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On (Semi)Topological BCC-algebras

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Abstract

In this paper, we introduce the notion of (semi)topological BCC-algebras and derive here conditions that imply a BCC-algebra to be a (semi)topological BCC-algebra. We prove that for each cardinal number α there is at least a (semi)topological BCC-algebra of order α. Also we study separation axioms on (semi)topological BCC-algebras and show that for any infinite cardinal number α there is a Hausdorff (semi)topological BCC-algebra of order α with nontrivial topology.

Keywords: (Semi)topological BCC-algebra; Ideal; Preideal; Hausdorff Space; Uryshon Space

MSC 2010 No.: 06F35, 22A26

1. Introduction

Imai and Iséki (1966) introduced a class of algebras of type $(2, 0)$ called BCK-algebras which generalizes on one hand the notion of algebra of sets with the set subtraction as the only fundamental non-nullary operation and on the other hand the notion of impliction algebra. Iséki posed an interesting problem whether the class of BCK-algebras form a variety. In connection with this problem, Komori (1983) introduced a notion of BCC-algebras which is a generalization of notion BCK-algebras and proved that the class of all BCC-algebras is not a variety. Dudek (1990) redefined the notion of BCC-algebras by using a dual form of the ordinary definition. Further study of BCC-algebras was continued by Dudek (1995, 1998, 1992, 1990). In recent years some mathematicians have endowed algebraic structures associated with logical systems with a topology and have studied some their propertises. For example, Borzooei et al. (2011, 2012) introduced (semi)topological BL-algebras and studied metrizability and separation axioms on them. Kouhestani and Borzooei (2014) introduced (semi)topological residuated lattices and studied separation axioms T_0, T_1, and T_2 on them. In Section 3 of this paper, we will define (left, right, semi)topological BCC-algebras and show that for each cardinal number α there is at least a topological BCC-algebra of order α. In Section 4, we study some topological results on BCC-algebras endowed with a topology. In Section 5, we will study the connection between (semi)topological BCC-algebras and T_i spaces, when $i = 0, 1, 2$. We prove that for any infinite cardinal number α there is a Hausdorff topological BCC-algebra of order α in which its topology is nontrivial.

2. Preliminary

In this section we present some of the basic information and notations that will be used in the text. Topological concepts used in this paper are from Bourbaki (1966) and BCC-algebras theory can be found in Dudek (1990, 1992, 1995, 1998, 1999, 2000).

Topological Space

Recall that a set A with a family $\mathcal{U}$ of its subsets is called a *topological space*, denoted by $(A,\mathcal{U})$, if $A, \emptyset \in \mathcal{U}$, the intersection of any finite numbers of members of $\mathcal{U}$ is in $\mathcal{U}$ and the arbitrary union of members of $\mathcal{U}$ is in $\mathcal{U}$. The members of $\mathcal{U}$ are called *open sets* of A and the complement of $U \in \mathcal{U}$, that is $A \setminus U$, is said to be a *closed set*. If B is a subset of A, the smallest closed set containing B is called the *closure* of B and denoted by $\overline{B}$ (or $cl_u B$). A subfamily $\{U_\alpha : \alpha \in I\}$ of $\mathcal{U}$ is said to be a *base* of $\mathcal{U}$ if for each $x \in U \in \mathcal{U}$, there exists an $\alpha \in I$ such that $x \in U_\alpha \subseteq U$, or equivalently, each U in $\mathcal{U}$ is the union of members of $\{U_\alpha\}$. A subset P of A is said to be a *neighborhood* of $x \in A$ if there exists an open set U such that $x \in U \subseteq P$. Let $\mathcal{U}_x$ denote the totality of all neighborhoods of x in A. Then a subfamily $\mathcal{V}_x$ of $\mathcal{U}_x$ is said to form a *fundamental system* of neighborhoods of x if for each U_x in $\mathcal{U}_x$ there exists a V_x in $\mathcal{V}_x$ such that $V_x \subseteq U_x$. A *directed set* I is a partially ordered set such that, for any i and j of I, there is a $k \in I$ with $k \geq i$ and $k \geq j$. If I is a directed set, then the subset $\{x_i : i \in I\}$ of A is called a *net*. A net $\{x_i : i \in I\}$ *converges* to $x \in A$ if, for each neighborhood U of x, there exists a $j \in I$ such that for all $i \geq j$, $x_i \in U$. If $B \subseteq A$ and $x \in \overline{B}$, then there is a net in B that converges to x.

A topological space $(A,\mathcal{U})$ is said to be a:
(1) T_0-space if for each $x \neq y \in A$, there is at least one in an open neighborhood excluding the

other.
(2) T_1-space if for each $x \neq y \in A$, each has an open neighborhood not containing the other.
(3) *Hausdorff* space if for each $x \neq y \in A$, there two disjoint open neighborhoods U, V of x and y, respectively.
(4) *Uryshon* space if for each $x \neq y \in A$, there are two open neighborhoods U, V of x and y, respectively, such that $\overline{U} \cap \overline{V} = \phi$.

BCC-Algebra

A BCC-algebra is a nonempty set X with a constant 0 and a binary operation $*$ satisfying the following axioms, for all $x, y, z \in X$:
(B1) $((x * y) * (z * y)) * (x * z) = 0$.
(B2) $0 * x = 0$.
(B3) $x * 0 = x$.
(B4) $x * y = 0$ and $y * x = 0$ imply $x = y$.

On any BCC-algebra X one define $x \leq y \Leftrightarrow x * y = 0$. It is not difficult to verify that this order is partial and 0 is its smallest element.

In BCC-algebra X, the following hold: for any $x, y, z \in X$,

(B5) $(x * y) * (z * y) \leq x * z$.
(B6) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$.
(B7) $(x * y) * z \leq x * z$.
(B8) $x * y \leq x$.
(B9) $(x * y) * z \leq x * (y * z)$.
(B10) $(x * y) * (x * z) \leq x * (x * z)$.
(B11) $(x * y) * z \leq (x * y) * (z * y)$.

Definition 2.1.

Let $(X, *, 0)$ be a BCC-algebra and $I \subseteq X$. I is called:

(1) *ideal* if $0 \in I$, and for each $x, y \in X$, $x * y \in I$ and $y \in I$ imply $x \in I$,
(2) *BCC-ideal* if $0 \in I$, and $y \in I$, $(x * y) * z \in I$, imply $x * z \in I$.

Let $(X, *, 0)$ be a BCC-algebra and I a subset of X. Then:
(1) if I is an ideal, then I is a subalgebra and if $x \in I$ and $y \leq x$, then $y \in I$,
(2) if I is a BCC-ideal, then it is an ideal,
(3) I is a BCC-ideal if and only if for each $x, y \in I$, and $z \in X$, $x * z$ and $z * ((z * x) * y)$, both, are in I, and
(4) if I is a BCC-ideal, then the following relation

$$x \equiv^I y \Leftrightarrow x * y \in I,\ y * x \in I$$

is a congruence relation on X, i.e. $\equiv^I$ is an equivalence relation and for each $a, b, x, y \in X$, if $x \equiv^I y$ and $a \equiv^I b$, then $a * x \equiv^I b * y$. For each $x \in X$, we consider $x/I = \{y : y \equiv^I x\}$ and $X/I = \{x/I : x \in X\}$.

3. Topological BCC-algebras

Definition 3.1.

Let $\mathcal{T}$ be a topology on a BCC-algebra $(X, *, 0)$. Then:

(1) $(X, *, \mathcal{T})$ is a *(right) left topological* BCC-algebra if $x * y \in U \in \mathcal{T}$, then there is a (V) $W \in \mathcal{T}$ such that $(x \in V)$ $y \in W$ and $(V * y \subseteq U)$ $x * W \subseteq U$. In this case, we also say that $*$ is *continous in (first) second variable*,

(2) $(X, *, \mathcal{T})$ is a *semitopological* BCC-algebra if it is a left and right topological BCC-algebra, i.e. if $x * y \in U \in \mathcal{T}$, then there are $V, W \in \mathcal{T}$ such that $x \in V, y \in W$ and $x * W \subseteq U$ and $V * y \subseteq U$. In this case we also say that $*$ is continuous in each variable separately,

(3) $(X, *, \mathcal{T})$ is a *topological* BCC-algebra if $*$ is continuous , i.e. if $x * y \subseteq U \in \mathcal{T}$, then there are two neighborhoods V, W of x, y, respectively, such that $V * W \subseteq U$.

Example 3.2.

(i) Let $X = \{0, 1, 2, 3\}$ be a BCC-algebra with the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	0
3	3	2	2	0.

Then, $\mathcal{T} = \{\{0, 1, 2\}, \{3\}, X, \phi\}$ is a topology on X such that $(X, *, \mathcal{T})$ is a right topological BCC-algebra that is not a topological BCC-algebra.

(ii) Let $X = [0, 1]$. Then X with the following operation is a BCC-algebra.

$$x * y = \begin{cases} 0, & \text{if } x \leq y \\ x, & \text{if } x > y. \end{cases}$$

It is easy to prove that $B = \{\{x\} : 0 \neq x\} \cup \{X\}$ is a subbase for a topology $\mathcal{T}$ and $(X, *, \mathcal{T})$ is a topological BCC-algebra.

Let $(X, *, 0)$ be a BCC-algebra. Then:
(1) a family Ω of subsets X is *prefilter* if for each $U, V \in \Omega$, there exists a $W \in \Omega$ such that $W \subseteq U \cap V$,
(2) for each $V \subseteq X$ and $x \in X$, we denote

$$V[x] = \{y \in X : y * x \in V\} \quad V(x) = \{y \in X : y * x, x * y \in V\}.$$

Theorem 3.3.

Let $\mathcal{I}$ be a prefilter of BCC-ideals in a BCC-algebra $(X, *, 0)$. Then there is a topology $\mathcal{T}$ on X such that $(X, *, \mathcal{T})$ is a topological BCC-algebra.

Proof:

Define $\mathcal{T} = \{U \subseteq X : \forall x \in U\ \exists I \in \mathcal{I}\ s.t\ x/I \subseteq U\}$. For each $x \in A$ and $I \in \mathcal{I}$, the set $x/I \in \mathcal{T}$ because if y is an arbitrary element of x/I, then $y/I \subseteq x/I$. It is easy to see that $\mathcal{T}$ is a topology on X. We prove that $*$ is continuous. For this, suppose $x * y \in U \in \mathcal{T}$, then for some $I \in \mathcal{I}$, $(x * y)/I \subseteq U$. Now x/I and y/I are two open neighborhoods of x and y, respectively, such that $x/I * y/I \subseteq (x * y)/I \subseteq U$. ■

Theorem 3.4.

Let I be an ideal in BCC-algebra $(X, *, 0)$. Then there is a topology $\mathcal{T}$ on X such that $(X, *, 0, \mathcal{T})$ is a right topological BCC-algebra. Moreover, if for each $x, y, z \in X$, $(x * y) * z = (x * z) * y$, then $(X, *, 0, \mathcal{T})$ is a topological BCC-algebra.

Proof:

Let $\mathcal{T} = \{U \subseteq X : \forall x \in U\ I[x] \subseteq U\}$. First we show that for any $x \in X$, $I[x] \in \mathcal{T}$. Suppose $x \in X$ and $y \in I[x]$. Then, $y * x \in I$. Take $z \in I[y]$. By (B1) $((z * x) * (y * x)) * (z * y) = 0 \in I$. Since I is ideal and $z * y$ and $y * x$ both are in I, $z * x$ is in I so. Hence, $I[y] \subseteq I[x]$. This implies that $I[x] \in \mathcal{T}$. Now we prove that $*$ is continuous in first variable. Let $x * y \in U \in \mathcal{T}$. If $z \in I[x]$, then $z * x \in I$. By (B5), $(z * y) * (x * y) \leq z * x$, hence $z * y \in I[x * y]$. This proves that $I[x] * y \subseteq I[x * y] \subseteq U$. Finally, let for each $x, y, z \in X$, $(x * y) * z = (x * z) * y$. We prove that $I[x] * I[y] \subseteq I[x * y]$, for each $x, y \in X$. For this, suppose $x, y \in X$ and $a \in I[x]$ and $b \in I[y]$.

Then,

$$[(b * x) * (x * y)] * (b * y) = [(b * x) * (b * y)] * (x * y) \leq (x * y) * (x * y) = 0.$$

Hence, $(b * x) * (x * y) \leq b * y$. Since $b * y \in I$, we conclude that $(b * x) * (x * y) \in I$. On the other hand, we have

$$[(a * b) * (x * y)] * (a * x) = [(a * b) * (a * x)] * (x * y) \leq (b * x) * (x * y).$$

This implies that $[(a * b) * (x * y)] * (a * x) \in I$. Since $a * x \in I$, $(a * b) * (x * y) \in I$. Hence, $a * b \in I[x * y]$ and so $I[x] * I[y] \subseteq I[x * y]$. This proves that $(X, *, 0, \mathcal{T})$ is a topological BCC-algebra. ■

Theorem 3.5.

Let $(X, *, 0, \mathcal{T})$ be a topological BCC-algebra and $a \notin X$. Suppose $X_a = X \cup \{a\}$ and $\mathcal{T}^* = \mathcal{T} \setminus \{\phi\}$. If $0 \in \cap \mathcal{T}^*$, then there are an operation $\otimes$ and a topology $\mathcal{T}_a$ on X_a such that $(X_a, \otimes, \mathcal{T}_a)$ is a topological BCC-algebra and $0 \in \cap \mathcal{T}_a^*$.

Proof:

Define the operation $\otimes$ on X_a by

$$x \otimes y = \begin{cases} x * y, & \text{if } x, y \in X \\ a, & \text{if } x = a, y \in X \\ 0, & \text{if } x \in X, y = a \\ 0, & \text{if } x = y = a. \end{cases}$$

Assume that $\mathcal{T}_a = \{U \cup \{a\} : U \in \mathcal{T}\} \cup \{\phi\}$. It is easy to verify that $(X_a, \otimes, 0)$ is a BCC-algebra and $\mathcal{T}_a$ is a topology on X_a. Let $x \otimes y \in U \cup \{a\}$. In the following cases we find two sets $V, W \in \mathcal{T}_a$ such that $x \in V$, $y \in W$ and $V \otimes W \subseteq U \cup \{a\}$.

Case 1. If $x, y \in X$, then $x * y = x \otimes y \in U$. Since $*$ is continuous, there are $V, W \in \mathcal{T}$ such that $x \in V$, $y \in W$ and $V * W \subseteq U$. If $z_1 \in V \cup \{a\}$ and $z_2 \in W \cup \{a\}$, then $z_1 \otimes z_2 \in \{z_1 * z_2, a, 0\} \subseteq U \cup \{a\}$. Hence, $V \cup \{a\} \otimes W \cup \{a\} \subseteq U \cup \{a\}$.

Case 2. If $x = a$ and $y \in X$, then $x = a \in \{a\} \in \mathcal{T}_a$, $y \in X_a \in \mathcal{T}_a$ and $\{a\} \otimes X_a = \{0, a\} \subseteq U \cup \{a\}$.

Case 3. If $x \in X$ and $y = a$, then $x \in X_a \in \mathcal{T}_a$, $y = a \in \{a\} \in \mathcal{T}_a$ and $X_a \otimes \{a\} = \{0\} \subseteq U \cup \{a\}$.

Case 4. If $x = y = a$, then $x = y = a \in \{a\} \in \mathcal{T}_a$ and $\{a\} \otimes \{a\} = \{0\} \subseteq U \cup \{a\}$.

Cases $1, 2, 3$ and 4 prove that $(X_a, \otimes, \mathcal{T}_a)$ is a topological BCC-algebra. But it is obvious that $0 \in \cap \mathcal{T}_a^*$. ■

Theorem 3.6.

For any integer $n \geq 4$ there exists a topological BCC-algebra of order n.

Proof:

We prove the theorem by using mathematical induction. If $n = 4$, then the set $X = \{0, 1, 2, 3\}$ is a BCC-algebra by the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	1
3	3	3	3	0.

If $\mathcal{T} = \{\{0, 1, 2\}, \{0, 1, 3\}, \{0, 1\}, X, \phi\}$, then $\mathcal{T}$ is a topology on X such that $(X, \mathcal{T})$ is a topological BCC-algebra and $0 \in \cap \mathcal{T}^*$. Take $(X, *, \mathcal{T})$ a topological BCC-algebra of order n such that $0 \in \cap \mathcal{T}^*$. Let $a \notin X$ and $X_a = X \cup \{a\}$. Then, by Theorem 3.5, there are the operation $\otimes$ and topology $\mathcal{T}_a$ on X_a such that $(X_a, \otimes, \mathcal{T}_a)$ is a topological BCC-algebra of order $n + 1$. ■

Theorem 3.7.

Let α be an infinite cardinal number. Then there is a topological BCC-algebra of order α.

Proof:

Let X be a set with cardinal number α. Consider $X_0 = \{x_0 = 0, x_1, x_2, ...\}$ as a countable subset of X and define the operation $*$ on X_0 by

$$x_i * x_j = \begin{cases} 0, & \text{if } i = j \\ x_i, & \text{if } i \neq j. \end{cases}$$

Then, $(X_0, *, 0)$ is a BCC-algebra. The set $I_n = \{0, x_1, ..., x_n\}$, for any $n \geq 1$ is a BCC-ideal of X_0. Since $B_0 = \{I_n : n \geq 1\}$ is a prefilter of BCC-ideals in X_0, by Theorem 3.3, there is a nontrivial topology $\mathcal{T}_0$ on X_0 such that $(X_0, *, \mathcal{T}_0)$ is a topological BCC-algebra. Now define the binary operation $\circ$ on X by

$$x \circ y = \begin{cases} x * y, & \text{if } x, y \in X_0 \\ 0, & \text{if } x \in X_0, y \notin X_0 \\ x, & \text{if } x \notin X_0, y \in X_0 \\ 0, & \text{if } x = y \notin X_0 \\ x, & \text{if } x \neq y, x, y \notin X_0. \end{cases}$$

It is routine to check that $(X, \circ, 0)$ is a BCC-algebra of order α and the set $B = \mathcal{T}_0 \cup \{\{x\} : x \notin X_0\}$ is a subbase for a topology $\mathcal{T}$ on X. Since $\{0\} \notin \mathcal{T}$, $\mathcal{T}$ is a nontrivial topology on X. In the following cases we will show that $(X, \circ, \mathcal{T})$ is a topological BCC-algebra. For this, let $x \circ y \in U \in B$.

Case 1. If $x, y \in X_0$, then $x \circ y = x * y \in U \in \mathcal{T}_0$. Since $*$ is continuous in $(X_0, \mathcal{T}_0)$, there are $V, W \in \mathcal{T}_0$ containing x, y, respectively, such that $V * W \subseteq U$. Hence $V \circ W \subseteq U$ which implies that $\circ$ is continuous in $(X, \mathcal{T})$.

Case 2. If $x \in X_0$ and $y \notin X_0$, then X_0 and $\{y\}$ are two elements of $\mathcal{T}$ such that $x \in X_0$, $y \in \{y\}$ and $X_0 \circ \{y\} = \{0\} \subseteq U$.

Case 3. If $x \notin X_0$ and $y \in X_0$, then $x \circ y = x \in U$. Now $\{x\}$ and X_0 both belong to $\mathcal{T}$ and $x \in \{x\}$, $y \in X_0$ and $\{x\} \circ X_0 = \{x\} \subseteq U$.

Case 4. If $x = y \notin X_0$, then $x \circ y = 0 \in U$. Then $\{x\}$ is an open set in $\mathcal{T}$ which contains x, y and $\{x\} \circ \{x\} = \{0\} \subseteq U$.

Case 5. If $x \notin X_0$ and $y \notin X_0$, then $x \in \{x\} \in \mathcal{T}$ and $y \in \{y\} \in \mathcal{T}$ and $\{x\} \circ \{y\} \subseteq U$.

Cases $1, 2, 3, 4$, and 5 show that the operation $\circ$ is continuous in $(X, \mathcal{T})$. ■

Theorem 3.8.

Let $(X, *, 0, \mathcal{T})$ be a topological BCC-algebra and α be a cardinal number. If $\alpha \geq |X|$, then there is a topological BCC-algebra $(Y, \circ, 0, \mathcal{U})$ such that $\alpha \leq |Y|$ and X is a subalgebra of Y.

Proof:

Suppose

$$\Gamma = \{(H, \circledast, 0, \mathcal{U}) : (H, \circledast, 0, \mathcal{U}) \text{ is a topological BCC-algebra, } X \subseteq H \ \ \circledast|_X = *\}.$$

The following relation is a partial order on Γ.

$$(H, \circledast, 0, \mathcal{U}) \leq (K, \odot, 0, \mathcal{V}) \Leftrightarrow H \subseteq K, \ \odot|_H = \circledast, \ \mathcal{U} \subseteq \mathcal{V}.$$

Let $\{(H_i, \circledast_i, 0, \mathcal{U}_i) : i \in I\}$ be a chain in Γ. Put $H = \cup H_i$ and $\mathcal{U} = \cup \mathcal{U}_i$. If x and y are two elements of H, then for some $i \in I$, $x, y \in H_i$. Define $x \circledast y = x \circledast_i y$. We prove that $\circledast$ is an operation on H. Suppose $x, y \in H_i \cap H_j$. Since $\{(H_i, \circledast_i, 0, \mathcal{U}_i) : i \in I\}$ is a chain, $H_i \subseteq H_j$ or $H_j \subseteq H_i$. Without the lost of generality, assume that $H_i \subseteq H_j$. Then, $\circledast_j|_{H_i} = \circledast_i$. So $x \circledast_j y = x \circledast_i y$. This proves that $\circledast$ is an operation on H. Now it is easy to see that $(H, \circledast, 0)$

is a BCC-algebra such that $\circledast|_X = *$. On the other hand, since $\{(H_i, \circledast_i, 0, \mathcal{U}_i) : i \in I\}$ is a chain, $\mathcal{U}$ is a topology on H. We prove that $(H, \circledast, \mathcal{U})$ is a topological BCC-algebra. Let $x \circledast y \in U \in \mathcal{U}$. Then there is an $i \in I$ such that $x \circledast y = x \circledast_i y \in U \in \mathcal{U}_i$. Since $\circledast_i$ is continuous in $(H_i, \mathcal{U}_i)$, there are $V, W \in \mathcal{U}_i$ such that $x \in V$, $y \in W$, and $V \circledast_i W \subseteq U$. This proves that $\circledast$ is continuous in $(H, \mathcal{U})$. Thus, $(H, \circledast, 0, \mathcal{U})$ is an upper bound for $\{(H_i, \circledast_i, 0, \mathcal{U}_i) : i \in I\}$ in Γ. By Zorn's Lemma, Γ has a maximal element. Suppose $(Y, \circ, 0, \mathcal{U})$ is a maximal element of Γ. We prove that $|Y| \geq \alpha$. If $|Y| < \alpha$, then for some nonempty set C, $|Y \cup C| = \alpha$. Take $a \in Y \setminus C$ and put $H = Y \cup \{a\}$. Then it is easy to claim that H with the following operation is a BCC-algebra.

$$x \circledast y = \begin{cases} x \circ y, & \text{if } x, y \in Y \\ 0, & \text{if } x \in Y, y = a \\ a, & \text{if } x = a, y \in Y \\ 0, & \text{if } x = y = a. \end{cases}$$

The set $B = \mathcal{U} \cup \{\{a\}\}$ is a subbase for a topology $\mathcal{V}$ on H. In the following cases we prove that $(H, \circledast, \mathcal{V})$ is a topological BCC-algebra. Let $x, y \in H$ and $x \circledast y \in U \in B$.

Case 1. If $U \in \mathcal{U}$, then or x, y, both, are in Y or $x \in Y$ and $y = a$ or $x = y = a$. If $x, y \in Y$, then since $\circ$ is continuous in $(Y, \mathcal{U})$, there are $V, W \in \mathcal{U} \subseteq B$ such that $x \in V$, $y \in W$ and $V \circledast W = V \circ W \subseteq U$. If $x \in Y$ and $y = a$, then Y and $\{a\}$ are two open sets in $\mathcal{V}$ containing x and y, respectively, such that $x \circledast y \in Y \circledast \{a\} = \{0\} \subseteq U$. If $x = y = a$, then $\{a\}$ is an open neighborhood of x, y in $\mathcal{V}$ such that $\{a\} \circledast \{a\} = \{0\} \subseteq U$.

Case 2. If $U = \{a\}$, then $x = a \in \{a\} \in B$ and $y \in Y \in B$ and $x \circledast y \in \{a\} \circledast Y \subseteq U$.

Thus by Cases 1 and 2, $(H, \circledast, \mathcal{V})$ is a topological BCC-algebra. But $(H, \circledast, \mathcal{V})$ is a member of Γ which $(Y, \circ, 0, \mathcal{U}) < (H, \circledast, 0, \mathcal{V})$, a contradiction. Therefore, $|Y| \geq \alpha$ and X is a subalgebra of Y. ■

Theorem 3.9.

Let α be an infinite cardinal number. Then there is a right topological BCC-algebra of order α which is not a topological BCC-algebra.

Proof:

Let X be a set with cardinal number α. Suppose $X_0 = \{x_0 = 0, x_1, x_2, ...\}$ is a countable subset of X. Define

$$x_i * x_j = \begin{cases} 0, & \text{if } i \leq j \\ x_i, & \text{if } i > j. \end{cases}$$

It is easy to prove that $(X_0, *, 0)$ is a BCC-algebra. If $U_i = \{x_i, x_{i+1}, x_{i+2}, ...\}$, then $B = \{U_i : i = 1, 2, 3, ...\}$ is a base for a topology $\mathcal{T}_0$ on X_0. We prove that $(X_0, *, \mathcal{T}_0)$ is a right topological BCC-algebra. Let $x_i * x_j \in U \in \mathcal{T}_0$. If $i \leq j$, then $x_i * x_j = 0 \in U$. Since X_0 is the only open neighborhood of 0, $U = X_0$. Cleraly, $x_i \in X_0$ and $X_0 * x_j \subseteq U$. If $i > j$, then $x_i * x_j = x_i$. Since B is a base for $\mathcal{T}_0$, $x_i \in U_i \subseteq U$. Since $i > j$, $U_i * x_j = U_i \subseteq U$. Therefore, $(X_0, *, \mathcal{T}_0)$ is a right topological BCC-algebra. But this space is not a topological BCC-algebra because $x_1 \in U_1$,

$x_2 \in U_2$ and $x_2 * x_1 = x_2 \in U_2$ but $U_2 * U_1 \nsubseteq U_2$. Consider X with the following operation

$$x \circ y = \begin{cases} x * y, & \text{if } x, y \in X_0 \\ 0, & \text{if } x \in X_0, y \notin X_0 \\ x, & \text{if } x \notin X_0, y \in X_0 \\ 0, & \text{if } x = y \notin X_0 \\ x, & \text{if } x \neq y, x, y \notin X_0, \end{cases}$$

then $(X, \circ, 0)$ is a BCC-algebra. By the proof of Theorem 3.7, we can claim that $B = \mathcal{T}_0 \cup \{\{x\} : x \notin X_0\}$ is a subbase for a topology $\mathcal{T}$ on X such that $(X, \circ, \mathcal{T})$ is a right topological BCC-algebra. But $\circ$ is not continuous in $(X, \mathcal{T})$ because $*$ is not continuous in $(X_0, \mathcal{T}_0)$. ■

Definition 3.10.

Let $(X, *, 0)$ be a BCC-algebra. A nonempty subset V on X is *preideal* if for each $x, y \in X$, $x \leq y,\ y \in V$ imply $x \in V$.

Clearly, ideals, BCC-ideals and initial segments are all preideal.

Proposition 3.11.

Let $(X, *, 0)$ be a BCC-algebra. Then:
(1) the arbitrary union and intersection of preideals in X is a preideal in X,
(2) if V is a preideal, then $0 \in V$,
(3) if V is a preideal, then for each $x \in X$ the set $V[x]$ is a preideal,
(4) if V is a preideal, then $V * X \subset V$.

Proof:

The proof is easy. ■

Theorem 3.12.

Let Ω be a family of preideals in a BCC-algebra $(X, *, 0)$ such that it is closed under intersection. If for each $x \in V \in \Omega$, there is a $U \in \Omega$ such that $U[x] \subseteq V$, then there is a topology $\mathcal{T}$ on X such that $(X, *, 0, \mathcal{T})$ is a right topological BCC-algebra.

Proof:

It is not difficult to prove that $\mathcal{T} = \{U \subseteq X : \forall x \in U,\ \exists V \in \Omega\ s.t\ V[x] \subseteq U\}$ is a topology on X. Let $U \in \Omega$ and $x \in X$. We show that $U[x] \in \mathcal{T}$. For this, suppose $y \in U[x]$; then $y * x \in U$. Consider $V \in \Omega$ such that $V[y * x] \subseteq U$. Let $z \in V[y]$. Since $(z * x) * (y * x) \leq z * y$ and $z * y \in V$, we get that $(z * x) * (y * x) \in V$. Hence $z * x \in V[y * x] \subseteq U$. This shows that $y \in V[y] \subseteq U[x]$. Therefore, $U[x]$ is an open set for each $U \in \Omega$ and $x \in X$. Also, obviously, the set $B = \{U[x] : U \in \Omega, x \in X\}$ is a base for $\mathcal{T}$. Now we prove that $*$ is continuous in the first variable. Let $x * y \in U[x * y] \in B$. If $z \in U[x]$, since $(z * y) * (x * y) \leq z * x$ and $z * x \in U$, we conclude that $(z * y) * (x * y) \in U$. So $z * y \in U[x * y]$. Thus, $U[x] * y \subseteq U[x * y]$. Therefore, $(X, *, 0, \mathcal{T})$ is right topological BCC-algebra. ■

Theorem 3.13.

Let Ω be a family of preideals in the BCC-algebra $(X, *, 0)$ such that it is closed under intersection. Let for each $x \in V \in \Omega$, there is a $U \in \Omega$ such that $U(x) \subseteq V$. If for each $x, y, z \in X$, $(x*y)*z = (x*z)*y$, then there is a topology $\mathcal{T}$ on X such that $(X, *, 0, \mathcal{T})$ is a semitopological BCC-algebra.

Proof:

Define $\mathcal{T} = \{U \subseteq X : \forall x \in U,\ \exists V \in \Omega \ \text{ s.t. } \ V(x) \subseteq U\}$. Easily, one can prove that $\mathcal{T}$ is a topology on X. At first we show that $B = \{U(x) : U \in \Omega, x \in X\}$ is a base for $\mathcal{T}$. Let $x \in U(a) \in B$. Then there exists a $V \in \Omega$ such that $V(x * a)$ and $V(a * x)$; both are the subsets of U. We show that $x \in V(x) \subseteq U(a)$. Let $y \in V(x)$. Then, $y * x$ and $x * y$ belong to V. Since

$$(y * a) * (x * a) \leq y * x, \quad (x * a) * (y * a) \leq x * y,$$

we get that $(y * a) * (x * a)$ and $(x * a) * (y * a)$ both belong to V. Hence, $y * a \in V(x * a) \subseteq U$. On the other hand, since

$$[(a * y) * (a * x)] * (x * y) = [(a * y) * (x * y)] * (a * x) \leq (a * x) * (a * x) = 0,$$

we have $(a * y) * (a * x) \leq x * y$. As $x * y \in V$, we get that $(a * y) * (a * x) \in V$. In a similar fashion, one can prove that $(a*x)*(a*y) \in V$. Hence, $a*y \in V(a*x) \subseteq U$. Since $a*y$ and $y*a$ both belong to U, we get that $y \in U(a)$. Thus we could show that $U(a) \in \mathcal{T}$, for each $a \in X$. Now it is easy to prove that B is a base for $\mathcal{T}$. In continuation we will prove that $*$ is continuous in first and second variable. Let $x * y \in V(x * y) \in B$. We show that $V(x) * y \subseteq V(x * y)$ and $x * V(y) \subseteq V(x * y)$. If $a \in V(x)$, then since

$$(a * y) * (x * y) \leq a * x, \quad (x * y) * (a * y) \leq x * a,$$

we get that $(a * y) * (x * y) \in V$ and $(x * y) * (a * y) \in V$. Hence, $a * y \in V(x * y)$ and so $V(x) * y \subseteq V(x * y)$. If $b \in V(y)$, since

$$(x * b) * (x * y) \leq y * b, \quad (x * y) * (x * b) \leq b * y,$$

we get that $x * b \in V(x * y)$. Hence $x * V(y) \subseteq V(x * y)$. ■

Example 3.14.

(i) An algebra $X = \{0, 1, 2, 3\}$ defined by the table

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	1	0.

is a BCC-algebra. Its preideals have the form $\{0, 1\}, \{0, 2\}, \{0, 1, 3\}$ and $\{0, 2, 3\}$. Let $U = \{0, 2\}$ and $V = \{0, 2, 3\}$. Then V is not an ideal because $1*3 = 0 \in V$ but $1 \notin V$. Also, $V[0], V[2], U[3]$, all, are the subsets of V and the sets $U[0], U[2]$, both, are the subsets of U. Hence, $\Omega = \{U, V\}$

satisfies Theorem 3.12. Therefore, $\mathcal{T} = \{W : \forall x \in W\ U[x] \subseteq W\ or\ V[x] \subseteq W\}$ is a topology on X such that $(X, *, 0, \mathcal{T})$ is a right topological BCC-algebra.

(ii) It is easy to verify that $X = [0, \infty)$ by

$$x * y = \begin{cases} 0, & \text{if } x \leq y \\ x, & \text{if } x > y, \end{cases}$$

is a BCC-algebra which is not proper. Let $I_n = [0, n]$, for any $n \geq 1$. Then:

$$I_n[x] = \begin{cases} I_n, & \text{if } x \leq n \\ \{x\}, & \text{if } x > n. \end{cases}$$

Now, if $\Omega = \{I_n : n \geq 1\}$, then Ω satisfies Theorem 3.13, and so there is a topology $\mathcal{T}$ on X such that $(X, *, 0, \mathcal{T})$ is a semitopological BCC-algebra.

4. Some results on (left, right) topological BCC-algeras

Proposition 4.1.

Let $(X, *, 0, \mathcal{T})$ be a topological BCC-algebra. If $U \in \mathcal{T}$, then:
(1) if $0 \in U$, then for each $x \in X$, there is an open neighborhood V of x such that $V * V \subseteq U$,
(2) if $x \in U$, then for some $V, W \in \mathcal{T}$, containing x and 0, respectively, $V * W \subseteq U$,
(3) if $0 \in U$, then for each $x \in X$, there are two open sets V and W containing x and 0, respectively, $V * W \subseteq U$,
(4) if $0 \in U$ and $x, y \in X$, then there are $V, W \in \mathcal{T}$ such that $x \in V$, $y \in W$ and $(V*W)*V \subseteq U$.

Proof:

We only prove (4). Let $0 \in U$ and $x, y \in X$. Since $(x * y) * x = 0$ and $*$ is continuous, there are two open neighborhoods V_1 and V_2 containing $x * y$ and x, respectively, such that $V_1 * V_2 \subseteq U$. Again, since $*$ is continuous, for some open neighborhood V_2 of x and an open set W containing y, we have $V_2 * W \subseteq V_1$. Suppose $V = V_1 \cap V_2$, then $x \in V \in \mathcal{T}$, $y \in W \in \mathcal{T}$ and

$$(V * W) * V \subseteq (V_2 * W) * V_2 \subseteq V_1 * V_2 \subseteq U.$$

■

Proposition 4.2.

Let $(X, *, 0, \mathcal{T})$ be a right topological BCC-algebra. If $0 \in U \in \mathcal{T}$, then for each $x \in X$, there are two open neighborhoods V and W of 0 and x, respectively, such that $V * x \subseteq U$, and $W * x \subseteq U$.

Proof:

The proof is easy. ■

Proposition 4.3.

Let $(X, *, 0, \mathcal{T})$ be a topological BCC-algebra. If $0 \in \cap\mathcal{T}$, then $B \subseteq X$ is open if and only if 0 is an interior point of B.

Proof:

If B is open, clearly 0 is an interior point of B. Let 0 be an interior point of B and $x \in B$. Since 0 is an interior point of B and $x * x = 0$, there is an open neighborhood V of 0 such that $x * x = 0 \in V \subseteq B$. Since $*$ is continuous, there exists an open set W containing x such that $W * W \subseteq V$. By hypothesis, $0 \in W$, hence $x \in W \subseteq W * W \subseteq V \subseteq B$. This proves that x is an interior point of B. ■

Proposition 4.4.

Let $(X, *, 0, \mathcal{T})$ be a left topological BCC-algebra which satisfies
(1) $x * y = 0 \Leftrightarrow x = y$,
(2) if net $\{x_i : i \in I\}$ converges to 0, then 0 belongs to it.
Then $B \subseteq X$ is closed if $0 \in B$.

Proof:

Let $0 \in B$ and $x \in \overline{B}$. If $x = 0$, then $x \in B$, so we assume that $x \neq 0$. Suppose $\{x_i : i \in I\}$ is a net in B which converges to x. Since $*$ is continuous in second variable, the net $\{x * x_i : i \in I\}$ converges to 0. By (2), there is an $i \in I$, such that $x * x_i = 0$. By (1), $x = x_i \in B$. ■

Proposition 4.5.

Let $(X, *, 0, \mathcal{T})$ be a left topological BCC-algebra and I be an ideal in X. Then I is closed if 0 is an interion point of I, or $0 \in \{x_j : j \in J\}$, for each net $\{x_j : j \in J\}$ which converges to 0.

Proof:

Let $x \in \overline{I}$, and $\{x_j : j \in J\}$ be a net in I which converges to x. Since $(X, *, \mathcal{T})$ is left topological BCC-algebra, the net $\{x * x_j : j \in J\}$ converges to 0. Now if 0 is an interior poin of I, then there is an open set U such that $0 \in U \subseteq I$. For some $j \in J$, $x * x_j \in U \subseteq I$. Since I is an ideal, $x \in I$. If $0 \in \{x_j : j \in J\}$, for each net $\{x_j : j \in J\}$ which converges to 0, then for some $j \in J$, $x * x_j = 0 \in I$ which implies that $x \in I$. ■

Proposition 4.6.

Let $(X, *, 0, \mathcal{T})$ be a semitopological BCC-algebra and I be an ideal in X. Then I is open and closed if 0 is an interior point of I.

Proof:

Let 0 be an interior point of I and $x \in I$. Then for some an open neighborhood U of 0, we have $x * x = 0 \in U \subseteq I$. Since $*$ is continuous in first variable, there is a $V \in \mathcal{T}$ such that $x \in V$ and $V * x \subseteq I$. Now for each $y \in V$, $y * x$ and x, both are in I, so $y \in I$. This shows that $x \in V \subseteq I$ and so I belongs to $\mathcal{T}$. Since $(X, \mathcal{T})$ is left topological BCC-algebra, by Proposition 4.5, I is closed. ■

Proposition 4.7.

Let $(X, *, 0, \mathcal{T})$ be a right topological BCC-algebra. If all of elements of X are atoms, then

$0 \in \overline{B}$, or B is closed, for each $B \subseteq X$ which $0 \notin B$.

Proof:

Let B be a subset of X which $0 \notin B$. Suppose B is not closed. Then there is a $x \in \overline{B}$ such that $x \notin B$. Let $\{b_i : i \in I\}$ be a net in B which converges to x. Since $*$ is continuous in first variable, the net $\{b_i * x : i \in I\}$ converges to 0. Since x and b_i, for each $i \in I$, are atoms and $0 \neq x \neq b_i$, $b_i * x = b_i$. Hence, the net $\{b_i : i \in I\}$ converges to 0, which implies that $0 \in \overline{B}$. ■

Proposition 4.8.

Let $(X, *, 0, \mathcal{T})$ be a topological BCC-algebra and I be a BCC-ideal in X. Then $\overline{I}$ is a BCC-ideal.

Proof:

Let $y \in \overline{I}$ and $x \in X$. Given a net $\{y_j : j \in J\}$ in I which converges to y. Since $*$ is continuous, the net $\{y_j * x : j \in J\}$ converges to $y * x$. Since I is a BCC-ideal and $y_j \in I$, for any $j \in J$, the net $\{y_j * x : j \in J\}$ is a subset of I, so $y * x \in \overline{I}$. Now suppose $y \in X$ and $x, z \in \overline{I}$. Then there are nets $\{x_j : j \in J\}$ and $\{z_j : j \in J\}$ in I which converge to x and z, respectively. Since $*$ is continuous, the net $\{y * ((y * x_j) * z_j) : j \in J\}$ converges to $y * ((y * x) * z)$. Since for any $j \in J$, $x_j, z_j \in I$ and I is a BCC-ideal, $y * ((y * x_j) * z_j)$ belongs to I. Hence $y * ((y * x) * z) \in \overline{I}$. ■

Proposition 4.9.

Let F and P be two disjoint subsetes of a topological BCC-algebra $(X, *, \mathcal{T})$. If F is compact and P is closed and for any $a \in X$, the map $l_a(x) = a * x$ is an open map of X into X, then there is an open neighborhood V of 0 such that $(F * V) \cap P = \phi$.

Proof:

Let $x \in F \subseteq X \setminus P$. Since $(x * 0) * 0 = x \in X \setminus P \in \mathcal{T}$ and $*$ is continuous, there exist $W, V_0 \in \mathcal{T}$ such that $x * 0 \in W$, $0 \in V_0$ and $W * V_0 \subseteq X \setminus P$. Also, there is an open neighborhood V_1 of 0 such that $x * V_1 \subseteq W$. If $V_x = V_0 \cap V_1$, then $(x * V_x) * V_x \subseteq W * V_0 \subseteq X \setminus P$. Since $C = \{x * V_x : x \in F\}$ is an open covering of the compact set F, there are $x_1 * V_{x_1}, ..., x_n * V_{x_n}$ in C such that $F \subseteq \bigcup_{i=1}^n x_i * V_{x_i}$. Suppose $V = \bigcap_{i=1}^n V_{x_i}$. Then V is an open neighborhood of 0 such that for each $y \in F$, $y \in x_i * V_{x_i}$, for some x_i, and $y * V \subseteq (x_i * V_{x_i}) * V \subseteq (x_i * V_{x_i}) * V_{x_i} \subseteq W * V_0 \subseteq X \setminus P$. This proves that $(F * V) \cap P = \phi$. ■

Recall in a BCC-algebra $(X, *, 0)$, $x \wedge y = y * (y * x)$ and it is *commutative* if $x \wedge y = y \wedge x$.

Proposition 4.10.

Let $(X, *, 0, \mathcal{T})$ be a left topological commutative BCC-algebra. Then $\wedge$ is continuous in first and second variable. Moreover, if $*$ is continuous, $\wedge$ is so.

Proof:

Let $x \wedge y \in U \in \mathcal{T}$. Since $*$ is continuous in second variable, there is a $V \in \mathcal{T}$ such that $y * x \in V$ and $y * V \subseteq U$. Again, since $*$ is continuous in second variable, there is an open neighborhood W of x such that $y * W \subseteq V$. Now W is an open set which contains x and $W \wedge y \subseteq U$. Hence, $\wedge$

is continuous in first variable. Since $\wedge$ is commutative, it is continuous in second variable. The proof of the other case is similar. ■

Proposition 4.11.

Let $(X, *, 0, \mathcal{T})$ be a semitopological BCC-algebra. Then for each $a \in X$, $V \in \mathcal{T}$ and $F \subseteq X$:
(1) the sets $V(a)$ and $V[a]$, both, are open,
(2) $\overline{F(a)} \subseteq \overline{F}(a)$ and $\overline{F[a]} \subseteq \overline{F}[a]$,
(3) if F is closed, then $F(a)$ and $F[a]$ are closed.

Proof:

(1) Let $x \in V(a)$. Then $x * a$ and $a * x$ belong to V. Since $(X, *, \mathcal{T})$ is semitopological BCC-algebra, there is an open set W containing x such that $W * a$ and $a * W$ are two subsets of V. Hence $x \in W \subseteq V(a)$. Similarly, we can show that $V[a]$ is open.

(2) Let $y \in \overline{F(a)}$. Then there is a net $\{x_i : i \in I\}$ in $F(a)$ which converges to y. Since $*$ is continuous in each variable separately, the net $\{x_i * a : i \in I\}$ converges to $y * a$ and the net $\{a * x_i : i \in I\}$ converges to $a * y$. Since the nets $\{x_i * a : i \in I\}$ and $\{a * x_i : i \in I\}$ are in F, we get that $a * y$ and $y * a$, both, are in $\overline{F}$. Hence, $y \in \overline{F}(a)$. Similarly, $\overline{F[a]} \subseteq \overline{F}[a]$.

(3) The proof is easy. ■

5. Separation axioms on topological BCC-algebra

Theorem 5.1.

Let $\mathcal{T}$ be a topology on a BCC-algebra $(X, *, 0)$. If for any $a \in X$ the map $l_a : X \hookrightarrow X$, by $l_a(x) = a * x$, is an open map, then $(X, \mathcal{T})$ is a T_0 space.

Proof:

Let $x \neq y \in X$ and U be an open neighborhood of 0. Then $x * U$ and $y * U$ are two open neighborhoods of x and y, respectively. If $x \in y * U$ and $y \in x * U$, then for some $a, b \in X$, $x = y * a \leq y$ and $y = x * b \leq x$. Hence $x = y$, which is a contradiction. Therefore, $x \notin y * U$ or $y \notin x * U$. This shows that $(X, \mathcal{T})$ is a T_0 space. ■

Theorem 5.2.

Let $(X, *, 0, \mathcal{T})$ be a right (left) topological BCC-algebra. Then $(X, \mathcal{T})$ is a T_0 space if and only if for any $x \neq 0$, there is a $U \in \mathcal{T}$ such that $x \in U$ and $0 \notin U$.

Proof:

Let for any $x \neq 0$, there is a $U \in \mathcal{T}$ such that $x \in U$ and $0 \notin U$. We prove that $(X, \mathcal{T})$ is a T_0 space. Given $x \neq y \in X$. Then $x * y \neq 0$ or $y * x \neq 0$. Suppose $x * y \neq 0$, then there exists a $U \in \mathcal{T}$ such that $x * y \in U$ and $0 \notin U$. Since $*$ is continuous in first variable, there is an open set

V containing x such that $V * y \subseteq U$. y is not in V because if $y \in V$, then $0 = y * y \in V * y \subseteq U$, which is a contradiction. Hence, $(X, \mathcal{T})$ is a T_0 space. The proof of converse is clear. ■

Theorem 5.3.

Let X be a BCC-algebra such that for any $a \in X \setminus \{0\}$, there is a $b \in X$ such that $0 < b < a$. Then there exists a nontrivial topology $\mathcal{T}$ on X such that $(X, *, 0, \mathcal{T})$ is a T_0 right topological BCC-algebra.

Proof:

Let $X_a = \{x \in X : a \leq x\}$, for any $a \in X$. Since $X = \bigcup_{a \in X} X_a$, the set $B = \{X_a : a \in X\}$ is a base of a topology $\mathcal{T}$ on X. Given $a \in X \setminus \{0\}$, then there is a $b \in X$ such that $0 < b < a$. Thus, $0 \notin X_b$ and $a, b \in X_b$. This implies that topology $\mathcal{T}$ is not a trivial topology. Now we prove that $*$ is continuous in first variable. Consider $x * y \in X_{x*y}$ and let $z \in X_x$. Since $x \leq z$, by (B_2), $x * y \leq z * y$. Hence $X_x * y \subseteq X_{x*y}$. This proves that $(X, *, 0, \mathcal{T})$ is a right topological BCC-algebra. To complete the proof we have to prove that this space is T_0. Let $x \neq 0$. Then there is a $b \in X$ such that $0 < b < x$. The set X_b is an open neighborhood of x such that $0 \notin X_b$. Hence by Theorem 5.2, $(X, \mathcal{T})$ is T_0. ■

Theorem 5.4.

If α is an infinite cardinal number, then there is a T_0 right topological BCC-algebra of order α.

Proof:

Let $(X_0, *, \mathcal{T}_0)$ and $(X, \circ, \mathcal{T})$ be right topological BCC-algebras in Theorem 3.9. Let $x \in X \setminus \{0\}$. If $x \in X_0$, then for some $i \geq 1$, $x \in U_i \in \mathcal{T}$ and $0 \notin U_i$. If $x \notin X_0$, then $x \in \{x\} \in \mathcal{T}$ and $0 \notin \{x\}$. Therefore, by Theorem 5.2, $(X, \circ, \mathcal{T})$ is a T_0 right topological BCC-algebra of order α. ■

Theorem 5.5.

If α is an infinite cardinal number, then there is a T_0 topological BCC-algebra of order α in which its topology is nontrivial.

Proof:

Let $(X_0, *, \mathcal{T}_0)$ and $(X, \circ, \mathcal{T})$ be topological BCC-algebras in Theorem 3.7. It is clear that $\mathcal{T}$ is nontrivial. Let $x \in X \setminus \{0\}$. If $x \in X_0$, then for some $n \geq 1$, $x \notin I_n$. Hence, $x \in x/I_n \in \mathcal{T}$ and $0 \notin x/I_n$. If $x \notin X_0$, then $x \in \{x\} \in \mathcal{T}$ and $0 \notin \{x\}$. Now by Theorem 5.2, $(X, \circ, \mathcal{T})$ is a T_0 topological BCC-algebra of order α. ■

Theorem 5.6.

Let $(X, *, 0, \mathcal{T})$ be a semitopological BCC-algebra. Then $(X, \mathcal{T})$ is a T_1 space if and only if for any $x \neq 0$, there are two open neighborhoods U and V of x and 0, respectively, such that $0 \notin U$ and $x \notin V$.

Proof:

If $(X, \mathcal{T})$ is T_1, then the proof is obvious. Conversely, let for any $x \neq 0$, there are two open neighborhoods U and V of x and 0, respectively, such that $0 \notin U$ and $x \notin V$. We prove that $(X, \mathcal{T})$ is a T_1 space. Given $x \neq y$, then $x * y \neq 0$ or $y * x \neq 0$. Without lost of generality, assume that $x * y \neq 0$. Then there are two open neighborhoods U and V of $x * y$ and 0, respectively, such that $x * y \notin V$ and $0 \notin U$. Since $*$ is continuous in each variable separately, there are W and W_1 belong to $\mathcal{T}$ such that $x \in W$, $y \in W_1$ and $W * y \subseteq U$ and $x * W_1 \subseteq U$. But $x \notin W_1$ because if $x \in W_1$, then $0 = x * x \in x * W_1 \subseteq U$, a contradiction. Similarly, $y \notin W$. Therefore, $(X, \mathcal{T})$ is a T_1 space. ■

Theorem 5.7.

Let $(X, *, 0, \mathcal{T})$ be a semitopological BCC-algebra. Then $(X, \mathcal{T})$ is a T_1 space if and only if it is a T_0 space.

Proof:

Let $(X, \mathcal{T})$ be a T_0 space and $x \neq y$. Then, $x * y \neq 0$ or $y * x \neq 0$. Without lost of generality, suppose $x * y \neq 0$. Then there is a $U \in \mathcal{T}$ such that $x * y \in U$ and $0 \notin U$ or $0 \in U$ and $x * y \notin U$. First assume that $x * y \in U$ and $0 \notin U$. Since $(X, *, \mathcal{T})$ is a semitopological BCC-algebra, there are two open neighborhoods V and W of x and y, respectively, such that $V * y \subseteq U$ and $x * W \subseteq U$. But $x \notin W$ because if $x \in W$, then $0 = x * x \in x * W \subseteq U$, a contradiction. Similarly, $y \notin V$. Now if $0 \in U$ and $x * y \notin U$, then since $x * x = y * y = 0 \in U$, there are open sets V and W such that $x \in V$, $y \in W$ and $x * V \subseteq U$ and $W * y \subseteq U$. If $x \in W$, then $x * y \in W * y \subseteq U$, a contradiction. Similarly, $y \notin V$. Therefore, $(X, \mathcal{T})$ is a T_1 space. If $(X, \mathcal{T})$ is T_1, clearly it is T_0. ■

Corollary 5.8.

If α is an infinite cardinal number, then there is a T_1 topological BCC-algebra of order α which its topology is nontrivial.

Proof:

By Theorems 5.5 and 5.7 the proof is clear. ■

Theorem 5.9.

Let $(X, *, 0, \mathcal{T})$ be a topological BCC-algebra. Then $(X, \mathcal{T})$ is Hausdorff if and only if for each $x \neq 0$, there are two disjoint open neighborhoods U and V of x and 0, respectively.

Proof:

If $(X, \mathcal{T})$ is Hausdorff, the proof is clear. Conversely, let for each $x \neq 0$, there are two disjoint open neighborhoods U and V of x and 0, respectively. We prove that $(X, \mathcal{T})$ is Hausdorff. For this, take $x \neq y$. Then $x * y \neq 0$ or $y * x \neq 0$. Without lost of generality, we suppose that $x * y \neq 0$. Then there are two disjoint open neighborhoods U and V of $x * y$ and 0, respectively. Since $*$ is continuous, there are two open sets W and W_1 such that $x \in W$, $y \in W_1$ and $W * W_1 \subseteq U$.

If $z \in W \cap W_1$, then $0 = z * z \in W * W_1 \subseteq U$, which is a contradiction. Hence, $W \cap W_1 = \phi$. Therefore, $(X, \mathcal{T})$ is Hausdorff. ■

Theorem 5.10.

Let $(X, *, 0, \mathcal{T})$ be a topological BCC-algebra. Then $(X, \mathcal{T})$ is a T_1 space if and only if it is a Hausdorff space.

Proof:

Let $(X, *, 0, \mathcal{T})$ be a T_1 topological BCC-algebra. Given $x \neq 0$. Then there are $U, V \in \mathcal{T}$ such that $x \in U$, $0 \in V$ and $x \notin V$ and $0 \notin U$. By Proposition 4.1 (2), there are two open neighborhoods W and W_1 such that $x \in W$, $0 \in W_1$ and $W * W_1 \subseteq U$. If $z \in W \cap W_1$, then $0 = z * z \in W * W_1 \subseteq U$, which is a contradiction. Hence $W \cap W_1 = \phi$. By Theorem 5.9, $(X, \mathcal{T})$ is Hausdorff. The converse is obvious. ■

Corollary 5.11.

If α is an infinite cardinal number, then there is a Hausdorff topological BCC-algebra of order α in which its topology is nontrivial.

Proof:

By Theorems 5.5, 5.7, and 5.10, the proof is clear. ■

Theorem 5.12.

Let $(X, *, \mathcal{T})$ be a topological BCC-algebra. Then $(X, \mathcal{T})$ is Hausdorff if and only if $\{0\}$ is closed.

Proof:

Let $\{0\}$ be closed. We show that for each $a \in X$ the set $\{a\}$ is closed. Take $a \in X$. By Proposition 4.9, $\wedge$ is continuous, hence $\wedge^{-1}(0) = \{(0, 0)\}$ is closed in $X \times X$. On the other hand, since $*$ is continuous, the map $h : X \hookrightarrow X \times X$ by $h(x) = (a * x, x * a)$ is continuous. Hence $h^{-1}(0, 0) = \{x : a * x = x * a = 0\} = \{a\}$ is closed in X. Thus, $(X, \mathcal{T})$ is T_1. By Theorem 5.10, $(X, \mathcal{T})$ is Hausdorff. ■

Theorem 5.13.

Let $\mathcal{N}$ be a fundamental system of neighborhoods of 0 in the topological BCC-algebra $(X, *, 0, \mathcal{T})$. The following conditions are equivalent.
(1) $(X, \mathcal{T})$ is T_0 space,
(2) $(X, \mathcal{T})$ is T_1 space,
(3) $(X, \mathcal{T})$ is Hausdorff space,
(4) $\cap \mathcal{N} = \{0\}$.

Proof:

By Theorems 5.7, 5.10, (1), (2) and (3) are equivalent. We prove that (ii) and (iv) are equivalent. If $(X, \mathcal{T})$ is T_1 space and $x \neq 0$, then by Theorem 5.6, there is a $U \in \mathcal{N}$ such that $x \notin U$, hence

$x \notin \cap\mathcal{N}$. Conversely, let $\cap\mathcal{N} = \{0\}$ and $x \neq 0$. Then there is a $V \in \mathcal{N}$ such that $x \notin V$. By Propsition 4.10 (1), $U(x)$ is an open neighborhood of x. But $0 \notin U(x)$ because $x \notin U$. Thus, U and $U(x)$ are two open sets containing $0, x$, respectively, such that $x \notin U$ and $0 \notin U(x)$. By Theorem 5.6, $(X, \mathcal{T})$ is T_1 space. ■

Theorem 5.14

The topological BCC-algebra $(X, *, 0, \mathcal{T})$ is an Uryshon space if and only if for any $x \neq 0$, there are two open sets U and V containing x and 0, respectively, such that $\overline{U} \cap \overline{V} = \phi$.

Proof:

Let for any $x \neq 0$, there are two open sets U and V containing x and 0, respectively, such that $\overline{U} \cap \overline{V} = \phi$. We prove that $(X, \mathcal{T})$ is Uryshon space. For this, suppose $x \neq y$. Then we can assume that $x * y \neq 0$. Take two open sets U and V such that $x * y \in U$ and $0 \in V$ and $\overline{U} \cap \overline{V} = \phi$. Since $*$ is continuous, there are open neighborhoods W and W_1 of x and y, respectively, $W * W_1 \subseteq U$. If $z \in \overline{W} \cap \overline{W_1}$, then there are two nets $\{x_i : i \in I\}$ and $\{y_i : i \in I\}$ in W and W_1, respectively, which converges to z. Now $\{x_i * y_i : i \in I\}$ is a net in U which converges to 0. Hence $0 \in \overline{U} \cap \overline{V} = \phi$, a contradiction. This proves that $(X, \mathcal{T})$ is Uryshon space. The converse is clear. ■

Theorem 5.15.

The topological BCC-algebra $(X, *, 0, \mathcal{T})$ is a Uryshon space if and only if it is Hausdorff.

Proof:

Let $(X, \mathcal{T})$ be Hausdorff space and $x \neq 0$. Then there are two disjoint open neighborhoods U and V of x and 0, respectively. By Proposition 4.1 (2), there are two open sets W and W_1 such that $x \in W$ and $0 \in W_1$ and $W * W_1 \subseteq U$. We prove that $\overline{W} \cap \overline{W_1} = \phi$. Let $z \in \overline{W} \cap \overline{W_1}$. Then there are two nets $\{x_i : i \in I\} \subseteq W$ and $\{y_i : i \in I\} \subseteq W_1$, which both converges to z. Thus, $\{x_i * y_i : i \in I\}$ is a net in U that converges to 0 which implies that $0 \in \overline{U}$. Since V is an open neighborhood of 0, $V \cap U \neq \phi$, a contradiction. Now by Theorem 5.14, $(X, \mathcal{T})$ is Uryshon space. If $(X, \mathcal{T})$ is Uryshon space, clearly it is Hausdorff. ■

Corollary 5.16.

If α is an infinite cardinal number, then there is an Uryshon topological BCC-algebra of order α in which its topology is nontrivial.

Proof:

By Theorem 5.15 and Corollary 5.11, the proof is obvious. ■

6. Conclusion

In this note, we have studied (semi)topological BCC-algebras. We have proved that for each infinite cardinal number α there exists at least a Hausdorff (semi)toplogical BCC-algebras of order α with nontrivial topology. In the future, we will investigate regularity, normality, and metrizability on (semi)topological BCC-algebras.

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