Applications and Applied Mathematics: An International Journal (AAM)

Volume 12 | Issue 1 Article 1

6-2017

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Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466
Vol. 12, Issue 1 (June 2017), pp. 1 – 22

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An International Journal
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An M/G/1 Retrial Queue with Single Working Vacation

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Received: August 11, 2016; Accepted: April 7, 2017

Abstract

We consider an $M/G/1$ retrial queue with general retrial times and single working vacation. During the working vacation period, customers can be served at a lower rate. Both service times in a vacation period and in a service period are generally distributed random variables. Using supplementary variable method we obtain the probability generating function for the number of customers and the average number of customers in the orbit. Furthermore, we carry out the waiting time distribution and some special cases of interest are discussed. Finally, some numerical results are presented.

Keywords: Poisson arrivals; Retrial queues; Vacation model; Working vacation; Supplementary variable method; Steady state orbit size distribution; Probability generating function; Idle state and waiting time process

MSC 2010 No.: 60K25, 60K30

1. Introduction

If the server is found to be busy, arriving customers join a retrial queue (called orbit) and retry for service after some random amount of time. In a telephone switching system we do have this type of application, and hence, in the last two decades, retrial queues have been investigated extensively. Moreover, retrial queues are also used as mathematical models for several computer systems: packet switching networks, shared bus local area networks operating under the carrier-sense multiple access protocol and collision avoidance star local area networks, etc. For more recent references, see the bibliographical overviews in Artalejo (1999). Further, a comprehensive comparison between retrial queues and their standard counterpart with classical waiting line can be found in Artalejo and Falin (2002).

A large number of researchers working in various fields have analyzed retrial queues. For a detailed review of literature on retrial queues one may refer Falin (1980, 1990), Falin and Templeton (1997), Yang and Templeton (1987), Artalejo and Gomez-Corral (1998), Choo and Conolly (1979), Gomez-corral (1999), Renganathan et al. (2002), and Kalyanaraman and Srinivasan (2003, 2004).

Recently, queueing systems with vacations have been studied extensively, along with a comprehensive and excellent study on the vacation models, including some applications such as production/inventory system, communication systems, and computer systems. As we know, there are mainly two vacation policies: classical vacation policy (also called ordinary vacation) and working vacation policy. The characteristic of a working vacation is that the server serves customers at a lower service rate during the vacation period, but in the case of classical vacation, the server stops the service completely during the vacation period.

In the literature of queueing systems with vacations has been discussed through a considerable amount of work in the recent past. Doshi (1990) has recorded prior work on vacation models and their applications in his survey paper. In recent years few authors who were concentrated on vacation queues are Madan and Gautam Choudhury (2005), Kalyanaraman and Pazhani Bala Murugan (2008) and Thangaraj and Vanitha (2010).

Servi and Finn (2002) studied an $M/M/1$ queue with multiple working vacation and obtained the probability generating function for the number of customers in the system and the waiting time distribution. Some other notable works were done by Wu and Takagi (2006), Tian et al. (2008), Aftab Begum (2011), and Santhi and Pazhani Bala Murugan (2013, 2014, 2015).

In this paper we study an Non-Markovian retrial queue with single working vacation. The organization of the paper is as follows. In Section 2, we describe the model. In Section 3, we obtain the steady state probability generating function. Particular cases are discussed in Section 4. Some performance measures are obtained in Section 5, and in Section 6 numerical study is presented. An example of applicability of our model is referred in the manufacturing system. Assume that a manufacturing plant has a machine shared by all units (called customers) of the plant. The machines is operated by a skilled worker and his apprentice. The apprentice only operates the machine to serve the units when the skilled worker is on vacation (called working vacation) and the service rate of apprentice is usually lower than that of the skilled worker. If the machine is busy, a new arrival unit will sign up on a waiting line, which corresponds to the retrial queue. Otherwise, the unit is served immediately. After the completion of a service, the skilled worker will contact the next unit on the list unless another external unit arrives before the contact is made. The contact time is assumed to be generally distributed (which is called general retrial time). When the skilled worker finds no units in waiting line, he will need to rest from his work, that is go on vacation. During the skilled worker's vacation period, his apprentice will serve the units if any and after his service completion if there are units in the waiting line, the skilled worker will interrupt his vacation to begin serving the next unit. Meanwhile, if there are no units when a vacation ends, the skilled worker begins another vacation, otherwise, he takes over his apprentice. To guarantee his service quality, the skilled worker will restart his service no matter how long the apprentice has served the unit.

2. The Model description

We consider an $M/G/1$ queueing system where the primary customers arrive according to a Poisson process with arrival rate $\lambda(>0)$. We assume that there is no waiting space and therefore if an arriving customer (external or repeated) finds the server occupied, he leaves the service area and joins a pool of blocked customers called orbit. We will assume that only the customer at the head of the orbit is allowed to reach the server at a service completion instant. The retrial time follows a general distribution, with distribution functions $B(x)$. Let $b(x)$ and $B^*(\theta)$ denote the probability density function and Laplace Stieltjes Transform of $B(x)$ respectively for regular service period and let $a(x)$, $A^*(\theta)$ denote the probability density function and Laplace Stieltjes Transform of $A(x)$ respectively for working vacation period. Just after the completion of a service, if any customer is in orbit the next customer to gain service is determined by a competition between the primary customer and the orbit customer. The service time is assumed to follow general distribution, with distribution function $S_b(x)$ and density function $s_b(x)$. Let $S_b^*(\theta)$ be the Laplace Stieltjes Transform (LST) of the service time S_b.

Whenever the orbit becomes empty at a service completion instant the server starts a working vacation and the duration of the vacation time follows an exponential distribution with rate η. At a vacation completion instant if there are customers in the system the server will start a new busy period. Otherwise he waits until a customer arrive. This type of vacation policy is called single working vacation. During the working vacation period, the server provides service with service time S_v which follows a general distribution with distribution function $S_v(x)$. Let $s_v(x)$ be the probability density function and $S_v^*(\theta)$ be the Laplace Stieltjes Transform of $S_v(x)$. Further, it is noted that the service interrupted at the end of a vacation is lost and it is restarted with different distribution at the beginning of the following service period. We assume that inter-arrival times, service times, working vacation times and a retrial times are mutually independent.

We define the following random variables:

$X(t)$ - the orbit size at time t,
$S_b^0(t)$ - the remaining service time in regular service period,
$S_v^0(t)$ - the remaining service time in WV period,
$A^0(t)$ - the remaining retrial time in WV period,
$B^0(t)$ - the remaining retrial time in regular service period,

$$Y(t) = \begin{cases} 0, & \text{if the server is on WV period at time } t \text{ but not occupied,} \\ 1, & \text{if the server is in regular service period at time } t \text{ but not occupied,} \\ 2, & \text{if the server is busy on WV period at time } t, \\ 3, & \text{if the server is busy in regular service period at time } t, \end{cases}$$

so that the supplementary variables $S_b^0(t), S_v^0(t), A^0(t)$ and $B^0(t)$ are introduced in order to obtain the bivariate Markov Process $\{N(t), \partial(t); t \geq 0\}$, where

$$\partial(t) = \begin{cases} A^0(t), & \text{if } Y(t) = 0, \\ B^0(t), & \text{if } Y(t) = 1, \\ S_v^0(t), & \text{if } Y(t) = 2, \\ S_b^0(t), & \text{if } Y(t) = 3. \end{cases}$$

We define the following limiting probabilities:

$$\begin{aligned} Q_{0,0} &= \lim_{t\to\infty} \Pr\{X(t) = 0, Y(t) = 0\}, \\ P_{0,0} &= \lim_{t\to\infty} Pr\{X(t) = 0, Y(t) = 1\}, \\ Q_{0,n} &= \lim_{t\to\infty} \Pr\{X(t) = n, Y(t) = 0, x < A^0(t) \leq x + dx\} \ ; \ n \geq 1, \\ P_{0,n} &= \lim_{t\to\infty} \Pr\{X(t) = n, Y(t) = 1, x < B^0(t) \leq x + dx\} \ ; \ n \geq 1, \\ Q_{1,n} &= \lim_{t\to\infty} \Pr\{X(t) = n, Y(t) = 2, x < S_v^0(t) \leq x + dx\} \ ; \ n \geq 0, \\ P_{1,n} &= \lim_{t\to\infty} \Pr\{X(t) = n, Y(t) = 3, x < S_b^0(t) \leq x + dx\} \ ; \ n \geq 0. \end{aligned}$$

We define the LST and the probability generating functions as follows:

$$\begin{aligned} S_b^*(\theta) &= \int_0^\infty e^{-\theta x} s_b(x) dx \ ; \qquad S_v^*(\theta) = \int_0^\infty e^{-\theta x} s_v(x) dx \ ; \\ A^*(\theta) &= \int_0^\infty e^{-\theta x} a(x) dx \ ; \qquad B^*(\theta) = \int_0^\infty e^{-\theta x} b(x) dx \ ; \\ Q_{0,n}^*(\theta) &= \int_0^\infty e^{-\theta x} Q_{0,n}(x) dx \ ; \qquad Q_{0,n}^*(0) = \int_0^\infty Q_{0,n}(x) dx \ ; \\ Q_{1,n}^*(\theta) &= \int_0^\infty e^{-\theta x} Q_{1,n}(x) dx \ ; \qquad Q_{1,n}^*(0) = \int_0^\infty Q_{1,n}(x) dx \ ; \\ P_{0,n}^*(\theta) &= \int_0^\infty e^{-\theta x} P_{0,n}(x) dx \ ; \qquad P_{0,n}^*(0) = \int_0^\infty P_{0,n}(x) dx \ ; \\ Q_0^*(z,\theta) &= \sum_{n=1}^\infty Q_{0,n}^*(\theta) z^n \ ; \qquad Q_0^*(z,0) = \sum_{n=1}^\infty Q_{0,n}^*(0) z^n \ ; \end{aligned}$$

$$
\begin{aligned}
Q_0(z,0) &= \sum_{n=1}^{\infty} Q_{0,n}(0)z^n \ ; & Q_1^*(z,\theta) &= \sum_{n=0}^{\infty} Q_{1,n}^*(\theta)z^n \ ; \\
Q_1^*(z,0) &= \sum_{n=0}^{\infty} Q_{1,n}^*(0)z^n \ ; & Q_1(z,0) &= \sum_{n=0}^{\infty} Q_{1,n}(0)z^n \ ; \\
P_0^*(z,\theta) &= \sum_{n=1}^{\infty} P_{0,n}^*(\theta)z^n \ ; & P_0^*(z,0) &= \sum_{n=1}^{\infty} P_{0,n}^*(0)z^n \ ; \\
P_0(z,0) &= \sum_{n=1}^{\infty} P_{0,n}(0)z^n \ ; & P_1^*(z,\theta) &= \sum_{n=0}^{\infty} P_{1,n}^*(\theta)z^n; \\
P_1^*(z,0) &= \sum_{n=0}^{\infty} P_{1,n}^*(0)z^n \ ; & P_1(z,0) &= \sum_{n=0}^{\infty} P_{1,n}(0)z^n \ .
\end{aligned}
$$

3. The Orbit Size Distribution for Single Working Vacation

By considering the steady state, we have the following system of the differential difference equations:

$$
\begin{aligned}
(\lambda+\eta)Q_{0,0} &= P_{1,0}(0)+Q_{1,0}(0), && (1)\\
-\frac{d}{dx}Q_{0,n}(x) &= -(\lambda+\eta)Q_{0,n}(x)+Q_{1,n}(0)a(x) \ ; \quad n\ge 1, && (2)\\
-\frac{d}{dx}Q_{1,0}(x) &= -(\lambda+\eta)Q_{1,0}(x)+Q_{0,1}(0)s_v(x)+\lambda Q_{0,0}s_v(x), && (3)\\
-\frac{d}{dx}Q_{1,n}(x) &= -(\lambda+\eta)Q_{1,n}(x)+\lambda Q_{1,n-1}(x)+Q_{0,n+1}(0)s_v(x) \\
&\quad +\lambda s_v(x)\int_0^{\infty} Q_{0,n}(x)dx \ ; \quad n\ge 1, && (4)\\
\lambda P_{0,0} &= \eta Q_{0,0}, && (5)\\
-\frac{d}{dx}P_{0,n}(x) &= -\lambda P_{0,n}(x)+P_{1,n}(0)b(x)+\eta b(x)\int_0^{\infty} Q_{0,n}(x)dx \ ; \quad n\ge 1, && (6)\\
-\frac{d}{dx}P_{1,0}(x) &= -\lambda P_{1,0}(x)+P_{0,1}(0)s_b(x)+\lambda P_{0,0}s_b(x)+\eta s_b(x)\int_0^{\infty} Q_{1,0}(x)dx, && (7)\\
-\frac{d}{dx}P_{1,n}(x) &= -\lambda P_{1,n}(x)+\lambda P_{1,n-1}(x)+P_{0,n+1}(0)s_b(x), \\
&\quad +\eta s_b(x)\int_0^{\infty} Q_{1,n}(x)dx+\lambda s_b(x)\int_0^{\infty} P_{0,n}(x)dx \ ; \quad n\ge 1. && (8)
\end{aligned}
$$

Taking the LST on both sides of the equations (2) to (4) and from (6) to (8), we get

$$-\int_0^\infty e^{-\theta x} dQ_{0,n}(x) = -(\lambda+\eta)\int_0^\infty e^{-\theta x} Q_{0,n}(x)dx + Q_{1,n}(0)\int_0^\infty e^{-\theta x} a(x)dx,$$

$$\theta Q^*_{0,n}(\theta) - Q_{0,n}(0) = (\lambda+\eta)Q^*_{0,n}(\theta) - Q_{1,n}(0)A^*(\theta) \; ; \quad n \geq 1. \tag{9}$$

$$-\int_0^\infty e^{-\theta x} dQ_{1,0}(x) = -(\lambda+\eta)\int_0^\infty e^{-\theta x} Q_{1,0}(x)dx + Q_{0,1}(0)\int_0^\infty e^{-\theta x} s_v(x)dx + \lambda Q_{0,0}\int_0^\infty e^{-\theta x} s_v(x)dx,$$

$$\theta Q^*_{1,0}(\theta) - Q_{1,0}(0) = (\lambda+\eta)Q^*_{1,0}(\theta) - Q_{0,1}(0)S^*_v(\theta) - \lambda Q_{0,0}S^*_v(\theta). \tag{10}$$

$$-\int_0^\infty e^{-\theta x} dQ_{1,n}(x) = -(\lambda+\eta)\int_0^\infty e^{-\theta x} Q_{1,n}(x)dx + \lambda\int_0^\infty e^{-\theta x} Q_{1,n-1}(x)dx + Q_{0,n+1}(0)\int_0^\infty e^{-\theta x} s_v(x)dx + \lambda\int_0^\infty Q_{0,n}(x)dx\int_0^\infty e^{-\theta x} s_v(x)dx,$$

$$\theta Q^*_{1,n}(\theta) - Q_{1,n}(0) = (\lambda+\eta)Q^*_{1,n}(\theta) - \lambda Q^*_{1,n-1}(\theta) - Q_{0,n+1}(0)S^*_v(\theta) - \lambda Q^*_{0,n}(0)S^*_v(\theta) \; ; \quad n \geq 1. \tag{11}$$

$$-\int_0^\infty e^{-\theta x} dP_{0,n}(x) = -\lambda\int_0^\infty e^{-\theta x} P_{0,n}(x)dx + P_{1,n}(0)\int_0^\infty e^{-\theta x} b(x)dx + \eta\int_0^\infty Q_{0,n}(x)dx\int_0^\infty e^{-\theta x} b(x)dx,$$

$$\theta P^*_{0,n}(\theta) - P_{0,n}(0) = \lambda P^*_{0,n}(\theta) - P_{1,n}(0)B^*(\theta) - \eta B^*(\theta)Q^*_{0,n}(0) \; ; \quad n \geq 1. \tag{12}$$

$$-\int_0^\infty e^{-\theta x} dP_{1,0}(x) = -\lambda\int_0^\infty e^{-\theta x} P_{1,0}(x)dx + P_{0,1}(0)\int_0^\infty e^{-\theta x} s_b(x)dx + \lambda P_{0,0}\int_0^\infty e^{-\theta x} s_b(x)dx + \eta\int_0^\infty e^{-\theta x} s_b(x)dx\int_0^\infty Q_{1,0}(x)dx,$$

$$\theta P^*_{1,0}(\theta) - P_{1,0}(0) = \lambda P^*_{1,0}(\theta) - P_{0,1}(0)S^*_b(\theta) - \lambda P_{0,0}S^*_b(\theta) - \eta S^*_b(\theta)Q^*_{1,0}(0). \tag{13}$$

$$
\begin{aligned}
-\int_0^\infty e^{-\theta x} dP_{1,n}(x) &= -\lambda \int_0^\infty e^{-\theta x} P_{1,n}(x)dx + \lambda \int_0^\infty e^{-\theta x} P_{1,n-1}(x)dx \\
&+ P_{0,n+1}(0) \int_0^\infty e^{-\theta x} s_b(x)dx + \eta \int_0^\infty e^{-\theta x} s_b(x)dx \int_0^\infty Q_{1,n}(x)dx \\
&+ \lambda \int_0^\infty e^{-\theta x} s_b(x)dx \int_0^\infty P_{0,n}(x)dx,
\end{aligned}
$$

$$
\begin{aligned}
\theta P^*_{1,n}(\theta) - P_{1,n}(0) &= \lambda P^*_{1,n}(\theta) - \lambda P^*_{1,n-1}(\theta) - P_{0,n+1}(0) S^*_b(\theta) \\
&- \eta S^*_b(\theta) Q^*_{1,n}(0) - \lambda S^*_b(\theta) P^*_{0,n}(0) \ ; \quad n \geq 1.
\end{aligned} \tag{14}
$$

Multiplying (9) with z^n and summed over n from 1 to ∞, we get

$$
\theta \sum_{n=1}^{\infty} Q^*_{0,n}(\theta) z^n - \sum_{n=1}^{\infty} Q_{0,n}(0) z^n = (\lambda + \eta) \sum_{n=1}^{\infty} Q^*_{0,n}(\theta) z^n - A^*(\theta) \sum_{n=1}^{\infty} Q_{1,n}(0) z^n,
$$

$$
[\theta - (\lambda + \eta)] Q^*_0(z, \theta) = Q_0(z, 0) - A^*(\theta) Q_1(z, 0) + A^*(\theta) Q_{1,0}(0). \tag{15}
$$

z^n times (11) summed over n from 1 to ∞ and added up with (10) gives

$$
\begin{aligned}
&\theta \sum_{n=0}^{\infty} Q^*_{1,n}(\theta) z^n - \sum_{n=0}^{\infty} Q_{1,n}(0) z^n = (\lambda + \eta) \sum_{n=0}^{\infty} Q^*_{1,n}(\theta) z^n - \lambda \sum_{n=1}^{\infty} Q^*_{1,n-1}(\theta) z^n \\
&- S^*_v(\theta) \sum_{n=1}^{\infty} Q_{0,n+1}(0) z^n - Q_{0,1}(0) S^*_v(\theta) - \lambda Q_{0,0} S^*_v(\theta) - \lambda S^*_v(\theta) \sum_{n=1}^{\infty} Q^*_{0,n}(0) z^n,
\end{aligned}
$$

and therefore,

$$
[\theta - (\lambda - \lambda z + \eta)] Q^*_1(z, \theta) = Q_1(z, 0) - \left[\frac{S^*_v(\theta)}{z}\right] Q_0(z, 0) - \lambda Q_{0,0} S^*_v(\theta) - \lambda S^*_v(\theta) Q^*_0(z, 0). \tag{16}
$$

Inserting $\theta = (\lambda + \eta)$ in (15), we get

$$
Q_0(z, 0) = A^*(\lambda + \eta)[Q_1(z, 0) - Q_{1,0}(0)]. \tag{17}
$$

Inserting $\theta = 0$ and substituting (17) in (15), we get

$$
Q^*_0(z, 0) = \frac{(1 - A^*(\lambda + \eta))(Q_1(z, 0) - Q_{1,0}(0))}{\lambda + \eta}. \tag{18}
$$

Inserting $\theta = (\lambda - \lambda z + \eta)$ and substituting (17) and (18) in (16), we get

$$
Q_1(z, 0) = \frac{S^*_v(\lambda - \lambda z + \eta)[\lambda z(\lambda + \eta) Q_{0,0} - (A^*(\lambda + \eta)(\lambda - \lambda z + \eta) + \lambda z) Q_{1,0}(0)]}{z(\lambda + \eta) - S^*_v(\lambda - \lambda z + \eta)(A^*(\lambda + \eta)(\lambda - \lambda z + \eta) + \lambda z)}. \tag{19}
$$

Substituting (19) in (17), we get

$$
Q_0(z, 0) = \frac{z A^*(\lambda + \eta)(\lambda + \eta)[\lambda S^*_v(\lambda - \lambda z + \eta) Q_{0,0} - Q_{1,0}(0)]}{z(\lambda + \eta) - S^*_v(\lambda - \lambda z + \eta)(A^*(\lambda + \eta)(\lambda - \lambda z + \eta) + \lambda z)}. \tag{20}
$$

Letting

$$f(z) = z(\lambda+\eta) - S_v^*(\lambda-\lambda z+\eta)(A^*(\lambda+\eta)(\lambda-\lambda z+\eta)+\lambda z),$$

we find $f(0) < 0$ and $f(1) > 0$. This implies that there exist a real root $z_1 \in (0,1)$ for the equation $f(z) = 0$. Hence, at $z = z_1$, the equation (20) becomes

$$Q_{1,0}(0) = \lambda S_v^*(\lambda-\lambda z_1+\eta)Q_{0,0}. \tag{21}$$

Substituting (21) in (19), we get

$$Q_1(z,0) = \frac{\lambda S_v^*(\lambda-\lambda z+\eta)\big[z(\lambda+\eta) - S_v^*(\lambda-\lambda z_1+\eta)\times\big(\lambda z + A^*(\lambda+\eta)(\lambda-\lambda z+\eta)\big)\big]}{z(\lambda+\eta) - S_v^*(\lambda-\lambda z+\eta)(A^*(\lambda+\eta)(\lambda-\lambda z+\eta)+\lambda z)}Q_{0,0}. \tag{22}$$

Substituting (21) in (20), we get

$$Q_0(z,0) = \frac{\big\{\lambda z A^*(\lambda+\eta)(\lambda+\eta)[S_v^*(\lambda-\lambda z+\eta) - S_v^*(\lambda-\lambda z_1+\eta)]\big\}Q_{0,0}}{z(\lambda+\eta) - S_v^*(\lambda-\lambda z+\eta)(A^*(\lambda+\eta)(\lambda-\lambda z+\eta)+\lambda z)}. \tag{23}$$

Substituting (21) and (22) in (18), we get

$$Q_0^*(z,0) = \frac{\big\{\lambda z(1-A^*(\lambda+\eta))[S_v^*(\lambda-\lambda z+\eta) - S_v^*(\lambda-\lambda z_1+\eta)]\big\}Q_{0,0}}{z(\lambda+\eta) - S_v^*(\lambda-\lambda z+\eta)(A^*(\lambda+\eta)(\lambda-\lambda z+\eta)+\lambda z)}. \tag{24}$$

Inserting $\theta = 0$ and substituting (22), (23) and (24) in (16), we get

$$Q_1^*(z,0) = \frac{\left\{\begin{array}{r}\lambda(1-S_v^*(\lambda-\lambda z+\eta))\big[z(\lambda+\eta) - S_v^*(\lambda-\lambda z_1+\eta)\\ \times(\lambda z + A^*(\lambda+\eta)(\lambda-\lambda z+\eta))\big]Q_{0,0}\end{array}\right\}}{\left\{\begin{array}{r}(\lambda-\lambda z+\eta)\big[z(\lambda+\eta) - S_v^*(\lambda-\lambda z+\eta)\\ \\ \times(A^*(\lambda+\eta)(\lambda-\lambda z+\eta)+\lambda z)\big]\end{array}\right\}}. \tag{25}$$

Multiplying (12) with z^n and summed over n from 1 to ∞, we get

$$\begin{aligned}\theta\sum_{n=1}^{\infty}P_{0,n}^*(\theta)z^n - \sum_{n=1}^{\infty}P_{0,n}(0)z^n &= \lambda\sum_{n=1}^{\infty}P_{0,n}^*(\theta)z^n - B^*(\theta)\sum_{n=1}^{\infty}P_{1,n}(0)z^n\\ &\quad -\eta B^*(\theta)\sum_{n=1}^{\infty}Q_{0,n}^*(0)z^n,\end{aligned}$$

$$(\theta-\lambda)P_0^*(z,\theta) = P_0(z,0) - B^*(\theta)[P_1(z,0) - P_{1,0}(0)] - \eta B^*(\theta)Q_0^*(z,0). \tag{26}$$

Substituting $Q_{1,0}(0) = \lambda S_v^*(\lambda - \lambda z_1 + \eta)Q_{0,0}$ in (1), we get

$$P_{1,0}(0) = [\lambda(1 - S_v^*(\lambda - \lambda z_1 + \eta)) + \eta]Q_{0,0}. \quad (27)$$

Inserting $\theta = \lambda$ and substituting $P_{1,0}(0) = [\lambda(1 - S_v^*(\lambda - \lambda z_1 + \eta)) + \eta]Q_{0,0}$ in (26),we get

$$P_0(z, 0) = B^*(\lambda)[P_1(z, 0) - [\lambda(1 - S_v^*(\lambda - \lambda z_1 + \eta)) + \eta]Q_{0,0} + \eta Q_0^*(z, 0)]. \quad (28)$$

z^n times (14) is summed over n from 1 to ∞ and added up with (13), we get

$$\begin{aligned} \theta\sum_{n=0}^{\infty} P_{1,n}^*(\theta)z^n - \sum_{n=0}^{\infty} P_{1,n}(0)z^n &= \lambda\sum_{n=0}^{\infty} P_{1,n}^*(\theta)z^n - S_b^*(\theta)\sum_{n=0}^{\infty} P_{0,n+1}(0)z^n - \lambda P_{0,0}S_b^*(\theta) \\ &\quad -\lambda\sum_{n=1}^{\infty} P_{1,n-1}^*(\theta)z^n - \eta S_b^*(\theta)\sum_{n=0}^{\infty} Q_{1,n}^*(0)z^n \\ &\quad -\lambda S_b^*(\theta)\sum_{n=1}^{\infty} P_{0,n}^*(0)z^n, \\ [\theta - (\lambda - \lambda z)]P_1^*(z, \theta) &= P_1(z, 0) - \frac{S_b^*(\theta)}{z}P_0(z, 0) - \lambda P_{0,0}S_b^*(\theta) \\ &\quad -\eta S_b^*(\theta)Q_1^*(z, 0) - \lambda S_b^*(\theta)P_0^*(z, 0). \end{aligned} \quad (29)$$

Inserting $\theta = 0$ and substituting (27) and $P_{1,0}(0) = [\lambda(1 - S_v^*(\lambda - \lambda z_1 + \eta)) + \eta]Q_{0,0}$ in (26), we get

$$P_0^*(z, 0) = \frac{(1 - B^*(\lambda))}{\lambda}[P_1(z, 0) - (\lambda(1 - S_v^*(\lambda - \lambda z_1 + \eta)) + \eta)Q_{0,0} + \eta Q_0^*(z, 0)]. \quad (30)$$

Inserting $\theta = (\lambda - \lambda z)$ and substituting (27) and (29) in (28) and also using (5), we get

$$P_1(z, 0) = \frac{\left\{\begin{array}{r} S_b^*(\lambda - \lambda z)\Big[\big[\eta z - (z + (1 - z)B^*(\lambda))(\lambda(1 - S_v^*(\lambda - \lambda z_1 + \eta)) + \eta)\big]Q_{0,0} \\ +\eta z Q_1^*(z, 0) + \eta(z + B^*(\lambda)(1 - z))Q_0^*(z, 0)\Big] \end{array}\right\}}{z - S_b^*(\lambda - \lambda z)(z + (1 - z)B^*(\lambda))}. \quad (31)$$

Substituting (24), (25) and (30) in (27), we get

$$\begin{aligned} P_0(z, 0) &= \frac{zB^*(\lambda)Q_{0,0}}{(\lambda - \lambda z + \eta)D_1(z)D_2(z)}\big\{(\lambda - \lambda z + \eta)[z(\lambda + \eta) \\ &\quad -S_v^*(\lambda - \lambda z + \eta)(A^*(\lambda + \eta)(\lambda - \lambda z + \eta) + \lambda z))] \\ &\quad \times(\eta(S_b^*(\lambda - \lambda z) - 1) - \lambda(1 - S_v^*(\lambda - \lambda z_1 + \eta))) \\ &\quad +\eta\lambda\big\{S_b^*(\lambda - \lambda z)(1 - S_v^*(\lambda - \lambda z + \eta)) \\ &\quad \times[z(\lambda + \eta) - S_v^*(\lambda - \lambda z_1 + \eta)(A^*(\lambda + \eta) \\ &\quad \times(\lambda - \lambda z + \eta) + \lambda z)] + z(\lambda - \lambda z + \eta) \\ &\quad \times(1 - A^*(\lambda + \eta))(S_v^*(\lambda - \lambda z + \eta) - S_v^*(\lambda - \lambda z_1 + \eta))\big\}\big\}, \end{aligned} \quad (32)$$

where

$$D_1(z) = z(\lambda + \eta) - S_v^*(\lambda - \lambda z + \eta)(A^*(\lambda + \eta)(\lambda - \lambda z + \eta) + \lambda z), \quad (33)$$

$$D_2(z) = z - S_b^*(\lambda - \lambda z)(z + (1-z)B^*(\lambda)). \tag{34}$$

Substituting (24), (25) and (30) in (29), we get

$$\begin{aligned} P_0^*(z,0) &= \frac{z(1-B^*(\lambda))Q_{0,0}}{\lambda(\lambda - \lambda z + \eta)D_1(z)D_2(z)}\Big\{(\eta(S_b^*(\lambda - \lambda z) - 1) \\ &\quad -\lambda(1 - S_v^*(\lambda - \lambda_1 + \eta))) \\ &\quad \times(\lambda - \lambda z + \eta)[z(\lambda+\eta) - S_v^*(\lambda - \lambda z + \eta) \\ &\quad \times(A^*(\lambda+\eta)(\lambda - \lambda z + \eta) + \lambda z)] \\ &\quad +\eta\lambda\big\{S_b^*(\lambda - \lambda z)(1 - S_v^*(\lambda - \lambda z + \eta))[z(\lambda+\eta) \\ &\quad -S_v^*(\lambda - \lambda z_1 + \eta)(A^*(\lambda+\eta)(\lambda - \lambda z + \eta) + \lambda z)] \\ &\quad +z(\lambda - \lambda z + \eta)(1 - A^*(\lambda+\eta))(S_v^*(\lambda - \lambda z + \eta) \\ &\quad -S_v^*(\lambda - \lambda z_1 + \eta))\big\}\Big\}, \end{aligned} \tag{35}$$

where $D_1(z)$ and $D_2(z)$ are given in (33) and (34) respectively. Substituting (24) and (25) in (30), we get

$$\begin{aligned} P_1(z,0) &= \frac{S_b^*(\lambda - \lambda z)Q_{0,0}}{(\lambda - \lambda z + \eta)D_1(z)D_2(z)}\Big\{\eta\lambda z\big\{(1 - S_v^*(\lambda - \lambda z + \eta)) \\ &\quad \times\big[z(\lambda+\eta) - S_v^*(\lambda - \lambda z_1 + \eta)(A^*(\lambda+\eta)(\lambda - \lambda z + \eta) + \lambda z)\big] \\ &\quad +(1 - A^*(\lambda+\eta))(\lambda - \lambda z + \eta)(z + B^*(\lambda)(1-z)) \\ &\quad \times(S_v^*(\lambda - \lambda z + \eta) - S_v^*(\lambda - \lambda z_1 + \eta))\big\} \\ &\quad -(\lambda - \lambda z + \eta)\big[z(\lambda+\eta) - S_v^*(\lambda - \lambda z + \eta) \\ &\quad \times(A^*(\lambda+\eta)(\lambda - \lambda z + \eta) + \lambda z)\big] \\ &\quad \times\big[\eta(1-z)B^*(\lambda) + \lambda(1 - S_v^*(\lambda - \lambda z_1 + \eta)) \\ &\quad \times(z + (1-z)B^*(\lambda))\big]\Big\}, \end{aligned} \tag{36}$$

where $D_1(z)$ and $D_2(z)$ are given in (33) and (34) respectively. Inserting $\theta = 0$ in (28) and also using (5), we get

$$-(\lambda - \lambda z)P_1^*(z,0) = P_1(z,0) - \frac{P_0(z,0)}{z} - \eta Q_{0,0} - \eta Q_1^*(z,0) - \lambda P_0^*(z,0). \tag{37}$$

Substituting (25), (31), (34) and (35) in (36), we get

$$\begin{aligned} P_1^*(z,0) &= \frac{(1 - S_b^*(\lambda - \lambda z))Q_{0,0}}{(\lambda - \lambda z)(\lambda - \lambda z + \eta)D_1(z)D_2(z)}\Big\{\eta\lambda z\big\{(\lambda - \lambda z + \eta) \times (1 - A^*(\lambda+\eta)) \\ &\quad \times(S_v^*(\lambda - \lambda z + \eta) - S_v^*(\lambda - \lambda z_1 + \eta)) \times (z + B^*(\lambda)(1-z)) \\ &\quad +(1 - S_v^*(\lambda - \lambda z + \eta)) \times \big[z(\lambda+\eta) - S_v^*(\lambda - \lambda z_1 + \eta) \times (\lambda z + (\lambda - \lambda z + \eta) \\ &\quad \times A^*(\lambda+\eta))\big]\big\} - (\lambda - \lambda z + \eta)\big[\eta B^*(\lambda)(1-z) + \lambda(z + B^*(\lambda)(1-z)) \\ &\quad (1 - S_v^*(\lambda - \lambda z_1 + \eta))\big] \times \big[z(\lambda+\eta) - S_v^*(\lambda - \lambda z + \eta)(A^*(\lambda+\eta) \\ &\quad \times(\lambda - \lambda z + \eta) + \lambda z)\big]\Big\}, \end{aligned} \tag{38}$$

where $D_1(z)$ and $D_2(z)$ are given in (33) and (34) respectively. We define

$$P_V(z) = Q_0^*(z,0) + Q_1^*(z,0) + Q_{0,0},$$

$$\begin{aligned} P_V(z) \;=\; & \frac{Q_{0,0}}{(\lambda-\lambda z+\eta)D_1(z)}\Big\{(\lambda-\lambda z+\eta)[\lambda z(1-A^*(\lambda+\eta))\times(S_v^*(\lambda-\lambda z+\eta) \\ & -S_v^*(\lambda-\lambda z_1+\eta))]+\lambda(1-S_v^*(\lambda-\lambda z+\eta))[z(\lambda+\eta)-S_v^*(\lambda-\lambda z_1+\eta) \\ & \times(\lambda z+(\lambda-\lambda z+\eta)A^*(\lambda+\eta))]+(\lambda-\lambda z+\eta)[z(\lambda+\eta)-S_v^*(\lambda-\lambda z+\eta) \\ & \times(\lambda z+A^*(\lambda+\eta)(\lambda-\lambda z+\eta))]\Big\} \end{aligned} \tag{39}$$

as the probability generating function for the number of customers in the orbit when the server is on working vacation period where $D_1(z)$ is given in (33), then $P_B(z) = P_0^*(z,0)+P_1^*(z,0)+P_{0,0}$ becomes

$$\begin{aligned} P_B(z) \;=\; & \frac{Q_{0,0}}{\lambda(\lambda-\lambda z)(\lambda-\lambda z+\eta)D_1(z)D_2(z)}\Big\{z(1-B^*(\lambda))(\lambda-\lambda z) \\ & \times\Big\{(\eta(S_b^*(\lambda-\lambda z)-1)-\lambda(1-S_v^*(\lambda-\lambda z_1+\eta))) \\ & \times(\lambda-\lambda z+\eta)[z(\lambda+\eta)-S_v^*(\lambda-\lambda z+\eta)(A^*(\lambda+\eta) \\ & \times(\lambda-\lambda z+\eta)+\lambda z)]+\eta\lambda\big\{S_b^*(\lambda-\lambda z)(1-S_v^*(\lambda-\lambda z+\eta)) \\ & \times[z(\lambda+\eta)-S_v^*(\lambda-\lambda z_1+\eta)(A^*(\lambda+\eta)(\lambda-\lambda z+\eta)+\lambda z)] \\ & +z(\lambda-\lambda z+\eta)(1-A^*(\lambda+\eta))(S_v^*(\lambda-\lambda z+\eta)-S_v^*(\lambda-\lambda z_1+\eta))\big\}\Big\} \\ & +\lambda(1-S_b^*(\lambda-\lambda z))\Big\{\eta\lambda z\big\{(\lambda-\lambda z+\eta)(1-A^*(\lambda+\eta)) \\ & \times(S_v^*(\lambda-\lambda z+\eta)-S_v^*(\lambda-\lambda z_1+\eta)) \\ & \times(z+B^*(\lambda)(1-z))+(1-S_v^*(\lambda-\lambda z+\eta)) \\ & \times\big[z(\lambda+\eta)-S_v^*(\lambda-\lambda z_1+\eta)(\lambda z+(\lambda-\lambda z+\eta)A^*(\lambda+\eta))\big]\big\} \\ & -(\lambda-\lambda z+\eta)\big[\eta B^*(\lambda)(1-z)+\lambda(z+B^*(\lambda) \\ & \times(1-z))(1-S_v^*(\lambda-\lambda z_1+\eta))\big]\big[z(\lambda+\eta)-S_v^*(\lambda-\lambda z+\eta) \\ & \times(A^*(\lambda+\eta)(\lambda-\lambda z+\eta)+\lambda z)\big]\Big\}+\eta(\lambda-\lambda z)(\lambda-\lambda z+\eta) \\ & \times[z(\lambda+\eta)-S_v^*(\lambda-\lambda z+\eta)(\lambda z+(\lambda-\lambda z+\eta)A^*(\lambda+\eta))] \\ & \times[z-S_b^*(\lambda-\lambda z)(z+(1-z)B^*(\lambda))]\Big\} \end{aligned} \tag{40}$$

as the probability generating function for the number of customers in the orbit when the server is regular service period where $D_1(z)$ and $D_2(z)$ are given in (33) and (34) respectively. Again, we define

$$P(z) = P_B(z) + P_V(z),$$

$$\begin{aligned}
P(z) &= \frac{Q_{0,0}}{\lambda(\lambda-\lambda z)(\lambda-\lambda z+\eta)D_1(z)D_2(z)}\Big\{z(1-B^*(\lambda))(\lambda-\lambda z)\\
&\times\Big\{(\eta(S_b^*(\lambda-\lambda z)-1)-\lambda(1-S_v^*(\lambda-\lambda z_1+\eta)))(\lambda-\lambda z+\eta)\\
&\times[z(\lambda+\eta)-S_v^*(\lambda-\lambda z+\eta)(A^*(\lambda+\eta)(\lambda-\lambda z+\eta)+\lambda z)]\\
&+\eta\lambda\big\{S_b^*(\lambda-\lambda z)(1-S_v^*(\lambda-\lambda z+\eta))[z(\lambda+\eta)\\
&-S_v^*(\lambda-\lambda z_1+\eta)(A^*(\lambda+\eta)(\lambda-\lambda z+\eta)+\lambda z)]+z(\lambda-\lambda z+\eta)\\
&\times(1-A^*(\lambda+\eta))(S_v^*(\lambda-\lambda z+\eta)-S_v^*(\lambda-\lambda z_1+\eta))\big\}\Big\}\\
&+\lambda(1-S_b^*(\lambda-\lambda z))\Big\{\eta\lambda z\big\{(\lambda-\lambda z+\eta)\\
&\times(1-A^*(\lambda+\eta))(S_v^*(\lambda-\lambda z+\eta)-S_v^*(\lambda-\lambda z_1+\eta))\\
&\times(z+B^*(\lambda)(1-z))+(1-S_v^*(\lambda-\lambda z+\eta))\big[z(\lambda+\eta)-S_v^*(\lambda-\lambda z_1+\eta)\\
&\times(\lambda z+(\lambda-\lambda z+\eta)A^*(\lambda+\eta))\big]\big\}-(\lambda-\lambda z+\eta)\big[\eta B^*(\lambda)(1-z)\\
&+\lambda(z+B^*(\lambda)(1-z))(1-S_v^*(\lambda-\lambda z_1+\eta))\big]\\
&\times\big[z(\lambda+\eta)-S_v^*(\lambda-\lambda z+\eta)(A^*(\lambda+\eta)(\lambda-\lambda z+\eta)+\lambda z)\big]\Big\}\\
&+\eta(\lambda-\lambda z)(\lambda-\lambda z+\eta)[z(\lambda+\eta)-S_v^*(\lambda-\lambda z+\eta)\\
&\times(\lambda z+(\lambda-\lambda z+\eta)A^*(\lambda+\eta))][z-S_b^*(\lambda-\lambda z)(z+(1-z)B^*(\lambda))]\Big\}\\
&+\Big\{(\lambda-\lambda z+\eta)[\lambda z(1-A^*(\lambda+\eta))(S_v^*(\lambda-\lambda z+\eta)-S_v^*(\lambda-\lambda z_1+\eta))]\\
&+\lambda(1-S_v^*(\lambda-\lambda z+\eta))[z(\lambda+\eta)-S_v^*(\lambda-\lambda z_1+\eta)\\
&\times(\lambda z+(\lambda-\lambda z+\eta)A^*(\lambda+\eta))]\\
&+(\lambda-\lambda z+\eta)[z(\lambda+\eta)-S_v^*(\lambda-\lambda z+\eta)(\lambda z+A^*(\lambda+\eta)(\lambda-\lambda z+\eta))]\Big\}\\
&\times\big\{\lambda(\lambda-\lambda z)[z-S_b^*(\lambda-\lambda z)(z+(1-z)B^*(\lambda))]\big\}\Big\}
\end{aligned}\tag{41}$$

as the probability generating function for the number of customers in the orbit where $D_1(z)$ and $D_2(z)$ are given in (33) and (34) respectively. We shall now use the normalizing condition $P(1) = 1$ to determine the unknown $Q_{0,0}$ which appears in (40). Substituting $z = 1$ in (40) and using L'Hospitals rule, we obtain

$$Q_{0,0} = \frac{1-\rho_b}{\left\{\dfrac{\left\{\begin{array}{c}(1-S_v^*(\eta)A^*(\lambda+\eta))[\eta^3B^*(\lambda)+\eta\lambda^2]+(1-S_v^*(\eta))\\ \times[\lambda^3+\eta^2\lambda B^*(\lambda)]+\eta\lambda[(\eta+\lambda)B^*(\lambda)-S_v^*(\eta)(\lambda+\eta A^*(\lambda+\eta))]\end{array}\right\}}{\big[\eta\lambda B^*(\lambda)[\lambda+\eta-S_v^*(\eta)(\lambda+\eta A^*(\lambda+\eta))]\big]}\right\}}$$

$$-\left[\frac{\lambda S_v^*(\lambda-\lambda z_1+\eta)}{\eta B^*(\lambda)}\right]+\left[\frac{S_v^*(\lambda-\lambda z_1+\eta)(\lambda+\eta A^*(\lambda+\eta))(1-B^*(\lambda))}{B^*(\lambda)[\lambda+\eta-S_v^*(\eta)(\lambda+\eta A^*(\lambda+\eta))]}\right]$$

$$-\left[\frac{\lambda E(S_b)S_v^*(\eta)[\lambda(1-S_v^*(\lambda-\lambda z_1+\eta))+\eta(1-A^*(\lambda+\eta)S_v^*(\lambda-\lambda z_1+\eta))]}{B^*(\lambda)[\lambda+\eta-S_v^*(\eta)(\lambda+\eta A^*(\lambda+\eta))]}\right], \tag{42}$$

where $\rho_b = \dfrac{\lambda E(S_b)}{B^*(\lambda)}, E(S_b)$ is the mean service time. From (41), we obtain the system stability condition as $\rho_b < 1$.

4. Particular Cases

Case i: Suppose that there is no retrial time in the system that is the retrial time is 0 (by setting $B^*(\lambda) = 1, A^*(\lambda + \eta) = 1$ in (40)) then our system is reduced to the $M/G/1$ queue with single working vacation (Julia Rose Mary (2010)).

$$P(z) = P_V(z) + P_B(z), \tag{43}$$

where

$$P_V(z) = \frac{\left\{\begin{array}{r}\lambda z(z - S_v^*(\lambda - \lambda z_1 + \eta))(1 - S_v^*(\lambda - \lambda z + \eta)) \\ +(\lambda - \lambda z + \eta)(z - S_v^*(\lambda - \lambda z + \eta))\end{array}\right\}}{(\lambda - \lambda z + \eta)(z - S_v^*(\lambda - \lambda z + \eta))} Q_{0,0},$$

$$P_B(z) = \frac{\left\{\begin{array}{r}\{\lambda z(1 - S_b^*(\lambda - \lambda z))\{\eta\lambda z(z - S_v^*(\lambda - \lambda z_1 + \eta))(1 - S_v^*(\lambda - \lambda z + \eta)) \\ -(\lambda(1 - S_v^*(\lambda - \lambda z_1 + \eta)) + \eta(1 - z))(\lambda - \lambda z + \eta)(z - S_v^*(\lambda - \lambda z + \eta))\} \\ +\eta(\lambda - \lambda z)(\lambda - \lambda z + \eta)(z - S_v^*(\lambda - \lambda z + \eta))(z - S_b^*(\lambda - \lambda z))\}\end{array}\right\}}{\lambda(\lambda - \lambda z)(\lambda - \lambda z + \eta)(z - S_v^*(\lambda - \lambda z + \eta))(z - S_b^*(\lambda - \lambda z))} Q_{0,0},$$

$$Q_{0,0} = \frac{1 - \lambda E(S_b)}{\left[\dfrac{(\lambda - \lambda S_v^*(\lambda - \lambda z_1 + \eta) + \eta)}{\eta} - \dfrac{\lambda E(S_b) S_v^*(\eta)(1 - S_v^*(\lambda - \lambda z_1 + \eta))}{(1 - S_v^*(\eta))} + \dfrac{\eta}{\lambda}\right]}.$$

Case ii: If the server never takes the vacation then taking limit $\eta \to \infty$ in (40), we get

$$P(z) = \left[\frac{(B^*(\lambda) - \lambda E(S_b))(1 - z) S_b^*(\lambda - \lambda z)}{B^*(\lambda)(1 - z) S_b^*(\lambda - \lambda z) - z(1 - S_b^*(\lambda - \lambda z))}\right]. \tag{44}$$

Equation (44) is well known probability generating function of the steady state system length distribution of an $M/G/1$ retrial queue (Equation (16) of Gomez-Corral (1999)) irrespective of the notations.

Case iii: If the server never takes the vacation and there is no retrial time in the system then taking limit $\eta \to \infty$ and putting $B^*(\lambda) = 1$ and $A^*(\lambda + \eta) = 1$ in (40), we get

$$P(z) = \left[\frac{S_b^*(\lambda - \lambda z)(1 - z)(1 - \lambda E(S_b))}{S_b^*(\lambda - \lambda z) - z}\right]. \tag{45}$$

Equation (45) is well known probability generating function of the steady state system length distribution of an $M/G/1$ queue (Medhi (1982)) irrespective of the notations.

5. Performance Measures

Mean Orbit Size

Let L_v and L_b denote the mean orbit size during the WV period and regular service period respectively and let W_v and W_b be the mean waiting time of the customer in the orbit during WV period and regular service period respectively.

$$\begin{aligned} L_v &= \frac{d}{dz}P_v(z)\Big|_{z=1}, \\ &= \frac{d}{dz}[Q_1^*(z,0)+Q_0^*(z,0)]\Big|_{z=1}, \\ &= \frac{d}{dz}\Big[\frac{A(z)}{(\lambda-\lambda z+\eta)D_1(z)}+\frac{B(z)}{D_1(z)}\Big]Q_{0,0}\Big|_{z=1}, \end{aligned}$$

$$= \Big[\frac{(\lambda-\lambda z+\eta)D_1(z)A'(z)-A(z)[(\lambda-\lambda z+\eta)D_1'(z)-\lambda D_1(z)]}{((\lambda-\lambda z+\eta)D_1(z))^2} + \frac{D_1(z)B'(z)-B(z)D_1'(z)}{(D_1(z))^2}\Big]Q_{0,0}\Big|_{z=1},$$

where

$$\begin{aligned} &A(z) = \lambda(1-S_v^*(\lambda-\lambda z+\eta))[z(\lambda+\eta)-S_v^*(\lambda-\lambda z_1+\eta)(A^*(\lambda+\eta)(\lambda-\lambda z+\eta)+\lambda z)], \\ &B(z) = \lambda z(1-A^*(\lambda+\eta))(S_v^*(\lambda-\lambda z+\eta)-S_v^*(\lambda-\lambda z_1+\eta)), \\ &D_1(z) = z(\lambda+\eta)-S_v^*(\lambda-\lambda z+\eta)(A^*(\lambda+\eta)(\lambda-\lambda z+\eta)+\lambda z), \end{aligned}$$

at $z=1$ the formula L_v becomes

$$L_v = \left[\frac{\eta D_1(1)A'(1)-A(1)[\eta D_1'(1)-\lambda D_1(1)]}{(\eta D_1(1))^2} + \frac{D_1(1)B'(1)-B(1)D_1'(1)}{(D_1(1))^2}\right]Q_{0,0}.$$

Using Little's formula, we get $W_v = \dfrac{L_v}{\lambda}$,

where

$$\begin{aligned} &A(1) = \lambda(1-S_v^*(\eta))[\lambda+\eta-S_v^*(\lambda-\lambda z_1+\eta)(\lambda+\eta A^*(\lambda+\eta))], \\ &A'(1) = \lambda^2 S_v^{*'}(\eta)[\lambda+\eta-S_v^*(\lambda-\lambda z_1+\eta)(\lambda+\eta A^*(\lambda+\eta))], \\ &\qquad + \lambda(1-S_v^*(\eta))[\lambda+\eta-S_v^*(\lambda-\lambda z_1+\eta)(\lambda-\lambda A^*(\lambda+\eta))], \\ &B(1) = \lambda(1-A^*(\lambda+\eta))(S_v^*(\eta)-S_v^*(\lambda-\lambda z_1+\eta)), \\ &B'(1) = \lambda(1-A^*(\lambda+\eta))[S_v^*(\eta)-S_v^*(\lambda-\lambda z_1+\eta)-\lambda S_v^{*'}(\eta)], \\ &D_1(1) = \lambda+\eta-S_v^*(\eta)(\lambda+\eta A^*(\lambda+\eta)), \end{aligned}$$

$$D_1'(1) = \lambda + \eta + \lambda S_v^{*'}(\eta)(\lambda + \eta A^*(\lambda+\eta)) - S_v^*(\eta)(\lambda - \lambda A^*(\lambda+\eta)),$$

$$\begin{aligned} L_b &= \frac{d}{dz}P_B(z)\big|_{z=1}, \\ &= \frac{d}{dz}[P_1^*(z,0) + P_0^*(z,0)]\big|_{z=1}, \\ &= \frac{d}{dz}\Big[\frac{N_1(z)N_2(z)}{D_1(z)D_2(z)D_3(z)} + \frac{N_3(z)N_4(z)}{D_1(z)D_2(z)\lambda(\lambda-\lambda z+\eta)}\Big]Q_{0,0}\big|_{z=1}, \end{aligned}$$

$$= \frac{\begin{bmatrix} D_1(z)D_2'(z)D_3'(z)(N_1''(z)N_2''(z) + N_1'(z)N_2''(z)) \\ -N_1'(z)N_2'(z)(2D_1'(z)D_2'(z)D_3'(z) \\ -D_1(z)D_2'(z)D_3''(z) - D_1(z)D_2''(z)D_3'(z)) \end{bmatrix}}{2(D_1(z)D_2'(z)D_3'(z))^2}Q_{0,0}\Big|_{z=1},$$

$$+ \frac{\begin{bmatrix} 2D_2'(z)N_4'(z)[\lambda(\lambda-\lambda z+\eta)(D_1(z)N_3'(z) \\ -N_3(z)D_1'(z)) + \lambda^2 N_3(z)D_1(z)] \\ +\lambda(\lambda-\lambda z+\eta)D_1(z)N_3(z)(D_2'(z)N_4''(z) - N_4'(z)D_2''(z)) \end{bmatrix}}{4[\lambda(\lambda-\lambda z+\eta)D_1(z)D_2'(z)]^2}Q_{0,0}\Big|_{z=1},$$

where

$$N_1(z) = (1 - S_b^*(\lambda - \lambda z)),$$

$$\begin{aligned} N_2(z) = \eta\lambda z\big\{&(\lambda-\lambda z+\eta)(1-A^*(\lambda+\eta))(S_v^*(\lambda-\lambda z+\eta) \\ &- S_v^*(\lambda-\lambda z_1+\eta))(z + B^*(\lambda)(1-z)) + (1 - S_v^*(\lambda-\lambda z+\eta))[z(\lambda+\eta) \\ &- S_v^*(\lambda-\lambda z_1+\eta)(\lambda z + (\lambda-\lambda z+\eta)A^*(\lambda+\eta))]\big\} - (\lambda-\lambda z+\eta)[\eta B^*(\lambda)(1-z) \\ &+ \lambda(z + B^*(\lambda)(1-z))(1 - S_v^*(\lambda-\lambda z_1+\eta))], \\ &\times [z(\lambda+\eta) - S_v^*(\lambda-\lambda z+\eta)(\lambda z + (\lambda-\lambda z+\eta)A^*(\lambda+\eta))], \end{aligned}$$

$$N_3(z) = z(1 - B^*(\lambda)),$$

$$\begin{aligned} N_4(z) = &(\eta(S_b^*(\lambda-\lambda z) - 1) - \lambda(1 - S_v^*(\lambda-\lambda z_1+\eta)))(\lambda-\lambda z+\eta)[z(\lambda+\eta) \\ &- S_v^*(\lambda-\lambda z+\eta)(\lambda z + A^*(\lambda+\eta)(\lambda-\lambda z+\eta))] \\ &+ \eta\lambda\big\{S_b^*(\lambda-\lambda z)(1 - S_v^*(\lambda-\lambda z+\eta))[z(\lambda+\eta) \\ &- S_v^*(\lambda-\lambda z_1+\eta)(\lambda z + (\lambda-\lambda z+\eta)A^*(\lambda+\eta))] \\ &+ z(\lambda-\lambda z+\eta)(1-A^*(\lambda+\eta))(S_v^*(\lambda-\lambda z+\eta) - S_v^*(\lambda-\lambda z_1+\eta))\big\}, \end{aligned}$$

$$\begin{aligned} D_1(z) &= z(\lambda+\eta) - S_v^*(\lambda-\lambda z+\eta)(\lambda z + (\lambda-\lambda z+\eta)A^*(\lambda+\eta)), \\ D_2(z) &= z - S_b^*(\lambda-\lambda z)(z + (1-z)B^*(\lambda)), \\ D_3(z) &= (\lambda-\lambda z)(\lambda-\lambda z+\eta). \end{aligned}$$

At $z = 1$ the formula L_b becomes

$$L_b = \frac{\begin{bmatrix} D_1(1)D_2'(1)D_3'(1)(N_1''(1)N_2''(1) + N_1'(1)N_2''(1)) \\ -N_1'(1)N_2'(1)(2D_1'(1)D_2'(1)D_3'(1) \\ -D_1(1)D_2'(1)D_3''(1) - D_1(1)D_2''(1)D_3'(1)) \end{bmatrix}}{2(D_1(1)D_2'(1)D_3'(1))^2} Q_{0,0}$$

$$+ \frac{\begin{bmatrix} 2D_2'(1)N_4'(1)[\lambda\eta(D_1(1)N_3'(1) - N_3(1)D_1'(1)) + \lambda^2 N_3(1)D_1(1)] \\ +\lambda\eta D_1(1)N_3(1)(D_2'(1)N_4''(1) - N_4'(1)D_2''(1)) \end{bmatrix}}{4[\lambda\eta D_1(1)D_2'(1)]^2} Q_{0,0},$$

using Little's formula $W_b = \dfrac{L_b}{\lambda}$,

where

$$N_1'(1) = -\lambda E(S_b),$$

$$N_1''(1) = -\lambda^2 E(S_b^2),$$

$$\begin{aligned} N_2'(1) = {} & \lambda^2(1 - S_v^*(\lambda - \lambda z_1 + \eta))[\lambda(1 - S_v^*(\eta)) - \eta(S_v^*(\eta)A^*(\lambda + \eta) - B^*(\lambda)(1 - S_v^*(\eta)))] \\ & + \eta\lambda(1 - S_v^*(\eta))[2\eta B^*(\lambda) - \eta B^*(\lambda)A^*(\lambda + \eta)S_v^*(\lambda - \lambda z_1 + \eta) + \lambda] \\ & + \eta^3 B^*(\lambda)(1 - A^*(\lambda + \eta)S_v^*(\eta)) - \eta^2\lambda A^*(\lambda + \eta)(S_v^*(\eta) - S_v^*(\lambda - \lambda z_1 + \eta)), \end{aligned}$$

$$\begin{aligned} N_2''(1) = {} & [2\lambda^2(1 - S_v^*(\lambda - \lambda z_1 + \eta)) + 2\eta B^*(\lambda)(\eta - \lambda S_v^*(\lambda - \lambda z_1 + \eta))] \\ & \times [\lambda S_v^{*'}(\eta)(\lambda + \eta A^*(\lambda + \eta)) - \lambda S_v^*(\eta)(1 - A^*(\lambda + \eta))] \\ & + 2(\lambda + \eta)(\lambda^2 + \eta^2 B^*(\lambda)) + 2\lambda^3(\eta B^*(\lambda)S_v^{*'}(\eta) - S_v^*(\lambda - \lambda z_1 + \eta)) \\ & + 2\lambda^3 B^*(\lambda)S_v^*(\lambda - \lambda z_1 + \eta)(1 - S_v^*(\eta)) \\ & + 2\eta\lambda^2 B^*(\lambda)S_v^*(\eta)A^*(\lambda + \eta)(1 - S_v^*(\lambda - \lambda z_1 + \eta)) \\ & - 2\eta\lambda^2 A^*(\lambda + \eta)S_v^*(\lambda - \lambda z_1 + \eta)(1 - B^*(\lambda)) \\ & - 2\eta\lambda^2 B^*(\lambda)S_v^*(\lambda - \lambda z_1 + \eta) + 2\eta^2\lambda^2 S_v^{*'}(\eta)(B^*(\lambda) + A^*(\lambda + \eta)) + 2\eta\lambda^3 S_v^{*'}(\eta) \\ & - 2\eta^2\lambda A^*(\lambda + \eta)(S_v^*(\eta) - S_v^*(\lambda - \lambda z_1 + \eta))(1 - B^*(\lambda)) + 2\lambda^2(1 - S_v^*(\eta))(\eta + \lambda) \\ & - 2\eta^2\lambda B^*(\lambda)S_v^*(\eta)(1 - A^*(\lambda + \eta)) - 2\eta\lambda^2 S_v^*(\eta) - 2\eta\lambda^2 B^*(\lambda)(1 - S_v^*(\eta)) \\ & + 2\lambda^2 S_v^*(\eta)S_v^*(\lambda - \lambda z_1 + \eta)(\lambda + \eta A^*(\lambda + \eta)), \end{aligned}$$

$$N_3(1) = 1 - B^*(\lambda),$$

$$N_3'(1) = 1 - B^*(\lambda),$$

$$\begin{aligned} N_4'(1) = {} & \lambda^2(1 - S_v^*(\lambda - \lambda z_1 + \eta))[\lambda(1 - S_v^*(\eta)) + \eta(1 - S_v^*(\eta)A^*(\lambda + \eta))] \\ & + \eta\lambda^2(1 - S_v^*(\eta))[+2\eta E(S_b) + \lambda E(S_b) - \lambda E(S_b)S_v^*(\lambda - \lambda z_1 + \eta) \\ & - \eta E(S_b)S_v^*(\lambda - \lambda z_1 + \eta)A^*(\lambda + \eta)] + \eta\lambda[\eta^2 E(S_b)(1 - S_v^*(\eta)A^*(\lambda + \eta)) \\ & + \lambda S_v^*(\lambda - \lambda z_1 + \eta) - \lambda S_v^*(\eta) - \eta A^*(\lambda + \eta)S_v^*(\eta) + \eta A^*(\lambda + \eta)S_v^*(\lambda - \lambda z_1 + \eta)], \end{aligned}$$

$$
\begin{aligned}
N_4''(1) = {} & 2\eta^2\lambda^2 E(S_b)S_v^*(\eta)A^*(\lambda+\eta) + 2\lambda^2[\lambda+\eta+\lambda^2 S_v^{*'}(\eta) + \eta\lambda S_v^{*'}(\eta)A^*(\lambda+\eta) \\
& - \lambda S_v^*(\eta) + \lambda S_v^*(\eta)A^*(\lambda+\eta)] - 2\lambda^4 S_v^*(\lambda-\lambda z_1+\eta)S_v^{*'}(\eta) \\
& - \lambda^3 S_v^*(\lambda-\lambda z_1+\eta)(1-S_v^*(\eta)) - \lambda^3 S_v^*(\lambda-\lambda z_1+\eta)S_v^*(\eta)A^*(\lambda+\eta) \\
& + \eta^2\lambda^2 E(S_b^2)[\lambda+\eta-\lambda S_v^*(\eta) - \eta S_v^*(\eta)A^*(\lambda+\eta)] \\
& - [\lambda+\eta+\eta\lambda S_v^{*'}(\eta)A^*(\lambda+\eta) - \lambda S_v^*(\eta) + \lambda S_v^*(\eta)A^*(\lambda+\eta)] \\
& \times (-\eta^2\lambda E(S_b) + \lambda^2 S_v^*(\lambda-\lambda z_1+\eta)) \\
& + \eta^2\lambda E(S_b)[\eta+\lambda^2 S_v^{*'}(\eta) + \eta\lambda S_v^{*'}(\eta)A^*(\lambda+\eta) - \lambda S_v^*(\eta) + \lambda S_v^*(\eta)A^*(\lambda+\eta)] \\
& - \eta\lambda^3 S_v^*(\lambda-\lambda z_1+\eta)S_v^{*'}(\eta)A^*(\lambda+\eta) \\
& + \eta\lambda^3 E(S_b^2)(1-S_v^*(\eta))[\lambda+\eta-\lambda S_v^*(\lambda-\lambda z_1+\eta) - \eta S_v^*(\lambda-\lambda z_1+\eta)A^*(\lambda+\eta)] \\
& + 2\eta\lambda^3 E(S_b)S_v^{*'}(\eta)[\lambda+\eta-\lambda S_v^*(\lambda-\lambda z_1+\eta) - \eta S_v^*(\lambda-\lambda z_1+\eta)A^*(\lambda+\eta)] \\
& + 2\eta\lambda^2 E(S_b)(1-S_v^*(\eta))[\lambda S_v^*(\lambda-\lambda z_1+\eta)A^*(\lambda+\eta) - \lambda S_v^*(\lambda-\lambda z_1+\eta)] \\
& + \eta^2\lambda^2 E(S_b)(1-S_v^*(\eta)) + 2\eta\lambda^3 S_v^{*'}(\eta) + \eta^2\lambda^2 S_v^{*'}(\eta) - \eta\lambda^2 S_v^*(\eta) \\
& + \eta\lambda^2 A^*(\lambda+\eta)(S_v^*(\eta) - S_v^*(\lambda-\lambda z_1+\eta)) + \eta^2\lambda^2 S_v^{*'}(\eta)A^*(\lambda+\eta),
\end{aligned}
$$

$$
\begin{aligned}
& D_1(1) = \lambda+\eta-S_v^*(\eta)(\lambda+\eta A^*(\lambda+\eta)), \\
& D_1'(1) = \lambda+\eta+\lambda S_v^{*'}(\eta)(\lambda+\eta A^*(\lambda+\eta)) - \lambda S_v^*(\eta)(1-A^*(\lambda+\eta)), \\
& D_2'(1) = B^*(\lambda) - \lambda E(S_b), \\
& D_2''(1) = -2\lambda E(S_b)(0)(1-B^*(\lambda)) - \lambda^2 E(S_b^2), \\
& D_3'(1) = -\eta\lambda, \quad D_3''(1) = 2\lambda^2.
\end{aligned}
$$

6. Numerical Result

Fixing the values of $\mu_v = 5, \mu_b = 9, \mu_{v_r} = 4, \mu_{b_r} = 8$ and ranging the values of λ from 0.4 to 0.8 insteps of 0.1 and varying the values of η from 0.50 to 0.54 insteps of 0.01, we calculated the corresponding values of L_b and W_b for single working vacation and tabulated in Table 1 and in Table 2 respectively. The corresponding graphs have been drawn for λ versus L_b and λ versus W_b and are shown in Figure 1 and in Figure 2 respectively. From the graphs it is seen that as λ increases both L_b and W_b increases for various values of η.

Table 1: Arrival rate (λ) versus mean orbit size (L_b) in regular service period

λ \ η	0.50	0.51	0.52	0.53	0.54
0.4	0.181584	0.169022	0.157262	0.146238	0.135894
0.5	0.346882	0.325493	0.305449	0.286644	0.268982
0.6	0.572823	0.539918	0.509056	0.480077	0.452838
0.7	0.862965	0.815787	0.771502	0.729887	0.690740
0.8	1.219647	1.155433	1.095109	1.038376	0.984969

Table 2: Arrival rate (λ) versus mean waiting time (W_b) in regular service period

λ \ η	0.50	0.51	0.52	0.53	0.54
0.4	0.453959	0.422556	0.393155	0.365595	0.339734
0.5	0.693764	0.650985	0.610898	0.573287	0.537964
0.6	0.954705	0.899864	0.848427	0.800128	0.754730
0.7	1.232807	1.165410	1.102146	1.042696	0.986771
0.8	1.524559	1.444291	1.368886	1.297970	1.231211

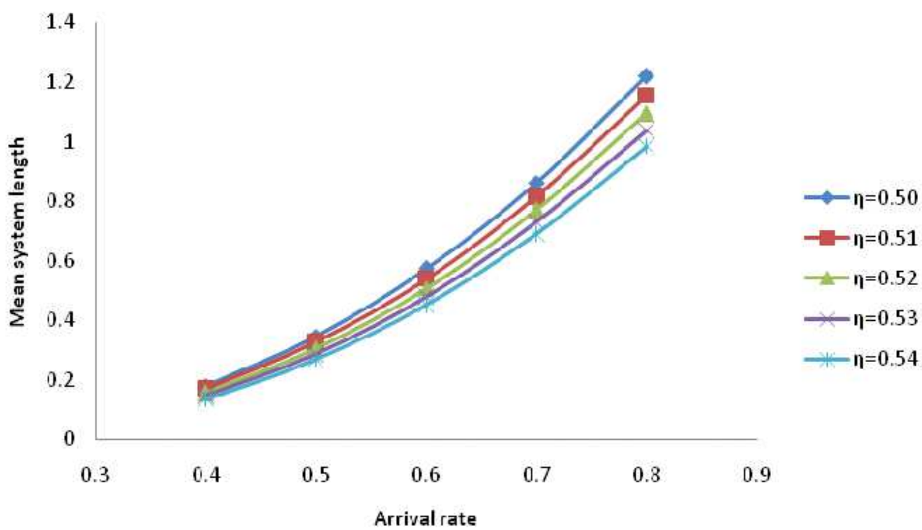

Figure 1: Arrival rate (λ) versus mean orbit size (L_b) in regular service period.

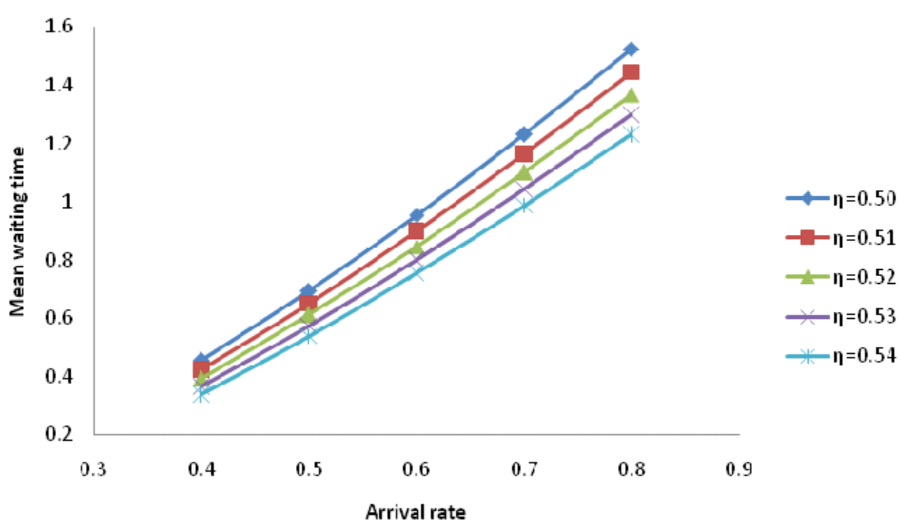

Figure 2: Arrival rate (λ) versus mean waiting time (W_b) in regular service period.

Again fixing the values of $\mu_v = 3, \mu_b = 5, \mu_{v_r} = 2, \mu_{b_r} = 4$ and ranging the values of λ from 0.3

to 0.7 insteps of 0.1 and varying the values of η from 2.5 to 2.9 insteps of 0.1, we calculated the values of L_v and W_v for swv and tabulated in Table 3 and in Table 4.

Table 3: Arrival rate (λ) versus mean orbit size (L_v) in WV period.

λ \ η	2.5	2.6	2.7	2.8	2.9
0.3	0.000441	0.000415	0.000390	0.000365	0.000343
0.4	0.000978	0.000919	0.000862	0.000808	0.000757
0.5	0.001776	0.001667	0.001563	0.001465	0.001373
0.6	0.002840	0.002665	0.002498	0.002340	0.002193
0.7	0.004152	0.003895	0.003650	0.003420	0.003205

Table 4: Arrival rate (λ) versus mean waiting time (W_v) in WV period.

λ \ η	2.5	2.6	2.7	2.8	2.9
0.3	0.001470	0.001383	0.001299	0.001218	0.001142
0.4	0.002444	0.002297	0.002155	0.002020	0.001893
0.5	0.003552	0.003335	0.003127	0.002930	0.002746
0.6	0.004734	0.004441	0.004163	0.003901	0.003655
0.7	0.005931	0.005564	0.005214	0.004886	0.004579

The corresponding graphs have been drawn for λ versus L_v and λ versus W_v and are shown in Figure 3 and in Figure 4 respectively. From the graphs it is seen that as λ increases both L_b and W_b increases for various values of η.

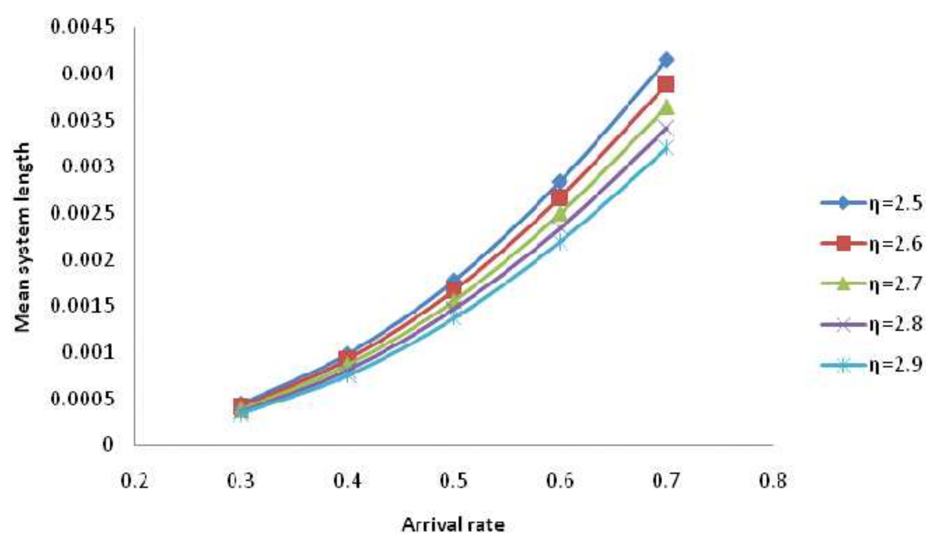

Figure 3: Arrival rate (λ) versus orbit size length (L_v) in WV period.

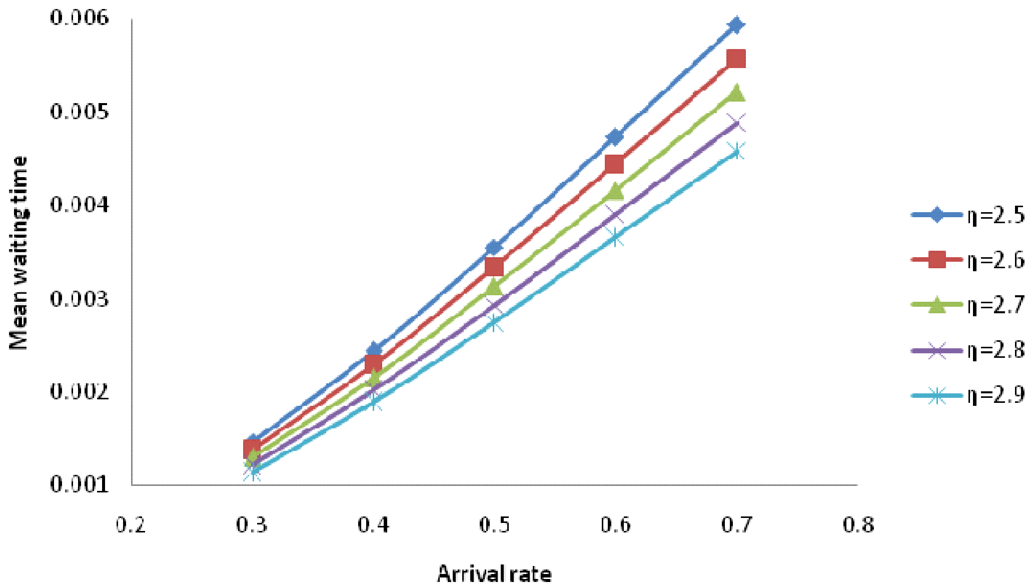

Figure 4: Arrival rate (λ) versus mean waiting time (W_v) in WV period.

7. Conclusion

In this paper we considered an $M/G/1$ Retrial Queue with Single Working Vacation.Using supplementary variable technique, we obtained the probability generating function for the number of customers in the system and we also calculated the mean orbit size size during the WV period and regular service period and the mean waiting time of the customer in the orbit during WV period and regular service period. Some particular cases were discussed. Numerical illustration is also given to see the utility of the model. Future work can also be done by including some parameters like bulk arrival, bulk service and multiple working vacation

Acknowledgments

We thank both referees and the Editor in Chief Professor Aliakbar Montazer Haghighi, for their valuable suggestions and constructive comments which helped us in revising the paper in the present form.

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