ t/T)}, \tag{16}$$

where G_0 and α are initialized at the beginning of the search, t is the current iteration and T is the total number of iterations.

$F_i^d(t)$ is the total force acting on i^{th} agent in dimension d and calculated as:

$$F_i^d(t) = \sum_{j=1, j\neq i}^{N} rand_j F_{ij}^d(t), \tag{17}$$

where $rand_j$ is a random number in the interval $[0,1]$.

Acceleration of the agent i is defined as follows:

$$a_i^d(t) = \frac{F_i^d(t)}{M_{ii}(t)}, \tag{18}$$

where M_{ii} is the inertial mass of the i^{th} agent.

Velocity and position of each agent at the next iteration are computed by the following recursive relations:

$$V_i(t+1) = rand_i V_i(t) + a_i(t), \tag{19}$$

$$X_i(t+1) = X_i(t) + V_i(t+1), \tag{20}$$

where $rand_i$ is a random number in the interval $[0,1]$. The process of changing agent's positions will continue until meeting an end criterion.

4.2. Particle Swarm Optimization

Particle swarm optimization (PSO) is a population based stochastic optimization technique developed by Kennedy and Eberhart in 1995, inspired by social behavior of bird flocking or fish schooling (Eberhart and Kennedy (1995), Kennedy and Eberhart (1995), Kennedy and Eberhart (2001)).

In PSO, each single solution is a bird in the search space, which is called a particle. Each particle has a fitness value which is evaluated by the fitness function to be optimized and velocities which directed the flying of the particles.

In PSO, each particle will change its position according to its personal experience and the experiences of the whole society. Social sharing information between particles has a series of evolutionary advantages, a hypothesis which is the basis of PSO algorithm.

PSO is initialized with a group of random particles or solutions in search space. In every iteration, each particle needs its best fitness which it has achieved so far. This value is called Pbest. Also another best value is needed which is the best value, obtained so far by any particle in the population. This best value is the global best value and is called Gbest.

Suppose we have m particles and each particle is treated as a point in D-dimensional searching space. We will show the position, velocity and the best position of i-th particle in searching space respectively by:

$$X_i = (x_i^1, x_i^2, ..., x_i^D), \ V_i = (v_i^1, v_i^2, ..., v_i^D) \text{ and } P_i = (p_i^1, p_i^2, ..., p_i^D) \text{ for } i = 1, ..., m,$$

and the global position in searching space by $P_g = (p_g^1, p_g^2, ..., p_g^D)$.

The velocity and position of each particle are updated in each time step by the recursive relations:

$$V_i(t+1) = wV_i(t) + c_1 r_1 (P_i - X_i(t)) + c_2 r_2 (P_g - X_i(t)), \tag{21}$$

and

$$X_i(t+1) = X_i(t) + V_i(t+1), \tag{22}$$

where w is a weighting function and, c_1 and c_2 are learning factors, and the recommended choice for them is 2 (Kennedy and Eberhart (1995)) and r_1, r_2 are two random numbers in

$[0,1]$. The process of changing particle's position will continue until meeting an end criterion.

4.3. Modified gravitational search algorithm (PSOGSA)

PSOGSA is a hybrid population-based algorithm which combines particle swarm optimization (PSO) and gravitational search algorithm (GSA) (Mirjalili and Mohd Hashim (2010)). In order to balance the ability of exploitation and exploration to find global optimum, PSOGSA uses the ability of social thinking (Gbest) in PSO and local search ability of GSA. In order to combine these algorithms the velocity of each agent in GSA is updated by the following relation (Mirjalili and Mohd Hashim (2010)):

$$V_i(t+1) = wV_i(t) + c_1' r a_i(t) + c_2' r(P_g - X_i(t)), \tag{23}$$

where c_1' and c_2' are weighting factors, w is a weighting function, r is a random number in $[0,1]$ and P_g is the best solution which has been obtained so far.

5. Hybrid pseudospectral method and PSOGSA for nonlinear FDEs

In this section, the implementation of pseudospectral method for solving nonlinear FDE (1) combined with (2) or (3) is presented.

The spectral methods for solving this class of equations is based on the expansion of the solution u for (1) and (2) as a finite sum in terms of smooth basis functions in the form:

$$\tilde{u}(x) \approx \sum_{i=0}^{N} a_i \phi_i(x),$$

in which $\{\phi_i\}_i$ represents a family of orthonormal polynomials on $[a,b]$. In this paper, we consider the first kind Chebyshev polynomials on $[-1,1]$.

Now we should compute the coefficients of the series:

$$\tilde{u}(x) \approx \sum_{i=0}^{N} a_i T_i(x), \tag{24}$$

where $\{T_i\}_{i=0}^{N}$ are Chebyshev polynomials as mentioned in section 2. By substituting (24) in (1) and its initial value conditions we define the residual function:

$$F(a_0, a_1, \ldots, a_N, x) = D^{\alpha} \sum_{i=0}^{N} a_i T_i(x) - f\left(x, \sum_{i=0}^{N} a_i T_i(x), D^{\beta_1} \sum_{i=0}^{N} a_i T_i(x), \ldots, D^{\beta_k} \sum_{i=0}^{N} a_i T_i(x)\right), \tag{25}$$

and the equations:

$$D^{j} \sum_{i=0}^{N} a_i T_i(a) = d_j, \qquad j = 0, \ldots, n. \tag{26}$$

In standard pseudospectral method, by considering the residual function and the initial conditions, and choosing $\{x_k\}_{k=1}^{M}$ as a set of collocation points, we obtain a nonlinear system with $N+1$ equations and $N+1$ unknown parameters (Hosseini (2007), Hosseini (2006)). Since solving a nonlinear system is facing many problems including the choice of a suitable starting point, we present a nonlinear unconstrained optimization problem for finding the coefficients of the Chebyshev expansion.

For $\{x_k\}_{k=1}^{M}$ as a set of collocation points, we define the general residual function by:

$$V=\sqrt{\frac{1}{M}\sum_{k=1}^{M}F^2(a_0,a_1,...,a_N,x_k)+\sum_{j=0}^{n}(D^j\sum_{i=0}^{N}a_iT_i(a)-d_j)^2}. \qquad (27)$$

And, according to (27), we define the nonlinear unconstrained optimization problem:

$$\begin{aligned}&\min\ V,\\ &s.t\ \ a_i\in R\end{aligned}. \qquad (28)$$

This optimization problem is solved by using PSOGSA algorithm, and the appropriate coefficients for the Chebyshev series are found.

We can use the same technique to solve fractional boundary value problems.

6. Numerical results

In this section, we present some interesting examples and use the proposed method in section 5 to solve them.

In our study, we choose an initial population for PSOGSA with 30 agents, where each agent is a random number for the coefficients in Chebyshev series, and also, we set $c_1'=0.5,\ c_2'=2, G_0=1,\ \alpha=20$ and w is a random number in $[0,1]$. It should be noted that N is the number of basis functions and M is the number of collocation points. We suppose $M=50$ for all examples, and also for all examples, we test the proposed method 20 times.

We define absolute error as

$$e(t)=|u(t)-\tilde{u}(t)|,$$

where $u(t)$ is the exact solution and $\tilde{u}(t)$ is the approximate solution.

Example 6.1.

Consider the following nonlinear initial FDE (Li (2010)):

$$\begin{aligned}&aD^2u+bD^{\alpha_2}u+cD^{\alpha_1}u+eu^3=\frac{2a}{\Gamma(2)}t+\frac{2b}{\Gamma(4-\alpha_2)}t^{3-\alpha_2}\\ &+\frac{2c}{\Gamma(4-\alpha_1)}t^{3-\alpha_1}+e(\frac{1}{3}t^3)^3,\ t\in(0,1).\end{aligned} \qquad (29)$$

For this problem, we should have $0 < \alpha_1 \leq 1$ and $1 < \alpha_2 < 2$, and the initial conditions are:

$$u(0) = u'(0) = 0, \tag{30}$$

with the exact solution

$$u(t) = \frac{1}{3}t^3. \tag{31}$$

We suppose $a = b = c = e = 1, \alpha_2 = 1.234, \alpha_1 = 0.333$ for solving this problem. Absolute error of mean and the best solution for different number of basis functions which have been obtained by the proposed method and Chebyshev wavelet method (Li (2010)) at given points and for different number of N are given in table 1.

Table 1. Results of absolute errors for Example 1

t	Absolute errors of mean solution value of the proposed method	Absolute errors of best solution value of the proposed method	Absolute errors of (Li (2010))		
	N=3	N=3	N=24	N=96	N=384
0.1	5.63×10^{-12}	1.42×10^{-13}	8.19×10^{-5}	5.25×10^{-6}	3.26×10^{-7}
0.2	2.13×10^{-13}	8.67×10^{-14}	2.05×10^{-4}	1.26×10^{-5}	7.92×10^{-7}
0.3	9.43×10^{-14}	3.89×10^{-14}	2.95×10^{-4}	1.85×10^{-5}	1.15×10^{-6}
0.4	8.74×10^{-15}	2.56×10^{-15}	3.05×10^{-4}	1.89×10^{-5}	1.18×10^{-6}
0.5	7.82×10^{-14}	1.96×10^{-14}	5.08×10^{-4}	3.17×10^{-5}	1.98×10^{-6}
0.6	9.59×10^{-14}	2.33×10^{-14}	4.29×10^{-4}	2.69×10^{-5}	1.68×10^{-6}
0.7	2.84×10^{-14}	5.27×10^{-15}	6.38×10^{-4}	3.97×10^{-5}	2.48×10^{-6}
0.8	1.68×10^{-13}	3.79×10^{-14}	7.11×10^{-4}	4.45×10^{-5}	2.78×10^{-6}
0.9	8.73×10^{-13}	1.10×10^{-13}	6.02×10^{-4}	3.74×10^{-5}	2.34×10^{-6}

Example 6.2.

Consider the following nonlinear fractional BVP (Li et al. (2016)):

$$D^{\alpha}u = \frac{\Gamma(6)}{\Gamma(6-\alpha)}t^{5-\alpha} + \frac{36}{\Gamma(5-\alpha)}t^{4-\alpha} - \Gamma(3+\alpha)t^2$$

$$-u^2 + (t^5 + \frac{3}{4}t^4 - 2t^{2+\alpha})^2, \qquad t \in (0,1). \tag{32}$$

Initial conditions for $1 < \alpha < 2$ are

$$u(0) = u'(0) = 0. \tag{33}$$

The exact solution of this problem is:

$$u(t) = t^5 + \frac{3}{4}t^4 - 2t^{2+\alpha}. \tag{34}$$

We suppose $\alpha = 1.25$. Absolute error of mean and the best solution for different number of basis functions which have been obtained by the proposed method and the absolute error obtained by methods in (Li et al. (2016)) are shown in table 2.

Table 2. Results of absolute errors at $t = 1$ for example 2

N	Absolute errors of mean solution value of the proposed method	Absolute errors of best solution value of the proposed method	N	Absolute errors of rectangular scheme (Li et al. (2016))	Absolute errors of trapezoidal scheme (Li et al. (2016))
4	2.29×10^{-2}	5.76×10^{-3}	80	4.99×10^{-2}	8.96×10^{-4}
7	5.29×10^{-5}	9.64×10^{-6}	160	2.53×10^{-2}	2.25×10^{-4}
10	5.31×10^{-6}	9.58×10^{-7}	320	1.28×10^{-2}	5.65×10^{-5}
12	9.51×10^{-7}	3.59×10^{-7}	640	6.41×10^{-3}	1.41×10^{-5}
			1280	3.21×10^{-3}	3.54×10^{-6}
			2560	1.61×10^{-3}	8.85×10^{-7}

The graphs of the absolute error of the best solution, which is obtained by the proposed method and the absolute error of standard pseudospectral method for $N = 12$ are shown in Figure 1.

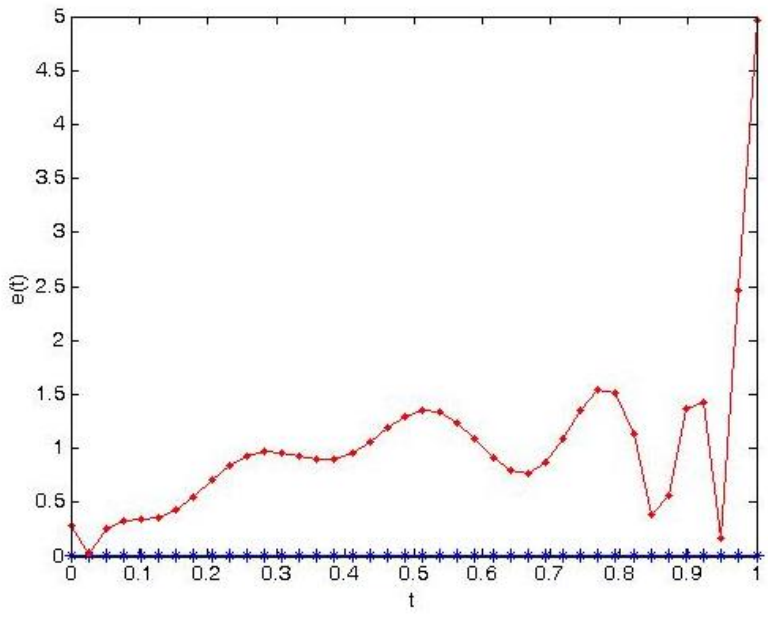

Figure 1. Absolute error of the best solution by the proposed method (-*) and standard pseudospectral method (.-) for $N = 12$, in example 2.

Example 6.3.

Consider the following nonlinear fractional BVP (Jia et al. (2016)):

$$D^{2.5}u + tu^2 = \frac{12\sqrt{t}}{\sqrt{\pi}} + t^7, \qquad t \in [0,1], \tag{35}$$

$$u(0) = u'(0) = 0, u(1) = 1. \tag{36}$$

We solve this example by the proposed method with $N = 3$. The problem is solved by the proposed method and we reach $u(t) = t^3$ as the exact solution for the above fractional BVP.

Example 6.4.

Consider the nonlinear boundary FDE (Jia et al. (2016)):

$$D^{1.5}u - u^3 = \frac{\Gamma(2.9)}{\Gamma(1.4)} t^{0.4} - (t^{1.9} - 1)^3, \qquad t \in [0,1], \tag{37}$$

$$u(0) = -1, u(1) = 0. \tag{38}$$

The exact solution is

$$u(t) = t^{1.9} - 1. \tag{39}$$

Absolute error of mean and the best solution for different number of basis functions which have been obtained by the proposed method and the absolute error obtained by standard pseudospectral method are shown in table 3.

Table 3. Results of absolute errors for example 4

N	Absolute errors of mean solution value of the proposed method	Absolute errors of best solution value of the proposed method	Absolute errors of standard pseudospectral method
5	6.52×10^{-3}	2.54×10^{-4}	12.97
10	5.06×10^{-5}	8.92×10^{-6}	8.67
15	9.08×10^{-6}	2.34×10^{-6}	4.20×10^{2}
20	9.38×10^{-7}	7.21×10^{-7}	1.42×10^{2}

Example 6.5.

Consider the following nonlinear FDE (Saeed (2017)):

$$D^{\alpha}u + u' + u + u^2u' = 2\cos(t) - \cos^3(t), \qquad 1 < \alpha \le 2, \tag{40}$$

$$u(0) = 0, \quad u'(0) = 1. \tag{41}$$

The exact solution of above problem when $\alpha = 2$, is

$$u(t) = \sin(t). \tag{42}$$

The exact solution of the problem and the approximate solutions for different values of α for $N = 7$ are shown in Figure 2.

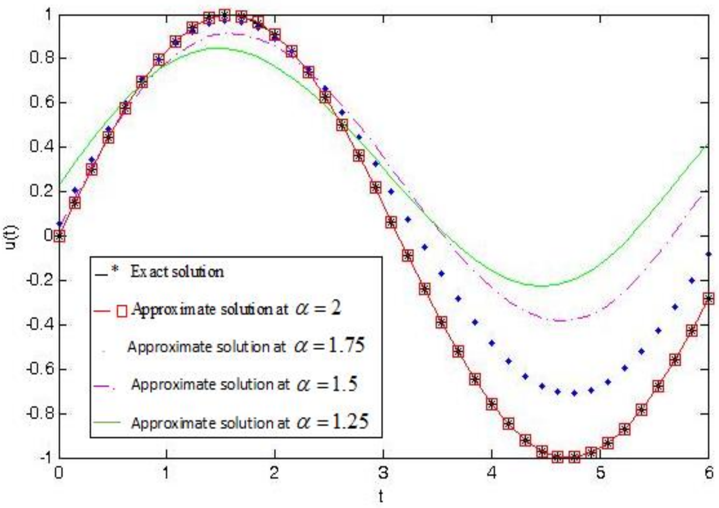

Figure 2. The exact solution of the example 5 at $\alpha = 2$ and the approximate solutions for different values of α.

As shown in figure 2, by increasing the values of α, approximate solutions converge to the exact solution at $\alpha = 2$.

The graphs of the absolute error of the best solution which is obtained by the proposed method and absolute error of standard pseudospectral method for $N = 7$ are shown in figure 3.

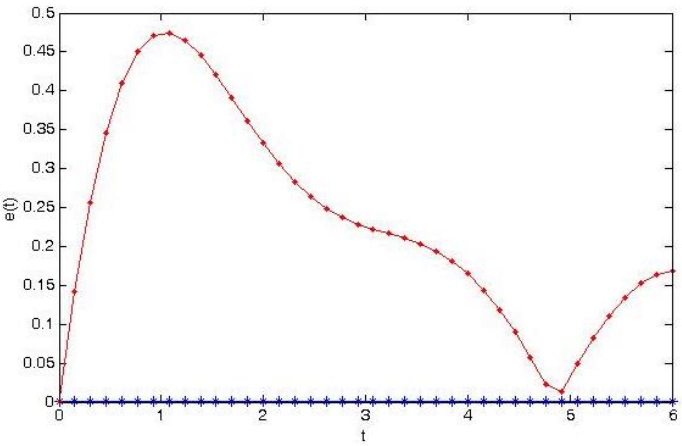

Figure 3. Absolute error of the best solution by the proposed method (-*) and standard pseudospectral method (.-) for $N = 7$, in example 5.

6. Conclusions

In this paper, we utilized pseudospectral method through the use of Chebyshev polynomials in solving nonlinear FDEs. Although, it appears that numerical solutions of nonlinear ordinary differential equations by spectral methods based on Chebyshev polynomials is arduous, as we must deal with nonlinear systems. To eliminate and overcome this difficulty, we define general residual function of the nonlinear FDE, and then, apply an appropriate unconstrained optimization model to the problem. Moreover, to solve this optimization problem we use PSOGSA algorithm. The novelty of this paper is to provide the possibility of achieving spectral accuracy in solving nonlinear differential equations. Using our method, one can easily solve initial and boundary value fractional problems. Finally, the numerical results of the above problems illustrate the high accuracy and efficiency of our proposed method.

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