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Generalized I of strongly Lacunary of χ^2 over $p-$metric spaces defined by Musielak Orlicz function

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Abstract

In this paper, we introduce generalized difference sequence spaces via ideal convergence, lacunary of χ^2 sequence spaces over $p-$metric spaces defined by Musielak function, and examine the Musielak-Orlicz function which satisfies uniform Δ_2 condition, and we also discuss some topological properties of the resulting spaces of χ^2 with respect to ideal structures which is solid and monotone. Hence, given an example of the space χ^2, this is not solid and not monotone. This theory is very useful for statistical convergence and also is applicable to rough convergence.

Keywords: Analytic sequence; double sequences; χ^2 space; difference sequence space; Musielak Orlicz function; $p-$ metric space; Lacunary sequence; ideal

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1. Introduction

Throughout w, χ and Λ denote the classes of all gai and analytic scalar valued single sequences, respectively. We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under coordinatewise addition and scalar multiplication. Some initial work on double sequence spaces is found in Bromwich (1965). Later on, they were investigated by Hardy (1917), Moricz (1991), Moricz and Rhoades (1988), Basarir and Solankan (1999), Tripathy (2003), Turkmenoglu (1999), Mishra et al. (2007, 2012) and many others.

We procure the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{ (x_{mn}) \in w^2 : sup_{m,n \in N} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$\mathcal{C}_p(t) := \left\{ (x_{mn}) \in w^2 : p - \lim_{m,n\to\infty} |x_{mn} - l|^{t_{mn}} = 1 \text{ for some } l \in \mathbb{C} \right\},$$

$$\mathcal{C}_{0p}(t) := \left\{ (x_{mn}) \in w^2 : p - \lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1 \right\},$$

$$\mathcal{L}_u(t) := \left\{ (x_{mn}) \in w^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t),$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - \lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}$; $\mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some documents related to the double sequence spaces. Gökhan and Colak (2004, 2005) have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter (2001) has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely (2003) and Tripathy (2003) have independently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Altay and Basar (2005) have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(\vartheta)-$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Basar and Sever (2009) have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Mishra (2010) have studied the space $\chi^2_M(p, q, u)$ of double sequences and gave some inclusion relations.

The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox (1986) as an extension of the definition of strongly Cesàro summable sequences. Connor (1989) further extended this definition to a definition of strong $A-$ summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong $A-$ summability, strong $A-$ summability with respect to a modulus, and $A-$ statistical convergence. In Pringsheim (1900) the notion of convergence of double sequences was presented by A. Pringsheim. Also, in Hamilton (1936, 1938), the four dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$ was studied extensively by Robison (1926) and Hamilton (1939).

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m,n \in \mathbb{N})$.

A sequence $x = (x_{mn})$ is said to be double analytic if $sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called a double gai sequence if $((m+n)! |x_{mn}|)^{1/m+n} \to 0$ as $m,n \to \infty$. The double gai sequences will be denoted by χ^2. Let $\phi = \{\text{all finite sequences}\}$.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\Im_{ij}$ for all $m,n \in \mathbb{N}$, where $\Im_{ij}$ denotes the double sequence whose only non zero term is $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i,j \in \mathbb{N}$.

An FK-space (or a metric space) X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X, or equivalently $x^{[m,n]} \to x$. An FDK-space is a double sequence space endowed with a complete metrizable locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})(m,n \in \mathbb{N})$ are also continuous.

Let M and Φ be mutually complementary modulus functions. Then, we have:
(i) For all $u, y \geq 0$,

$$uy \leq M(u) + \Phi(y), \text{(Young's inequality) (see Kampthan et al. (1981))}, \tag{1}$$

(ii) for all $u \geq 0$,

$$u\eta(u) = M(u) + \Phi(\eta(u)), \tag{2}$$

and

(iii) for all $u \geq 0$ and $0 < \lambda < 1$,

$$M(\lambda u) \leq \lambda M(u). \tag{3}$$

Lindenstrauss and Tzafriri (1971) used the idea of Orlicz function to construct the Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}.$$

The space ℓ_M with the norm

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p\,(1 \leq p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.

A sequence $f = (f_{mn})$ of modulus function is called a Musielak-modulus function. A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = \sup\left\{ |v|\, u - (f_{mn})(u) : u \geq 0 \right\}, m, n = 1, 2, \ldots$$

is called the complementary function of a Musielak-modulus function f. For a given Musielak modulus function f, the Musielak-modulus sequence space t_f and its subspace h_f are defined as follows:

$$t_f = \left\{ x \in w^2 : I_f\left(|x_{mn}|\right)^{1/m+n} \to 0 \text{ as } m, n \to \infty \right\},$$

$$h_f = \left\{ x \in w^2 : I_f\left(|x_{mn}|\right)^{1/m+n} \to 0 \text{ as } m, n \to \infty \right\},$$

where I_f is a convex modular defined by

$$I_f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn}\left(|x_{mn}|\right)^{1/m+n}, x = (x_{mn}) \in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d(x, y) = \sup_{mn} \left\{ \inf\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn}\left(\frac{|x_{mn}|^{1/m+n}}{mn} \right) \right) \leq 1 \right\}.$$

If X is a sequence space, we give the following definitions:

(1) X'= the continuous dual of X,
(2) $X^{\alpha} = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn} x_{mn}| < \infty, \text{for each } x \in X \right\}$,
(3) $X^{\beta} = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn} x_{mn} \text{ is convegent, for each } x \in X \right\}$,
(4) $X^{\gamma} = \left\{ a = (a_{mn}) : sup_{mn} \geq 1 \left| \sum_{m,n=1}^{M,N} a_{mn} x_{mn} \right| < \infty, \text{ for each } x \in X \right\}$,
(5) let X be an FK-space $\supset \phi$; then $X^f = \left\{ f(\Im_{mn}) : f \in X' \right\}$,

(6) $X^{\delta} = \left\{ a = (a_{mn}) : \sup_{mn} |a_{mn}x_{mn}|^{1/m+n} < \infty, \text{ for each } x \in X \right\},$

where X^{α}, X^{β} and X^{γ} are called $\alpha-$(or Köthe-Toeplitz) dual of X, $\beta-$ (or generalized-Köthe-Toeplitz) dual of X, $\gamma-$ dual of X, and $\delta-$ dual of X, respectively. X^{α} is defined by Gupta and Kamptan (1981). It is clear that $X^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\beta} \subset X^{\gamma}$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\},$$

for $Z = c, c_0$ and ℓ_{∞}, where $\Delta x_k = x_k - x_{k+1}$, for all $k \in \mathbb{N}$.

Here, c, c_0 and ℓ_{∞} denote the classes of convergent, null,and bounded scalar valued single sequences, respectively. The difference sequence space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \leq p \leq \infty$ by Başar and Altay and in the case $0 < p < 1$ by Altay and Başar (2005). The spaces $c(\Delta), c_0(\Delta), \ell_{\infty}(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + \sup_{k \geq 1} |\Delta x_k| \text{ and } \|x\|_{bv_p} = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}, (1 \leq p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\},$$

where $Z = \Lambda^2, \chi^2$, and

$\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}, \ \forall m, n \in \mathbb{N}.$

The generalized difference double notion has the following representation: $\Delta^m x_{mn} = \Delta^{m-1}x_{mn} - \Delta^{m-1}x_{mn+1} - \Delta^{m-1}x_{m+1n} + \Delta^{m-1}x_{m+1n+1}$, and also this generalized difference double notion has the following binomial representation:

$$\Delta^m x_{mn} = \sum_{i=0}^{m} \sum_{j=0}^{m} (-1)^{i+j} \binom{m}{i} \binom{m}{j} x_{m+i,n+j}.$$

2. Definition and Preliminaries

Let $n \in \mathbb{N}$ and X be a real vector space of dimension w, where $n \leq w$. A real valued function $d_p(x_1, \ldots, x_n) = \|(d_1(x_1), \ldots, d_n(x_n))\|_p$ on X satisfes the following four conditions:

(1) $\|(d_1(x_1), \ldots, d_n(x_n))\|_p = 0$ if and and only if $d_1(x_1), \ldots, d_n(x_n)$ are linearly dependent,

(2) $\|(d_1(x_1), \ldots, d_n(x_n))\|_p$ is invariant under permutation,
(3) $\|(\alpha d_1(x_1), \ldots, d_n(x_n))\|_p = |\alpha| \, \|(d_1(x_1), \ldots, d_n(x_n))\|_p, \alpha \in \mathbb{R}$,
(4) $d_p\left((x_1, y_1), (x_2, y_2) \cdots (x_n, y_n)\right) = \left(d_X(x_1, x_2, \cdots x_n)^p + d_Y(y_1, y_2, \cdots y_n)^p\right)^{1/p}$, for $1 \leq p < \infty$, or
(5) $d\left((x_1, y_1), (x_2, y_2), \cdots (x_n, y_n)\right) := \sup\left\{d_X(x_1, x_2, \cdots x_n), d_Y(y_1, y_2, \cdots y_n)\right\}$, for $x_1, x_2, \cdots x_n \in X, y_1, y_2, \cdots y_n \in Y$ is called the p product metric.

A trivial example of the p product metric of n metric space is the p norm space $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space with the p norm:

$$\begin{aligned}\|(d_1(x_1), \ldots, d_n(x_n))\|_E &= \sup\left(|\det(d_{mn}(x_{mn}))|\right) \\ &= \sup\left(\begin{vmatrix} d_{11}(x_{11}) & d_{12}(x_{12}) & \ldots & d_{1n}(x_{1n}) \\ d_{21}(x_{21}) & d_{22}(x_{22}) & \ldots & d_{2n}(x_{1n}) \\ . & & & \\ . & & & \\ . & & & \\ d_{n1}(x_{n1}) & d_{n2}(x_{n2}) & \ldots & d_{nn}(x_{nn}) \end{vmatrix}\right),\end{aligned}$$

where $x_i = (x_{i1}, \cdots x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots n$.

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the $p-$ metric. Any complete $p-$ metric space is said to be a $p-$ Banach metric space.

Let X be a linear metric space. A function $w : X \to \mathbb{R}$ is called paranorm, if

(1) $w(x) \geq 0$, for all $x \in X$,
(2) $w(-x) = w(x)$, for all $x \in X$,
(3) $w(x + y) \leq w(x) + w(y)$, for all $x, y \in X$, and
(4) If (σ_{mn}) is a sequence of scalars with $\sigma_{mn} \to \sigma$ as $m, n \to \infty$ and (x_{mn}) is a sequence of vectors with $w(x_{mn} - x) \to 0$ as $m, n \to \infty$, then $w(\sigma_{mn} x_{mn} - \sigma x) \to 0$ as $m, n \to \infty$.

A paranorm w for which $w(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, w) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see Wilansky (1984), Theorem 10.4.2, p. 183).

Let X be a non-empty set. Then a family of sets $I \subset 2^X$ (the class of all subsets of X) is called an ideal if and only if for each $A, B \in I$, we have $A \bigcup B \in I$ and for each $A \in I$ and each $B \subset A$, we have $B \in I$. A non-empty family of sets $F \subset 2^X$ is a filter on X if and only if $\phi \notin F$, for each $A, B \in F$, we have $A \bigcap B \in F$ and each $A \in F$ and $A \subset B$, we have $B \in F$. An ideal I is called non-trivial ideal if $I \neq \phi$ and $X \neq I$. Clearly $I \subset 2^X$ is a non-trivial ideal if and only $F = F(I) = \{X/A : A \in I\}$ is a filter on X. A non-trivial ideal $I \subset 2^X$ is called admissible if and only if $\{\{x\} \cdot x \in X\} \subset I$. A sequence $(x_{mn})_{m,n \in \mathbb{N}}$ in X is said to be $I-$ convergent to $0 \in X$, if for each $\epsilon > 0$ the set $A(\epsilon) = \{m, n \in \mathbb{N} : \|(d_1(x_1), \ldots, d_n(x_n)) - 0\|_p \geq \epsilon\}$ belongs to I. Further details on ideals of 2^X can be found in Kopstyrko et al. (2001). The notion was further investigated by Salat et al. (2004) and others.

By the convergence of a double sequence we mean the convergence on the Pringsheim sense, that is, a double sequence $x = (x_{mn})$ has Prinsheim limit L (denoted by $P - \lim x = L$) provided that given $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $|x_{mn} - L| < \epsilon$ whenever $m, n > n$. We shall write this more briefly as $P-$ convergent.

The double sequence $\theta_{rs} = \{(m_r, n_s)\}$ is called a double lacunary sequence if there exist two increasing integers such that

$$m_0 = 0, \varphi_r = m_r - m_{r-1} \to \infty \text{ as } r \to \infty \text{ and}$$

$$n_0 = 0, \varphi_s = n_s - n_{s-1} \to \infty \text{ as } s \to \infty.$$

Notations: $m_{rs} = m_r n_s, h_{rs} = \varphi_r \bar{\varphi}_s, \theta_{rs}$ are determined by

$$I_{rs} = \{(m, n) : m_{r-1} < m \leq m_r \text{ and } n_{s-1} < n \leq n_s\},$$

$$q_r = \tfrac{k_r}{k_{r-1}}, \bar{q}_s = \tfrac{n_s}{n_{s-1}} \text{ and } q_{rs} = q_r \bar{q}_s.$$

The notion of $\lambda-$ double gai and double analytic sequences are as follows. Let $\lambda = (\lambda_{mn})_{m,n=0}^{\infty}$ be a strictly increasing sequence of positive real numbers tending to infinity, that is,

$$0 < \lambda_{00} < \lambda_{11} < \cdots \text{ and } \lambda_{mn} \to \infty \text{ as } m, n \to \infty$$

and said that a sequence $x = (x_{mn}) \in w^2$ is $\lambda-$ convergent to 0 is called $\lambda-$ limit of x, if

$$\mu_{m\ n}(x) \to 0 \text{ as } m, n \to \infty,$$

where
$$\mu_{m\ n}(x) = \frac{1}{\varphi_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} (\Delta^{m-1}\lambda_{m,n} - \Delta^{m-1}\lambda_{m,n+1} - \Delta^{m-1}\lambda_{m+1,n} + \Delta^{m-1}\lambda_{m+1,n+1})\, |x_{mn}|^{1/m+n}.$$

The sequence $x = (x_{mn}) \in w^2$ is $\lambda-$ double analytic if $\sup_{uv} |\mu_{mn}(x)| < \infty$. If $\lim_{mn} x_{mn} = 0$ in the ordinary sense of convergence, then

$$\lim_{mn} \frac{1}{\varphi_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} (\Delta^{m-1}\lambda_{m,n} - \Delta^{m-1}\lambda_{m,n+1} - \Delta^{m-1}\lambda_{m+1,n} + \Delta^{m-1}\lambda_{m+1,n+1}) \left((m+n)!\, |x_{mn} - 0|\right)^{1/m+n} = 0.$$

This implies that

$$\begin{aligned} &\lim_{mn} |\mu_{mn}(x) - 0| \\ &= \lim_{mn} \frac{1}{\varphi_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} (\Delta^{m-1}\lambda_{m,n} - \Delta^{m-1}\lambda_{m,n+1} - \Delta^{m-1}\lambda_{m+1,n} \\ &\qquad + \Delta^{m-1}\lambda_{m+1,n+1}) \left((m+n)!\, |x_{mn} - 0|\right)^{1/m+n} \\ &= 0, \end{aligned}$$

which yields that $\lim_{uv} \mu_{mn}(x) = 0$ and hence $x = (x_{mn}) \in w^2$ is $\lambda-$ convergent to 0.

Let I^2- be an admissible ideal of $2^{\mathbb{N}\times\mathbb{N}}, \theta_{rs}$ be a double lacunary sequence, $f = (f_{mn})$ be a Musielak-modulus function, and $\left(X, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)$ be a $p-$metric space, $q = (q_{mn})$ be double analytic sequence of strictly positive real numbers. By $w^2(p - X)$ we denote the space of all sequences defined over $\left(X, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)$. The following inequality will be used throughout the paper.

If $0 \leq q_{mn} \leq \sup q_{mn} = H, K = \max\left(1, 2^{H-1}\right)$, then

$$|a_{mn} + b_{mn}|^{q_{mn}} \leq K\left\{|a_{mn}|^{q_{mn}} + |b_{mn}|^{q_{mn}}\right\}, \tag{4}$$

for all m, n and $a_{mn}, b_{mn} \in \mathbb{C}$.

Also $|a|^{q_{mn}} \leq \max\left(1, |a|^H\right)$ for all $a \in \mathbb{C}$.

In the present paper we define the following sequence spaces:

$$\left[\chi^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^2}_{\theta_{rs}} = \left\{r, s \in I_{rs} : \left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} \geq \epsilon\right\} \in I^2,$$

and

$$\left[\Lambda^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^2}_{\theta_{rs}} = \left\{r, s \in I_{rs} : \left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} \geq K\right\} \in I^2.$$

If we take $f_{mn}(x) = x$, we get

$$\left[\chi^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^2}_{\theta_{rs}} = \left\{r, s \in I_{rs} : \left[\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} \geq \epsilon\right\} \in I^2,$$

and

$$\left[\Lambda^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^2}_{\theta_{rs}} = \left\{r, s \in I_{rs} : \left[\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} \geq K\right\} \in I^2.$$

If we take $q = (q_{mn}) = 1$, we get

$$\left[\chi^{2}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I}_{\theta_{rs}} = \left\{r, s \in I_{rs} : \left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right] \geq \epsilon\right\} \in I^2,$$

and

$$\left[\Lambda^2_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^2}_{\theta_{rs}} = \left\{r, s \in I_{rs} : \left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right] \geq K\right\} \in I^2.$$

In the present paper we plan to study some topological properties and inclusion relations between the above defined sequence spaces,

$$\left[\chi^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^2}_{\theta_{rs}} \text{ and } \left[\Lambda^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^2}_{\theta_{rs}},$$

which we shall discuss in this paper.

3. Main Results

Theorem 1.

Let $f = (f_{mn})$ be a Musielak-Orlicz function and $q = (q_{mn})$ be a double analytic sequence of strictly positive real numbers. The sequence spaces

$$\left[\chi^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^2}_{\theta_{rs}}$$

and

$$\left[\Lambda^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^2}_{\theta_{rs}}$$

are linear spaces.

Proof:

It is routine verification. Therefore the proof is omitted.

Theorem 2.

Let $f = (f_{mn})$ be a Musielak-Orlicz function and $q = (q_{mn})$ be a double analytic sequence of strictly positive real numbers. The sequence space

$$\left[\chi^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^2}_{\theta_{rs}}$$

is a paranormed space with respect to the paranorm defined by

$$g(x) = \inf\left\{\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} \leq 1\right\}.$$

Proof:

Clearly $g(x) \geq 0$ for $x = (x_{mn}) \in \left[\chi^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^2}_{\theta_{rs}}$.

Since $f_{mn}(0) = 0$, we get $g(0) = 0$.

Conversely, suppose that $g(x) = 0$.

Then, $\inf\left\{\left[f_{mn}\left(\left\|\mu_{mn}\left(x\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}\leq 1\right\}$.

Suppose that $\mu_{mn}\left(x\right)\neq 0$ for each $m,n\in\mathbb{N}$. Then, $\left\|\mu_{mn}\left(x\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p^{\varphi}\to\infty$.

It follows that

$\left(\left[f_{mn}\left(\left\|\mu_{mn}\left(x\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H}\to\infty,$

which is a contradiction. Therefore, $\mu_{mn}\left(x\right)=0$.

Let $\left(\left[f_{mn}\left(\left\|\mu_{mn}\left(x\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H}\leq 1$

and

$\left(\left[f_{mn}\left(\left\|\mu_{mn}\left(y\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H}\leq 1.$

Then, by using Minkowski's inequality, we have

$$\begin{aligned}
&\left(\left[f_{mn}\left(\left\|\mu_{mn}\left(x+y\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H}\\
\leq&\left(\left[f_{mn}\left(\left\|\mu_{mn}\left(x\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H}\\
+&\left(\left[f_{mn}\left(\left\|\mu_{mn}\left(y\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H}.
\end{aligned}$$

So we have

$$\begin{aligned}
g(x+y)&=\inf\left\{\left[f_{mn}\left(\left\|\mu_{mn}\left(x+y\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}\leq 1\right\}\\
&\leq\inf\left\{\left[f_{mn}\left(\left\|\mu_{mn}\left(x\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}\leq 1\right\}\\
&+\inf\left\{\left[f_{mn}\left(\left\|\mu_{mn}\left(y\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}\leq 1\right\}.
\end{aligned}$$

Therefore,

$$g\left(x+y\right)\leq g\left(x\right)+g\left(y\right).$$

Finally, to prove that the scalar multiplication is continuous, let λ be any complex number.

By definition, $g\left(\lambda x\right)=\inf\left\{\left[f_{mn}\left(\left\|\mu_{mn}\left(\lambda x\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}\leq 1\right\}$.

Then, $g\left(\lambda x\right)=\inf\left\{((|\lambda|\,t)^{q_{mn}/H}:\left[f_{mn}\left(\left\|\mu_{mn}\left(\lambda x\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}\leq 1\right\}$, where $t=\frac{1}{|\lambda|}$. Since $|\lambda|^{q_{mn}}\leq\max\left(1,|\lambda|^{\sup p_{mn}}\right)$, we have

$g\left(\lambda x\right)\leq\max\left(1,|\lambda|^{\sup p_{mn}}\right)\inf\left\{t^{q_{mn}/H}:\left[f_{mn}\left(\left\|\mu_{mn}\left(\lambda x\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}\leq 1\right\}.$

This completes the proof. □

Theorem 3.

(i) If the sequence (f_{mn}) satisfies uniform Δ_2- condition, then

$$\left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^{2\alpha}}_{\theta_{rs}} = \left[\chi^{2q\mu}_{g}, \|\mu_{uv}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^{2}}_{\theta_{rs}}.$$

(ii) If the sequence (g_{mn}) satisfies uniform Δ_2- condition, then

$$\left[\chi^{2q\mu}_{g}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^{2\alpha}}_{\theta_{rs}} = \left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^{2}}_{\theta_{rs}}.$$

Proof:

Let the sequence (f_{mn}) satisfy uniform Δ_2- condition. We get

$$\left[\chi^{2q\mu}_{g}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^{2}}_{\theta_{rs}} \subset \left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^{2\alpha}}_{\theta_{rs}}.$$

To prove the inclusion

$$\left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^{2\alpha}}_{\theta_{rs}} \subset \left[\chi^{2q\mu}_{g}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^{2}}_{\theta_{rs}},$$

let $a \in \left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^{2\alpha}}_{\theta_{rs}}$. Then for all $\{x_{mn}\}$ with $(x_{mn}) \in \left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^{2}}_{\theta_{rs}}$ we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |x_{mn}a_{mn}| < \infty. \tag{5}$$

Since the sequence (f_{mn}) satisfies uniform Δ_2- condition, then

$$(y_{mn}) \in \left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^{2}}_{\theta_{rs}},$$

and we get $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left|\frac{\varphi_{rs}y_{mn}a_{mn}}{\Delta^m\lambda_{mn}(m+n)!}\right| < \infty$ by (5). Thus

$$(\varphi_{rs}a_{mn}) \in \left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^{2}}_{\theta_{rs}} = \left[\chi^{2q\mu}_{g}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^{2}}_{\theta_{rs}}$$

and hence

$$(a_{mn}) \in \left[\chi_g^{2q\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]_{\theta_{rs}}^{I^2}.$$

This gives that

$$\left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]_{\theta_{rs}}^{I^{2\alpha}} \subset \left[\chi_g^{2q\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]_{\theta_{rs}}^{I^2}.$$

From above we have,

$$\left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]_{\theta_{rs}}^{I^{2\alpha}} = \left[\chi_g^{2q\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]_{\theta_{rs}}^{I^2}.$$

(ii) Similarly, one can prove that

$$\left[\chi_g^{2q\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]_{\theta_{rs}}^{I^{2\alpha}} \subset \left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]_{\theta_{rs}}^{I^2}$$

if the sequence (g_{mn}) satisfies uniform Δ_2- condition. $\square$

Proposition 1.

If $0 < q_{mn} < p_{mn} < \infty$ for each m and m, then,

$$\left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]_{\theta_{rs}}^{I^2} \subseteq \left[\Lambda_{f\mu}^{2p}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]_{\theta_{rs}}^{I^2}.$$

Proof:

The proof is standard, so we omit it.

Proposition 2.

(i) If $0 < \inf q_{mn} \leq q_{mn} < 1$, then,

$$\left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]_{\theta_{rs}}^{I^2} \subset \left[\Lambda_{f\mu}^{2}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]_{\theta_{rs}}^{I^2}.$$

(ii) If $1 \leq q_{mn} \leq \sup q_{mn} < \infty$, then,

$$\left[\Lambda^2_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^2}_{\theta_{rs}} \subset \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^2}_{\theta_{rs}}.$$

Proof:

The proof is standard, so we omit it.

Proposition 3.

Let $f^{'} = \left(f^{'}_{mn}\right)$ and $f^{''} = \left(f^{''}_{mn}\right)$ are sequences of Musielak functions, we have

$$\left[\Lambda^{2q}_{f^{'}\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^2}_{\theta_{rs}}$$
$$\bigcap \left[\Lambda^{2q}_{f^{''}\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^2}_{\theta_{rs}}$$
$$\subseteq \left[\Lambda^{2q}_{f^{'}+f^{''}\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^2}_{\theta_{rs}}.$$

Proof:

The proof is easy so we omit it.

Proposition 4.

For any sequence of Musielak functions $f = (f_{mn})$ and $q = (q_{mn})$ be double analytic sequence of strictly positive real numbers. Then,

$$\left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^2}_{\theta_{rs}} \subset \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^2}_{\theta_{rs}}.$$

Proof:

The proof is easy so we omit it.

Proposition 5.

The sequence space $\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^2}_{\theta_{rs}}$ is solid.

Proof:

Let $x = (x_{mn}) \in \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^2}_{\theta_{rs}}$, i.e.

$$\sup_{mn} \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^2}_{\theta_{rs}} < \infty.$$

Let (α_{mn}) be double sequence of scalars such that $|\alpha_{mn}| \leq 1$ for all $m, n \in \mathbb{N} \times \mathbb{N}$. Then we get

$$\sup_{mn} \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(\alpha x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}} \leq \sup_{mn} \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}}.$$

This completes the proof. □

Proposition 6.

The sequence space $\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}}$ is monotone.

Proof:

The proof follows from Proposition 5.

Proposition 7.

If $f = (f_{mn})$ be any Musielak function. Then,

$$\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^*}_{p}\right]^{I^2}_{\theta_{rs}} \subset \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^{**}}_{p}\right]^{I^2}_{\theta_{rs}},$$

if and only if, $\sup_{r,s\geq 1} \frac{\varphi^*_{rs}}{\varphi^{**}_{rs}} < \infty$.

Proof:

Let $x \in \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^*}_{p}\right]^{I^2}_{\theta_{rs}}$ and $N = \sup_{r,s\geq 1} \frac{\varphi^*_{rs}}{\varphi^{**}_{rs}} < \infty$. Then we get

$$\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^{**}_{rs}}_{p}\right]^{I^2}_{\theta_{rs}} = N \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^*_{rs}}_{p}\right]^{I^2}_{\theta_{rs}} = 0.$$

Thus $x \in \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^{**}}_{p}\right]^{I^2}_{\theta_{rs}}$. Conversely, suppose that

$$\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^*}_{p}\right]^{I}_{N_\theta} \subset \left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^{**}}_{p}\right]^{I^2}_{\theta_{rs}}$$

and $x \in \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^*}_{p}\right]^{I^2}_{\theta_{rs}}$. Then
$\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^*}_{p}\right]^{I^2}_{\theta_{rs}} < \epsilon$, for every $\epsilon > 0$.

Suppose that $\sup_{r,s\geq 1}\frac{\varphi^{*}_{rs}}{\varphi^{**}_{rs}} = \infty$, then there exists a sequence of members (rs_{jk}) such that $\lim_{j,k\to\infty}\frac{\varphi^{*}_{jk}}{\varphi^{**}_{jk}} = \infty$. Hence, we have

$$\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^{*}_{rs}}_{p}\right]^{I^2}_{\theta_{rs}} = \infty.$$

Therefore

$$x \notin \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^{**}}_{p}\right]^{I^2}_{\theta_{rs}},$$

which is a contradiction. This completes the proof. □

Proposition 8.

If $f = (f_{mn})$ be any Musielak function. Then

$$\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^{*}}_{p}\right]^{I^2}_{\theta_{rs}} = \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^{**}}_{p}\right]^{I^2}_{\theta_{rs}},$$

if and only if $\sup_{r,s\geq 1}\frac{\varphi^{*}_{rs}}{\varphi^{**}_{rs}} < \infty, \sup_{r,s\geq 1}\frac{\varphi^{**}_{rs}}{\varphi^{*}_{rs}} > \infty$.

Proof:

It is easy to prove so we omit.

Proposition 9.

The sequence space $\left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}}$ is not solid.

Proof:

The result follows from the following example.

Example.

Consider

$$x = (x_{mn}) = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ . & & & \\ . & & & \\ . & & & \\ 1 & 1 & \ldots & 1 \end{pmatrix} \in \left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}}.$$

Let

$$\alpha_{mn} = \begin{pmatrix} -1^{m+n} & -1^{m+n} & \ldots & -1^{m+n} \\ -1^{m+n} & -1^{m+n} & \ldots & -1^{m+n} \\ . & & & \\ . & & & \\ . & & & \\ -1^{m+n} & -1^{m+n} & \ldots & -1^{m+n} \end{pmatrix}, \text{ for all } m, n \in \mathbb{N}.$$

Then $\alpha_{mn}x_{mn} \notin \left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}}$. Hence $\left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}}$ is not solid.

Proposition 10.

The sequence space $\left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}}$ is not monotone.

Proof:

The proof follows from Proposition 9.

4. Conclusion

We introduce generalized difference sequence spaces via ideal convergence, lacunary of χ^2 sequence spaces over $p-$ metric spaces defined by Musielak-Orlicz function and also discuss some topological properties of our proved results on these spaces. The growing interest in this field is strongly stimulated by the treatment of recent problems in elasticity, fluid dynamics, calculus of variations, and differential equations. One can extend our results for more general spaces.

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