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M. Mursaleen
Aligarh Muslim University

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Generalized statistical summability of double sequences and Korovkin type approximation theorem

M. Mursaleen

Department of Mathematics
Aligarh Muslim University, Aligarh-202002, India
mursaleenm@gmail.com

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Abstract

In this paper, we introduce the notion of statistical (λ, μ)-summability and find its relation with (λ, μ)-statistical convergence. We apply this new method to prove a Korovkin type approximation theorem for functions of two variables. Furthermore, we provide an example in support to show that our result is stronger than the previous ones.

Keywords: Double sequence; density; statistical convergence; (λ, μ)-statistical convergence; (λ, μ)-summability; positive linear operator; Korovkin type approximation theorem

MSC 2010 No.: 41A10, 41A25, 41A36, 40A30, 40G15

1. Introduction

The concept of statistical convergence for sequences of real numbers was introduced by Fast (1951) and further studied many others.

Let $K \subseteq \mathbb{N}$ and $K_n = \{k \leq n : k \in K\}$. Then, the *natural density* of K is defined by $\delta(K) = \lim_n n^{-1}|K_n|$ if the limit exists, where $|K_n|$ denotes the cardinality of K_n.

A sequence $x = (x_k)$ of real numbers is said to be *statistically convergent* to L provided that for every $\epsilon > 0$ the set $K_\epsilon := \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}$ has natural density zero, i.e., for each $\epsilon > 0$,

$$\lim_n \frac{1}{n}|\{j \leq n : |x_j - L| \geq \epsilon\}| = 0.$$

By the convergence of a double sequence we mean the convergence in the Pringsheim's sense (1900). A double sequence $x = (x_{jk})$ is said to be *Pringsheim's convergent* (or P-*convergent*) if for given $\epsilon > 0$ there exists an integer N such that $|x_{jk} - \ell| < \epsilon$ whenever $j, k > N$. In this case, ℓ is called the Pringsheim limit of $x = (x_{jk})$ and it is written as $P - \lim x = \ell$. For our convenience, we will write $\lim x$ instead of $P - \lim x$.

A double sequence $x = (x_{jk})$ is said to be *bounded* if there exists a positive number M such that $|x_{jk}| < M$ for all j, k.

Note that, in contrast to the case for single sequences, a convergent double sequence need not be bounded.

The idea of statistical convergence for double sequences was introduced by Mursaleen and Edely (2003).

Let $K \subseteq \mathbb{N} \times \mathbb{N}$ be a two-dimensional set of positive integers and let $K_{m,n} = \{(j,k) : j \leq m, k \leq n\}$. Then, the two-dimensional analogue of natural density can be defined as follows.

In case the sequence $(K(m,n)/mn)$ has a limit in Pringsheim's sense, then we say that K has a *double natural density* and is defined as

$$P - \lim_{m,n} \frac{K(m,n)}{mn} = \delta^{(2)}\{K\}.$$

For example, let $K = \{(i^2, j^2) : i, j \in \mathbb{N}\}$. Then,

$$\delta^{(2)}\{K\} = P - \lim_{m,n} \frac{K(m,n)}{mn} \leq P - \lim_{m,n} \frac{\sqrt{m}\sqrt{n}}{mn} = 0,$$

i.e., the set K has double natural density zero, while the set $\{(i, 2j) : i, j \in \mathbb{N}\}$ has double natural density $\frac{1}{2}$.

A real double sequence $x = (x_{jk})$ is said to be *statistically convergent* to the number L if for each $\epsilon > 0$, the set

$$\{(j,k), j \leq m \text{ and } k \leq n : | x_{jk} - L | \geq \epsilon\}$$

has double natural density zero. In this case we write $st^{(2)}\text{-}\lim_{j,k} x_{jk} = L$.

Remark 1.

Note that if $x = (x_{jk})$ is P-convergent then it is statisically convergent but not conversely. See the following example.

Example 2.

The double sequence $x = (x_{jk})$ defined by

$$x_{jk} = \begin{cases} 1 \ , & \text{if } \ j \ \text{ and } \ k \text{ are squares;} \\ 0 \ , & \text{otherwise.} \end{cases} \tag{1.1}$$

Then, x is statistically convergent to zero but not P-convergent.

Moricz (2003) introduced the idea of statistical summability $(C,1,1)$.

We say that a double sequence $x=(x_{jk})$ is *statistically summability* $(C,1,1)$ to some number L, if $st^{(2)}\text{-}\lim\limits_{m,n}\sigma_{mn}=L$, where

$$\sigma_{mn}=\frac{1}{mn}\sum_{j=1}^{m}\sum_{k=1}^{n}x_{jk}.$$

In this case, we write $st_{(C,1,1)}\text{-}\lim x=L$. It is trivial that $st^{(2)}\text{-}\lim\limits_{j,k}x_{jk}=L$ implies $st^{(2)}\text{-}\lim\limits_{m,n}\sigma_{mn}=L$.

Mursaleen et. al. (2010) defined the (λ,μ)-statistical convergence, and further studied in Kumar and Mursaleen (2011) as follows:

We define the following.

Let $\lambda=(\lambda_m)$ and $\mu=(\mu_n)$ be two non-decreasing sequences of positive real numbers such that each tending to ∞ and

$$\lambda_{m+1}\leq\lambda_m+1,\ \ \lambda_1=0,$$

and

$$\mu_{n+1}\leq\mu_n+1,\ \ \mu_1=0.$$

Let $K\subseteq\mathbb{N}\times\mathbb{N}$ be a two-dimensional set of positive integers. Then, the (λ,μ)-*density* of K is defined as

$$\delta_{\lambda,\mu}(K)=P-\lim_{m,n}\frac{1}{\lambda_m\mu_n}|\{m-\lambda_m+1\leq j\leq m,n-\mu_n+1\leq k\leq n:(j,k)\in K\}|,$$

provided that the limit on the right hand-side exists.

A double sequence $x=(x_{jk})$ is said to be (λ,μ)-*statistically convergent* to ℓ if $\delta_{\lambda,\mu}(E)=0$, where $E=\{j\in J_m,k\in I_n:|x_{jk}-\ell|\geq\epsilon\}$, i.e., if for every $\epsilon>0$, $\lim\limits_{m,n}\frac{1}{\lambda_m\mu_n}|\{j\in J_m,k\in I_n:|x_{jk}-\ell|\geq\epsilon\}|=0$.

In this case, we write $st_{(\lambda,\mu)}\text{-}\lim\limits_{j,k}x_{jk}=\ell$,and we denote the set of all (λ,μ)-statistically convergent double sequences by $S_{\lambda,\mu}$.

In case $\lambda_m=m,\mu_n=n$, the (λ,μ)-density reduces to the natural double density. Also, since $(\lambda_m/m)\leq1,(\mu_n/n)\leq1$, then $\delta_2(K)\leq\delta_{\lambda,\mu}(K)$, for every $K\subseteq\mathbb{N}\times\mathbb{N}$.

2. Statistically (λ,μ)-summability

We define the generalized double de la Valée-Pousin mean by

$$t_{mn}=\frac{1}{\lambda_m\mu_n}\sum_{j\in J_m}\sum_{k\in I_n}x_{jk},$$

where $J_m=[m-\lambda_m+1,m]$ and $I_n=[n-\mu_n+1,n]$.

A double sequence $x=(x_{jk})$ is said to be (V,λ,μ)-*summable* to a number ℓ, if

$$P\text{-}\lim_{m,n} t_{m,n}=\ell.$$

A double sequence $x=(x_{jk})$ is said to be *statistically* (λ,μ)-*summable* to ℓ, if the sequence (t_{mn}) is statistically convergent to ℓ. In this case, we write $(\lambda,\mu)_{st}\text{-}\lim_{j,k} x_{jk}=\ell$.

Theorem 3.

If a sequence $x=(x_{jk})$ is bounded and (λ,μ)-statistically convergent to L, then it is statistically (λ,μ)-summable to L but not conversely.

Proof:

Let x be bounded and (λ,μ)-statistically convergent to L, and $K(\epsilon):=\{j\in J_m, k\in I_n : |x_{jk}-\ell|\geq\epsilon\}$. Then,

$$\begin{aligned}
|t_{mn}-L| &= \left|\frac{1}{\lambda_m\mu_n}\sum_{j\in J_m}\sum_{k\in I_n}x_{jk}-L\right| = \left|\frac{1}{\lambda_m\mu_n}\sum_{j=m-\lambda_m+1}^{m}\sum_{k=n-\mu_n+1}^{n}(x_{jk}-L)\right| \\
&\leq \left|\frac{1}{\lambda_m\mu_n}\sum_{(j,k)\in K(\epsilon)}(x_{jk}-L)\right| \\
&\leq \frac{1}{\lambda_m\mu_n}\left(\sup_{j,k}|x_{jk}-L|\right) \mid K(\epsilon)\mid \\
&\to 0 \quad \text{as } m,n\to\infty.
\end{aligned}$$

Thus, x is (λ,μ)-convergent to L, and hence statistically (λ,μ)-summable to L.

For converse, consider the case $\lambda_m=m$, $\mu_n=n$ and the sequence $x=(x_{jk})$ defined by (1.1). Then, of course this sequence is not (λ,μ)-statistically convergent. On the other hand, x is (V,λ,μ)-summable to 0, and hence statistically (λ,μ) -summable to 0. This completes the proof of the theorem. □

3. Korovkin type theorem

Let $C[a,b]$ be the space of all functions f continuous on $[a,b]$. We know that $C[a,b]$ is a Banach space with norm

$$\|f\|_{C[a,b]}:=\sup_{x\in[a,b]}|f(x)|,\ \ f\in C[a,b].$$

The classical Korovkin approximation theorem states as follows (Korovkin (1960)):

Let (T_n) be a sequence of positive linear operators from $C[a,b]$ into $C[a,b]$. Then,

$$\lim_{n}\|T_n(f,x)-f(x)\|_{C[a,b]}=0, \forall f\in C[a,b]$$

if and only if

$$\lim_n \|T_n(e_i, x) - e_i(x)\|_{C[a,b]} = 0, \quad i = 0, 1, 2;$$

where

$$e_0(x) = 1,$$

$$e_1(x) = x,$$

and

$$e_2(x) = x^2.$$

Quite recently, such type of approximation theorems have been established for functions of one and/ or two variables, by using statistical convergence [Dirik and Demirci (2010), Gadjiev and Orhan (2002)]; generalized statistical convergence [Aktuğlu (2014), Belen and Mohiuddine (2013), Braha et. al. (2014), Edely et. al.(2010), Srivastava et. al. (2012)]; A-statistical convergence (Dirik and Demirci (2010)); statistical A-summability (Belen et. al. (2012), Demirci and Karakuş (2013)); statistically summability $(C, 1)$ (Mohiuddine et. al. (2012)); weighted statistical convergence (Braha et. al. (2015), Kadak (2016), Mohiuddine (2016), Özarslan and Aktuğlu) and almost convergence (Mohiuddine (2011)). In this paper, we extend the result of (Taşdelen and Erençin (2007)) by using the notion of statistical summability $(C, 1, 1)$ and show that our result is stronger than those proved by Taşdelen and Erençin (2007) and Dirik and Demirci (2010).

Let $I = [0, A],\ J = [0, B],\ A, B \in (0, 1)$ and $K = I \times J$. We denote by $C(K)$, the space of all continuous real valued functions on K. This space is a equipped with norm

$$\|f\|_{C(K)} := \sup_{(x,y)\in K} |f(x, y)|,\ \ f \in C(K).$$

Let $H_\omega(K)$ denote the space of all real valued functions f on K such that

$$| f(s, t) - f(x, y) | \leq \omega\left(f; \sqrt{\left(\frac{s}{1-s} - \frac{x}{1-x}\right)^2 + \left(\frac{t}{1-t} - \frac{y}{1-y}\right)^2}\ \right),$$

where ω is the modulus of continuity, i.e.,

$$\omega(f; \delta) = \sup_{(s,t),(x,y)\in K} \{|f(s, t) - f(x, y)| : \sqrt{(s-x)^2 + (t-y)^2} \leq \delta\}.$$

It is to be noted that any function $f \in H_\omega(K)$ is continuous and bounded on K.

The following result was given by Taşdelen and Erençin (2007).

Theorem A.

Let $(T_{j,k})$ be a double sequence of positive linear operators from $H_\omega(K)$ into $C(K)$. Then, for all $f \in H_\omega(K)$,

$$P\text{-}\lim_{j,k\to\infty} \left\| T_{j,k}(f; x, y) - f(x, y) \right\|_{C(K)} = 0, \tag{1}$$

if and only if

$$P\text{-}\lim_{j,k\to\infty} \left\| T_{j,k}(f_i; x, y) - f_i \right\|_{C(K)} = 0 \ (i = 0, 1, 2, 3), \tag{2}$$

where

$$f_0(x,y) = 1,$$
$$f_1(x,y) = \frac{x}{1-x},$$
$$f_2(x,y) = \frac{y}{1-y},$$

and

$$f_3(x,y) = \left(\frac{x}{1-x}\right)^2 + \left(\frac{y}{1-y}\right)^2.$$

Recently, Dirik and Demirci (2010) proved the following theorem.

Theorem B.

Let $(T_{j,k})$ be a double sequence of positive linear operators from $H_\omega(K)$ into $C(K)$. Then, for all $f \in H_\omega(K)$

$$st^{(2)}\text{-}\lim_{j,k\to\infty}\left\|T_{j,k}(f;x,y) - f(x,y)\right\|_{C(K)} = 0, \qquad (1)'$$

if and only if

$$st^{(2)}\text{-}\lim_{j,k\to\infty}\left\|T_{j,k}(f_i;x,y) - f_i\right\|_{C(K)} = 0 (i = 0,1,2,3), \qquad (2)'$$

Now, we prove the following result.

Theorem 4.

Let $(T_{j,k})$ be a double sequence of positive linear operators from $H_\omega(K)$ into $C(K)$. Then, for all $f \in H_\omega(K)$,

$$(\lambda,\mu)_{st} - \text{-}\lim\left\|T_{j,k}(f;x,y) - f(x,y)\right\|_{C(K)} = 0, \qquad (3.1)$$

if and only if

$$(\lambda,\mu)_{st} - \text{-}\lim\left\|T_{j,k}(1;x,y) - 1\right\|_{C(K)} = 0, \qquad (3.2)$$

$$(\lambda,\mu)_{st} - \text{-}\lim\left\|T_{j,k}\left(\frac{s}{1-s};x,y\right) - \frac{x}{1-x}\right\|_{C(K)} = 0, \qquad (3.3)$$

$$(\lambda,\mu)_{st} - \text{-}\lim\left\|T_{j,k}(\frac{t}{1-t};x,y) - \frac{y}{1-y}\right\|_{C(K)} = 0, \qquad (3.4)$$

$$st_{(C,1,1)}\text{-}\lim\left\|T_{j,k}\left(\left(\frac{s}{1-s}\right)^2 + \left(\frac{t}{1-t}\right)^2;x,y\right) - \left(\left(\frac{x}{1-x}\right)^2 + \left(\frac{y}{1-y}\right)^2\right)\right\|_{C(K)} = 0. \qquad (3.5)$$

Proof:

Since each $1, \frac{x}{1-x}, \frac{y}{1-y}, (\frac{x}{1-x})^2+(\frac{y}{1-y})^2$ belong to $H_\omega(K)$, conditions (3.2)–(3.5) follow immediately from (3.1). Let $f\in H_\omega(K)$ and $(x,y)\in K$ be fixed. Then, after using the properties of f, a simple calculation gives that

$$\begin{aligned}
\mid T_{j,k}(f;x,y)-f(x,y)\mid &\le T_{j,k}(\mid f(s,t)-f(x,y)\mid;x,y)+\mid f(x,y)\mid\mid T_{j,k}(f_0;x,y)-f_0(x,y)\mid\\
&\le \varepsilon+(\varepsilon+N+\frac{2N}{\delta^2})\mid T_{j,k}(f_0;x,y)-f_0(x,y)\mid+\frac{4N}{\delta^2}\mid T_{j,k}(f_1;x,y)-f_1(x,y)\mid\\
&\quad+\frac{4N}{\delta^2}\mid T_{j,k}(f_2;x,y)-f_2(x,y)\mid+\frac{2N}{\delta^2}\mid T_{j,k}(f_3;x,y)-f_3(x,y)\mid\\
&\le \varepsilon+M\{\mid T_{j,k}(f_0;x,y)-f_0(x,y)\mid+\mid T_{j,k}(f_1;x,y)-f_1(x,y)\mid\\
&\quad+\mid T_{j,k}(f_2;x,y)-f_2(x,y)\mid+\mid T_{j,k}(f_3;x,y)-f_3(x,y)\mid\},
\end{aligned}$$

where $N=\| f\|_{C(K)}$ and

$$M=\max\left\{\varepsilon+N+\frac{2N}{\delta^2}\left(\left(\frac{A}{1-A}\right)^2+\left(\frac{B}{1-B}\right)^2\right),\frac{4N}{\delta^2}\left(\frac{A}{1-A}\right),\frac{4N}{\delta^2}\left(\frac{B}{1-B}\right),\frac{2N}{\delta^2}\right\}.$$

Now, replacing $T_{j,k}(f;x,y)$ by $\frac{1}{\lambda_m\mu_n}\sum_{j\in J_m}\sum_{k\in I_n}T_{j,k}(f;x,y)$ and taking $\sup_{(x,y)\in K}$, we get

$$\begin{aligned}
\left\|\frac{1}{\lambda_m\mu_n}\sum_{j\in J_m}\sum_{k\in I_n}T_{j,k}(f;x,y)-f(x,y)\right\|_{C(K)} &\le \varepsilon+M\left(\left\|\frac{1}{\lambda_m\mu_n}\sum_{j\in J_m}\sum_{k\in I_n}T_{j,k}(f_0;x,y)-f_0(x,y)\right\|_{C(K)}\right.\\
&+\left\|\frac{1}{\lambda_m\mu_n}\sum_{j\in J_m}\sum_{k\in I_n}T_{j,k}(f_1;x,y)-f_1(x,y)\right\|_{C(K)}+\left\|\frac{1}{\lambda_m\mu_n}\sum_{j\in J_m}\sum_{k\in I_n}T_{j,k}(f_2;x,y)-f_2(x,y)\right\|_{C(K)}\\
&\left.+\left\|\frac{1}{\lambda_m\mu_n}\sum_{j\in J_m}\sum_{k\in I_n}T_{j,k}(f_3;x,y)-f_3(x,y)\right\|_{C(K)}\right). \qquad (3.6)
\end{aligned}$$

For a given $r>0$, choose $\varepsilon>0$ such that $\varepsilon<r$. Define the following sets:

$$D:=\left\{(m,n), m\le p \text{ and } n\le q:\left\|\frac{1}{\lambda_m\mu_n}\sum_{j\in J_m}\sum_{k\in I_n}T_{j,k}(f;x,y)-f(x,y)\right\|_{C(K)}\ge r\right\},$$

$$D_1:=\left\{(m,n), m\le p \text{ and } n\le q:\left\|\frac{1}{\lambda_m\mu_n}\sum_{j\in J_m}\sum_{k\in I_n}T_{j,k}(f_0;x,y)-f_0(x,y)\right\|_{C(K)}\ge \frac{r-\varepsilon}{4K}\right\},$$

$$D_2:=\left\{(m,n), m\le p \text{ and } n\le q:\left\|\frac{1}{\lambda_m\mu_n}\sum_{j\in J_m}\sum_{k\in I_n}T_{j,k}(f_1;x,y)-f_1(x,y)\right\|_{C(K)}\ge \frac{r-\varepsilon}{4K}\right\},$$

$$D_3:=\left\{(m,n), m\le p \text{ and } n\le q:\left\|\frac{1}{\lambda_m\mu_n}\sum_{j\in J_m}\sum_{k\in I_n}T_{j,k}(f_2;x,y)-f_2(x,y)\right\|_{C(K)}\ge \frac{r-\varepsilon}{4K}\right\},$$

$$D_4:=\left\{(m,n), m\le p \text{ and } n\le q:\left\|\frac{1}{\lambda_m\mu_n}\sum_{j\in J_m}\sum_{k\in I_n}T_{j,k}(f_3;x,y)-f_3(x,y)\right\|_{C(K)}\ge \frac{r-\varepsilon}{4K}\right\}.$$

Then, from (3.6), we see that $D \subset D_1 \cup D_2 \cup D_3 \cup D_4$, and therefore $\delta_{\lambda,\mu}\{D\} \leq \delta_{\lambda,\mu}\{D_1\} + \delta_{\lambda,\mu}\{D_2\} + \delta_{\lambda,\mu}\{D_3\} + \delta_{\lambda,\mu}\{D_4\}$. Hence, conditions (3.2)–(3.5) imply the condition (3.1).

This completes the proof of the theorem. □

4. Example

We show that the following double sequence of positive linear operators satisfies the conditions of Theorem 3.1 but does not satisfy the conditions of Theorem A and Theorem B.

Example 5.

Consider the following Meyer-König and Zeller (1960) operators:

$$B_{m,n}(f;x,y) := (1-x)^{m+1}(1-y)^{n+1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} f\left(\frac{j}{j+m+1},\frac{k}{k+n+1}\right)\binom{m+j}{j}\binom{n+k}{k}x^j y^k, \tag{4.1}$$

where $f \in H_\omega(K)$, and $K=[0,A]\times[0,B],\ A,B\in(0,1)$. Since, for $x\in[0,A],\ A\in(0,1)$,

$$\frac{1}{(1-x)^{m+1}} = \sum_{j=0}^{\infty}\binom{m+j}{j}x^j,$$

it is easy to see that

$$B_{m,n}(f_0;x,y) = f_0(x,y).$$

Also, we obtain

$$\begin{aligned}B_{m,n}(f_1;x,y) &= (1-x)^{m+1}(1-y)^{n+1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{j}{m+1}\binom{m+j}{j}\binom{n+k}{k}x^j y^k\\ &= (1-x)^{m+1}(1-y)^{n+1}x\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{m+1}\frac{(m+j)!}{m!(j-1)!}\binom{n+k}{k}x^{j-1}y^k\\ &= (1-x)^{m+1}(1-y)^{n+1}x\frac{1}{(1-x)^{m+2}}\frac{1}{(1-y)^{n+1}} = \frac{x}{(1-x)},\end{aligned}$$

and similarly

$$B_{m,n}(f_2;x,y) = \frac{y}{(1-y)}.$$

Finally, we get

$$\begin{aligned}B_{m,n}(f_3;x,y) &= (1-x)^{m+1}(1-y)^{n+1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left\{\left(\frac{j}{m+1}\right)^2+\left(\frac{k}{n+1}\right)^2\right\}\binom{m+j}{j}\binom{n+k}{k}x^j y^k\\ &= (1-x)^{m+1}(1-y)^{n+1}\frac{x}{m+1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{j}{m+1}\frac{(m+j)!}{m!(j-1)!}\binom{n+k}{k}x^{j-1}y^k\\ &\quad+(1-x)^{m+1}(1-y)^{n+1}\frac{y}{n+1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{k}{n+1}\binom{m+j}{j}\frac{(n+k)!}{n!(k-1)!}x^j y^{k-1}\end{aligned}$$

$$= (1-x)^{m+1}(1-y)^{n+1}\frac{x}{m+1}\Big\{x\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(m+j+1)!}{(m+1)!(j-1)!}\binom{n+k}{k}x^{j-1}y^k$$

$$+\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{m+j+1}{j}\binom{n+k}{k}x^jy^k\Big\}$$

$$+(1-x)^{m+1}(1-y)^{n+1}\frac{y}{n+1}\Big\{y\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(n+k+1)!}{(n+1)!(k-1)!}\binom{m+j}{j}x^jy^{k-1}$$

$$+\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{n+k+1}{k}\binom{m+j}{j}x^jy^k\Big\}$$

$$= \frac{m+2}{m+1}\Big(\frac{x}{1-x}\Big)^2+\frac{1}{m+1}\frac{x}{1-x}+\frac{n+2}{n+1}\Big(\frac{y}{1-y}\Big)^2+\frac{1}{n+1}\frac{y}{1-y}$$

$$\to \Big(\frac{x}{1-x}\Big)^2+\Big(\frac{y}{1-y}\Big)^2.$$

Therefore, the conditions of Theorem A are satisfied, and we get for all $f \in H_\omega(K)$ that

$$P\text{-}\lim_{j,k\to\infty}\Big\|T_{j,k}(f;x,y)-f(x,y)\Big\|_{C(K)}=0.$$

Now, define $w=(w_{mn})$ by $w_{mn}=(-1)^m$ for all n. Take $\lambda_n=n$, $\mu_m=m$. Then, this sequence is neither P–convergent nor (λ,μ)-statistically convergent but it is statistically (λ,μ)-summable to 0 (since $(C,1,1)$–$\lim w=0$). Let $L_{m,n}:H_\omega(K)\to C(K)$ be defined by

$$L_{m,n}(f;x,y)=(1+w_{mn})B_{m,n}(f;x,y).$$

It is easy to see that the sequence $(L_{m,n})$ satisfies the conditions (3.2)–(3.5). Hence by Theorem 3.1, we have

$$(\lambda,\mu)_{st}\text{-}\text{-}\lim_{m,n\to\infty}\|L_{m,n}(f;x,y)-f(x,y)\|=0.$$

On the other hand, the sequence $(L_{m,n})$ does not satisfy the conditions of Theorem A and Theorem B, since $(L_{m,n})$ is neither P–convergent nor (λ,μ)-statistically convergent. That is, Theorem A and Theorem B do not work for our operators $L_{m,n}$. Hence, our Theorem 3.1 is stronger than Theorem A and Theorem B.

5. Conclusion

We introduced a new method of summability, namely, statistical (λ,μ)-summability and obtained its relation with (λ,μ)-statistical convergence. As an application of our method, we have used it to prove a Korovkin type approximation theorem for functions of two variables. Through Meyer-König and Zeller operators, we have shown that our result is stronger than the previous results proved for P–convergence and statistical convergence.

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Author's Biographical Note:

The author is a full Professor & Chairman, Department of Mathematics, Aligarh Muslim University, India. His research interests are in the areas of pure and applied mathematics including Approximation Theory, Summability Theory, Measures of Noncompactness, Fixed Point Theory, Operator Theory etc. He has published more than 260 research articles in reputed international journals. He is member of several scientific committees, advisory boards as well as member of editorial board of a number of scientific journals. He has visited a number of foreign universities/institutions as a visiting scientist/ visiting professor and delivered a number of talks. Recently, he has been awarded the Outstanding Researcher of the Year 2014 of Aligarh Muslim University.