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Abstract

A problem of reducing intuitionistic fuzzy matrices is examined and some useful properties are obtained with respect to nilpotent intutionistic fuzzy matrices. First, reduction of irreflexive and transitive intuitionistic fuzzy matrices are considered, and then the properties are applied to nilpotent intutionistic fuzzy matrices. Nilpotent intuitionistic fuzzy matrices are intuitionistic fuzzy matrices which signify acyclic graphs, and the graphs are used to characterize consistent systems. The properties are handy for generalization of various systems with intuitionistic fuzzy transitivity.

Keywords: Intuitionistic fuzzy sets; Intuitionistic fuzzy matrix; Intuitionistic fuzzy transitive matrix; Intuitionistic fuzzy irreflexive matrix; Intuitionistic fuzzy nilpotent matrix

MSC 2010 No.: 03E72, 15B15

1. Introduction

Many theories have been put forth over the years to deal with the various types of uncertainties. These theories are put into practice and when found to be wanting are improved upon, paving the way for new theories to handle the tricky uncertainties. In 1965, Zadeh (1965) came out with the concept of fuzzy set which is indeed an extension of the classical notion of set. However, it often falls short of the expected standard when describing the neutral state. As a result, a new concept namely Intuitionistic Fuzzy Set (IFS) was introduced by Atanassov (1983) and represented it as $A = [\langle x, \mu_A(x), \nu_A(x)\rangle /x \in E$, where E denotes a universal set] in which $\mu_A : E \to [0,1]$ and $\nu_A : E \to [0,1]$ denote membership and non-membership functions of A, respectively, and its sum is less than or equal to one. In short we write the elements of IFS as $\langle x, x'\rangle$ such that $x + x' \leq 1$. The ideas of IFS were developed later in Atanassov (1998, 1999, 2005a, 2005b, 2005c). Xu and Yager (2006) developed some geometric operators based on IFSs.

The notion Intuitionistic Fuzzy Matrix (IFM) was introduced by Atanssov (1987). The index matrix representation of the intuitionistic fuzzy graphs has been studied in Atanssov (1994). Pal et al. (2002), Meenakshi and Gandhimathi (2010), and Sriram and Murugadas (2011, 2010) studied IFM for finding intuitionistic fuzzy linear relation equation, g-inverse and intuitionistic fuzzy linear transformation and others. Meenakshi (2008) studied minus ordering, space ordering and schur complement of fuzzy matrix and block fuzzy matrix. Shyamal and Pal (2002) have studied the distance between IFM. Bhowmik and Pal (2008a, 2008b) examined circulant IFM and generalized IFMs. Im (2006) studied the determinant of square IFMs. In (2014) Atanassov studied intuitionistic fuzzy index matrix on the basis of index matrix and extended intuitionistic fuzzy index matrix. Several authors (Adak et al. (2011, 2012), Lee and Jeong (2005), Mondal and Pal (2013, 2014), Murugadas and Padder (2015, 2016a, 2016b), Pradhan and Pal (2012, 2013, 2014)) worked on IFMs and obtained various interesting results, which are very useful in handling uncertainty problems in our daily life.

Hashimoto (1982a, 1984, 2005) used implication operators in fuzzy matrices and studied some properties. Murugadas (2011) and Murugadas and Lalitha (2012, 2014a, 2014b, 2014c, 2016) used implication operators in IFM for obtaining g-inverse, sub-inverse and decomposition of an IFM. Tan (2005) obtained some important properties of reduction of nilpotent fuzzy matrix. Lur et al. (2003) studied simultaneously nilpotent fuzzy matrices. Lur et al. (2004) obtained some properties of nilpotent matrices in terms of eigenvalues. Han et al. (2005) studied some important properties of the reduction of nilpotent incline matrices over a additively residuated incline. Hashimoto (1982b) studied the reduction of fuzzy matrices and obtained some results of nilpotent fuzzy matrices. The purpose of this paper is to study the reduction of nilpotent IFM by using the implication operator.

2. Preliminaries

Throughout the paper, matrix means IFM. Atanassov introduced the following operations on IFS for $\langle x, x'\rangle, \langle y, y'\rangle \in IFS$, $\langle x, x'\rangle \vee \langle y, y'\rangle = \langle max\{x, y\}, min\{x', y'\}\rangle$ and $\langle x, x'\rangle \wedge \langle y, y'\rangle = \langle min\{x, y\}, max\{x', y'\}\rangle$. $\langle x, x'\rangle \geq \langle y, y'\rangle \Rightarrow x \geq y$ and $x' \leq y'$, therefore in this case we say $\langle x, x'\rangle$ and $\langle y, y'\rangle$ are comparable.

For any two comparable elements $\langle x, x'\rangle$, $\langle y, y'\rangle \in$ IFS, the operation $\langle x, x'\rangle \leftharpoondown \langle y, y'\rangle$ is defined as

$$\langle x, x' \rangle \leftharpoondown \langle y, y' \rangle = \begin{cases} \langle x, x' \rangle & \text{if } \langle x, x' \rangle > \langle y, y' \rangle, \\ \langle 0, 1 \rangle & \text{if } \langle x, x' \rangle \leq \langle y, y' \rangle. \end{cases}$$

Definition 2.1. (Pal et al. (2002))

Let $X = \{x_1, x_2, ...x_m\}$ be a set of alternatives and $Y = \{y_1, y_2, ...y_n\}$ be the attribute set of each element of X. An Intuitionistic Fuzzy Matrix (IFM) is defined by

$$A = (\langle (x_i, y_j), \mu_A(x_i, y_j), \nu_A(x_i, y_j) \rangle),$$

for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$, where $\mu_A : X \times Y \to [0, 1]$ and $\nu_A : X \times Y \to [0, 1]$ satisfies the condition $0 \leq \mu_A(x_i, y_j) + \nu_A(x_i, y_j) \leq 1$. For simplicity we denote an intuitionistic fuzzy matrix (IFM) is a matrix of pairs $A = (\langle a_{ij}, a'_{ij} \rangle)$ of a non negative real numbers satisfying $a_{ij} + a'_{ij} \leq 1$ for all i, j. We denote the set of all IFM of order $m \times n$ by $\mathscr{F}_{mn}$.

Some usual matrix operations on IFMs are listed below.

For $n \times n$ IFMs $Q = (\langle q_{ij}, q'_{ij} \rangle)$ and $S = (\langle s_{ij}, s'_{ij} \rangle)$ some operations are defined as follows:

$Q \vee S = (\langle q_{ij} \vee s_{ij}, q'_{ij} \wedge s'_{ij} \rangle)$,
$Q \wedge S = (\langle q_{ij} \wedge s_{ij}, q'_{ij} \vee s'_{ij} \rangle)$,
$Q \overset{c}{\leftharpoondown} S = (\langle q_{ij}, q'_{ij} \rangle \overset{c}{\leftharpoondown} \langle s'_{ij}, s'_{ij} \rangle)$ (component wise),
$Q \times S = (\langle q_{i1} \wedge s_{1j}, q'_{i1} \vee s'_{1j} \rangle \vee \langle q_{i2} \wedge s_{2j}, q'_{i2} \vee s'_{2j} \rangle \vee \ldots \vee \langle q_{in} \wedge s_{nj}, q'_{in} \vee s'_{nj} \rangle)$,
$Q/S = Q \overset{c}{\leftharpoondown} (Q \times S)$,
$Q^1 = Q$,
$Q^{k+1} = Q^k \times Q, \quad k = 1, 2, 3...$,
$Q^+ = Q \vee Q^2 \vee ... \vee Q^n$, and
$Q \leq S (S \geq Q)$ if and only if $\langle q_{ij}, q'_{ij} \rangle \leq \langle s_{ij}, s'_{ij} \rangle$.

A zero matrix is a matrix whose entries are $\langle 0, 1 \rangle$.

An IFM Q is called transitive if $Q^2 \leq Q$. An IFM Q is said to be irreflexive if all of its diagonal elements are zero i.e $\langle 0, 1 \rangle$. An IFM Q is said to be nilpotent if there exists a natural number n, such that $Q^n = (\langle 0, 1 \rangle)$.

The transitive closure of a matrix Q is Q^+ and closure properties are familiar. A transitive matrix Q is a matrix which signifies a transitive relation. Transitive relations are used in several applications (Ovchinnikov (1981), Zadeh (1971)).

If a matrix is irreflexive and transitive the matrix is nilpotent, which implies if Q is nilpotent then Q is irreflexive and there is a permutation matrix P such that $P \times Q \times P^T$ is an upper triangular or lower triangular with $\langle 0, 1 \rangle$ diagonal elements, where P^T is the transpose of P.

From definition $Q/Q = Q \overset{c}{\leftharpoondown} (Q \times Q)$, the (i,j) entry of Q/Q is either $\langle q_{ij}, q'_{ij} \rangle$ or $\langle 0, 1 \rangle$.

3. Reduction of Intuitionistic Fuzzy Nilpotent Matrix (IFNM)

During the present study, we examine the reduction of irreflexive and transitive IFMs and obtain some theorems. Then we use these theorems to nilpotent IFMs in order to obtain some properties of reduction of nilpotent IFMs.

Theorem 3.1.

If Q is an $n \times n$ irreflexive and transitive IFM, then

$$(Q/Q)^{+} = Q.$$

Proof:

Since $S = (\langle s_{ij}, s'_{ij}\rangle) = Q/Q$, $S^k = (\langle s_{ij}^{(k)}, s_{ij}'^{(k)}\rangle)$. Let Q and S are nilpotent.

Since, clearly,

$$(Q/Q)^{+} = (Q/Q) \vee (Q/Q)^2 \vee ... \vee (Q/Q)^{n-1} \leq Q,$$

we have to prove that

$$Q \leq (Q/Q) \vee (Q/Q)^2 \vee ... \vee (Q/Q)^{n-1}.$$

That is, $\langle q_{ij}, q'_{ij}\rangle \leq \langle s_{ij}^{(k)}, s_{ij}'^{(k)}\rangle$ for some $k(1 \leq k \leq n-1)$.

Suppose that $\langle q_{ij}, q'_{ij}\rangle \geq \langle s_{ij}^{(k)}, s_{ij}'^{(k)}\rangle$ for every $k = 1, 2, ..., n-1$.

(1) (a) Since $\langle s_{ij}, s'_{ij}\rangle \leq \langle q_{ij}, q'_{ij}\rangle$, we get

$$\langle s_{ij}, s'_{ij}\rangle = \langle q_{ij}, q'_{ij}\rangle \leftharpoondown \langle \bigvee_{k=1}^{n} (q_{il} \wedge q_{lj}), \bigwedge_{k=1}^{n} (q'_{il} \vee q'_{lj})\rangle = \langle 0, 1\rangle.$$

That is,

$$\langle q_{ij}, q'_{ij}\rangle \leq \langle \bigvee_{l=1}^{n} (q_{il} \wedge q_{lj}), \bigwedge_{k=1}^{n} (q'_{il} \vee q'_{lj})\rangle.$$

Thus, we can find l_{11} such that $\langle q_{il_{11}}, q'_{il_{11}}\rangle \geq \langle q_{ij}, q'_{ij}\rangle$ and $\langle q_{l_{11}i}, q'_{l_{11}i}\rangle \geq \langle q_{ij}, q'_{ij}\rangle$.

It follows that $\langle q_{ij}^{(2)}, q_{ij}'^{(2)}\rangle \geq \langle q_{ij}, q'_{ij}\rangle > \langle 0, 1\rangle$. Therefore, $\langle q_{ij}^{(2)}, q_{ij}'^{(2)}\rangle = \langle q_{ij}, q'_{ij}\rangle$, since Q is transitive.

We now prove that $\langle s_{il_{1_{p(1)}}}, s'_{il_{1_{p(1)}}}\rangle \geq \langle q_{ij}, q'_{ij}\rangle$ and $\langle q_{l_{1_{p(1)}}}, q'_{l_{1_{p(1)}}}\rangle \geq \langle q_{ij}, q'_{ij}\rangle$ for a few $l_{1_{p(1)}}$.

(b) If, $\langle s_{il_{11}}, s'_{il_{11}}\rangle \geq \langle q_{ij}, q'_{ij}\rangle$, then we put $p(1) = 1$.

If, $\langle s_{il_{11}}, s'_{il_{11}}\rangle = \langle 0,1\rangle$, that is, if

$$\langle q_{ij}, q'_{ij}\rangle \leftharpoondown \langle \bigvee_{k=1}^{n}(q_{il}\wedge q_{ll_{11}}), \bigwedge_{k=1}^{n}(q'_{il}\vee q'_{ll_{11}})\rangle = \langle 0,1\rangle,$$

then, $\langle q_{il_{11}}, q'_{il_{11}}\rangle \leq \langle \bigvee_{k=1}^{n}(q_{il}\wedge q_{ll_{11}}), \bigwedge_{k=1}^{n}(q'_{il}\vee q'_{ll_{11}})\rangle$.

Since $\langle q_{il_{11}}, q'_{il_{11}}\rangle \geq \langle q_{ij}, q'_{ij}\rangle$, we get

$$\langle q_{il_{12}}, q'_{il_{12}}\rangle \geq \langle q_{ij}, q'_{ij}\rangle, \langle q_{l_{12}l_{11}}, q'_{l_{12}l_{11}}\rangle \geq \langle q_{ij}, q'_{ij}\rangle \text{ and}$$

$$\langle q^{(3)}_{ij}, q'^{(3)}_{ij}\rangle = \langle q_{ij}, q'_{ij}\rangle > \langle 0,1\rangle \text{ for a few } l_{12}.$$

Further, since

$$(\langle q_{l_{12}l_{11}}\wedge q_{l_{11}j}, q'_{l_{12}l_{11}}\vee q'_{l_{11}j}\rangle) \leq \langle q_{l_{12}j}, q'_{l_{12}j}\rangle \text{ and } \langle q_{l_{11}j}, q'_{l_{11}j}\rangle \geq \langle q_{ij}, q'_{ij}\rangle,$$

we have $\langle q_{l_{12}j}, q'_{l_{12}j}\rangle \geq \langle q_{ij}, q'_{ij}\rangle$.

(c) Moreover, if $\langle s_{il_{11}}, s'_{il_{11}}\rangle \geq \langle q_{ij}, q'_{ij}\rangle$, then we put $p(1) = 2$.

If, $\langle s_{il_{12}}, s'_{il_{12}}\rangle = \langle 0,1\rangle$ that is, $\langle q_{il_{12}}, q'_{il_{12}}\rangle \overset{c}{\leftharpoondown} \langle \bigvee_{k=1}^{n}(q_{il}\wedge q_{ll_{12}}), \bigwedge_{k=1}^{n}(q'_{il}\vee q'_{ll_{12}})\rangle = \langle 0,1\rangle$,

then,

$$\langle q_{il_{12}}, q'_{il_{12}}\rangle \leq \langle \bigvee_{k=1}^{n}(q_{il}\wedge q_{ll_{12}}), \bigwedge_{k=1}^{n}(q'_{il}\vee q'_{ll_{12}})\rangle = \langle 0,1\rangle.$$

Since $\langle q_{il_{12}}, q'_{il_{12}}\rangle \geq \langle q_{ij}, q'_{ij}\rangle$, we get

$$\langle q_{il_{12}}, q'_{il_{12}}\rangle \geq \langle q_{ij}, q'_{ij}\rangle, \langle q_{l_{13}l_{12}}, q'_{l_{13}l_{12}}\rangle \geq \langle q_{ij}, q'_{ij}\rangle \text{ and}$$

$$\langle q^{(4)}_{ij}, q'^{(4)}_{ij}\rangle = \langle q_{ij}, q'_{ij}\rangle > \langle 0,1\rangle \text{ for some } l_{13}.$$

Thus, since

$$(\langle q_{l_{13}l_{12}}\wedge q_{l_{12}j}, q'_{l_{13}l_{12}}\vee q'_{l_{12}j}\rangle) \leq \langle q_{l_{13}j}, q'_{l_{13}j}\rangle \text{ and } \langle q_{l_{12}j}, q'_{l_{12}j}\rangle \geq \langle q_{ij}, q'_{ij}\rangle,$$

we have $\langle q_{l_{13}j}, q'_{l_{13}j}\rangle \geq \langle q_{ij}, q'_{ij}\rangle$.

(d) By repeating the same process, since Q is nilpotent for a few $l_{ip(1)}$ such that $p(1) < n-1$, we get

$$\langle s_{il_{1p(1)}}, s'_{il_{1p(1)}}\rangle \geq \langle q_{ij}, q'_{ij}\rangle, \langle q_{il_{1p(1)}}, q'_{il_{1p(1)}}\rangle \geq \langle q_{ij}, q'_{ij}\rangle.$$

(2) Next, since $\langle s_{ij}^{(2)}, s_{ij}'^{(2)}\rangle < \langle q_{ij}, q_{ij}'\rangle$, we get $\langle s_{l_{1p(1)j}}, s'_{l_{1p(1)j}}\rangle < \langle q_{ij}, q_{ij}'\rangle$.

Then, by $\langle q_{l_{1p(1)j}}, q'_{il_{1p(1)j}}\rangle \geq \langle q_{ij}, q_{ij}'\rangle$ it follows that

$$\langle q_{l_{1p(1)j}}, q'_{l_{1p(1)j}}\rangle \leftharpoondown \langle \bigvee_{k=1}^{n} (q_{l_{1p(1)}l} \wedge q_{ij}), \bigwedge_{k=1}^{n} (q'_{l_{1p(1)}l} \vee q'_{ij})\rangle = \langle 0, 1\rangle,$$

that is,

$$\langle q_{l_{1p(1)j}}, q'_{l_{1p(1)j}}\rangle \leq \langle \bigvee_{k=1}^{n} (q_{l_{1p(1)}l} \wedge q_{lj}), \bigwedge_{k=1}^{n} (q'_{l_{1p(1)}l} \vee q'_{lj})\rangle.$$

Since $\langle q_{ij}, q_{ij}'\rangle \leq \langle q_{l_{1p(1)j}}, q'_{l_{1p(1)j}}\rangle$, we have

$$\langle q_{l_{1p(1)}l_{21}}, q'_{l_{1p(1)}l_{21}}\rangle \geq \langle q_{ij}, q_{ij}'\rangle, \langle q_{l_{21}j}, q'_{l_{21}j}\rangle \geq \langle q_{ij}, q_{ij}'\rangle, \text{ and}$$

$$\langle q^{(2)}_{l_{1p(1)j}}, q'^{(2)}_{l_{1p(1)j}}\rangle \geq \langle q_{ij}, q_{ij}'\rangle > \langle 0, 1\rangle \text{ for a few } l_{21}.$$

By the same process as in (1) it follows as below

$$\langle s_{l_{1p(1)}l_{2p(2)}}, s'_{l_{1p(1)}l_{2p(2)}}\rangle \geq \langle q_{ij}, q_{ij}'\rangle, \langle q_{l_{2p(2)}j}, q'_{l_{2p(2)}j}\rangle \geq \langle q_{ij}, q_{ij}'\rangle, \text{ and}$$

$$\langle s^{(2)}_{il_{2p(2)}}, s'^{(2)}_{il_{2p(2)}}\rangle \geq \langle q_{ij}, q_{ij}'\rangle > \langle 0, 1\rangle \text{ for some } l_2p(2).$$

(3) By continuing the same process as above,

$$\langle s_{l_{n-1p(n-1)}l_{np(n)}}, s'_{l_{n-1p(n-1)}l_{np(n)}}\rangle \geq \langle q_{ij}, q_{ij}'\rangle, \langle q_{l_{np(n)}j}, q'_{l_{np(n)}j}\rangle \geq \langle q_{ij}, q_{ij}'\rangle, \text{ and}$$

$$\langle s^{(n)}_{il_{np(n)}}, s^{(n)}_{il_{np(n)}}\rangle \geq \langle q_{ij}, q_{ij}'\rangle > \langle 0, 1\rangle.$$

So, a contradiction arises with S nilpotent.

Thus, we get $\langle s_{ij}^{(k)}, s_{ij}^{(k)}\rangle \geq \langle q_{ij}, q_{ij}'\rangle$ for a few $k (1 \leq k \leq n-1)$, so that,

$$S^{+} = Q.$$

The following example shows that Q/Q is the reduced form of Q and it is enough if we calculate the transitive closure of Q/Q instead of calculating transitive closure of Q.

Example 3.1.

Let Q be the following irreflexive and transitive intuitionistic fuzzy matrix

$$Q = \begin{pmatrix} \langle 0.0, 1.0\rangle & \langle 0.4, 0.3\rangle & \langle 0.0, 1.0\rangle \\ \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle \\ \langle 0.4, 0.3\rangle & \langle 0.4, 0.3\rangle & \langle 0.0, 1.0\rangle \end{pmatrix} \text{ (irreflexive)},$$

$$Q^2 = \begin{pmatrix} \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle \\ \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle \\ \langle 0.0, 1.0\rangle & \langle 0.4, 0.3\rangle & \langle 0.0, 1.0\rangle \end{pmatrix} \text{ (Transitive } Q^2 \leq Q\text{)}.$$

Now,

$$Q/Q = Q \overset{c}{\leftharpoondown} (Q \times Q),$$

$$Q/Q = \begin{pmatrix} \langle 0.0, 1.0\rangle & \langle 0.4, 0.3\rangle & \langle 0.0, 1.0\rangle \\ \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle \\ \langle 0.4, 0.3\rangle & \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle \end{pmatrix},$$

$$(Q/Q)^+ = (Q/Q) \vee (Q/Q)^2 \vee (Q/Q)^3,$$

$$(Q/Q)^+ = \begin{pmatrix} \langle 0.0, 1.0\rangle & \langle 0.4, 0.3\rangle & \langle 0.0, 1.0\rangle \\ \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle \\ \langle 0.4, 0.3\rangle & \langle 0.4, 0.3\rangle & \langle 0.0, 1.0\rangle \end{pmatrix}.$$

Therefore,

$$Q = (Q/Q)^+ = (Q/Q) \vee (Q/Q)^2 \vee (Q/Q)^3. \tag{1}$$

Remark 3.2.

In the above theorem irreflexivity and transitivity are essential. If any one of the above two conditions fail, then the result will fail. It is illustrated through the following example.

Example 3.2.

Let $Q = \begin{pmatrix} \langle 0.0, 1.0\rangle & \langle 0.2, 0.5\rangle \\ \langle 0.0, 1.0\rangle & \langle 0.1, 0.7\rangle \end{pmatrix}$. Clearly Q is not irreflexive.

$Q^2 = \begin{pmatrix} \langle 0.0, 1.0\rangle & \langle 0.1, 0.7\rangle \\ \langle 0.0, 1.0\rangle & \langle 0.1, 0.7\rangle \end{pmatrix} \Rightarrow (Q^2 \leq Q)$. So Q is transitive.

Now,

$$Q/Q = Q \overset{c}{\leftharpoondown} (Q \times Q),$$

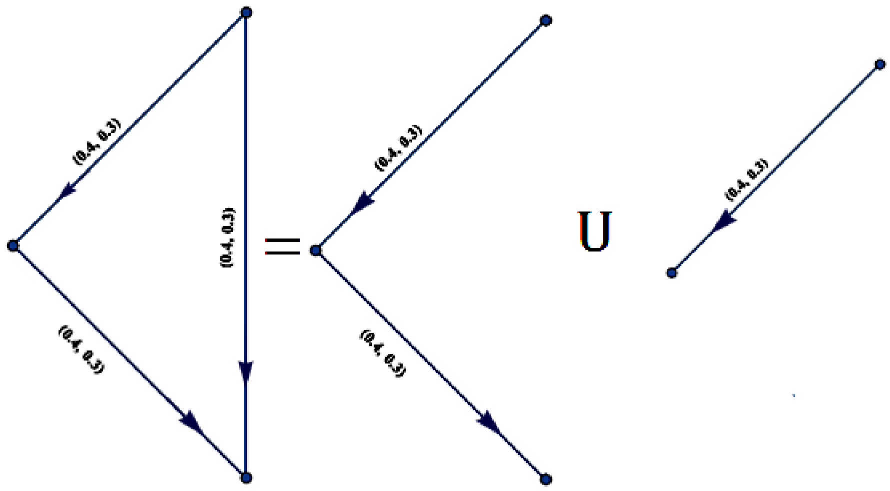

Figure 1: The graphical representation of equation (1)

$$Q/Q = \begin{pmatrix} \langle 0.0, 1.0\rangle & \langle 0.2, 0.5\rangle \\ \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle \end{pmatrix},$$

$$(Q/Q)^2 = \begin{pmatrix} \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle \\ \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle \end{pmatrix},$$

$$(Q/Q)^+ = (Q/Q) \vee (Q/Q)^2,$$

$$(Q/Q)^+ = \begin{pmatrix} \langle 0.0, 1.0\rangle & \langle 0.2, 0.5\rangle \\ \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle \end{pmatrix}.$$

Therefore,

$$Q \neq (Q/Q)^+ = (Q/Q) \vee (Q/Q)^2. \tag{2}$$

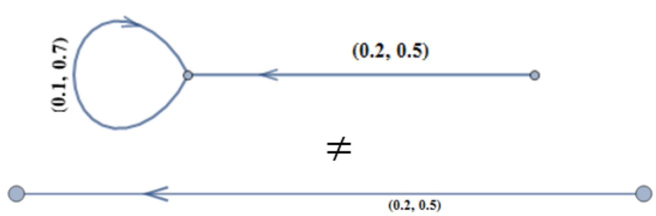

Figure 2: The graphical representation of equation (2)

Example 3.3.

Let

$Q = \begin{pmatrix} \langle 0.0, 1.0\rangle & \langle 0.4, 0.3\rangle & \langle 0.2, 0.5\rangle \\ \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle \\ \langle 0.4, 0.3\rangle & \langle 0.4, 0.3\rangle & \langle 0.0, 1.0\rangle \end{pmatrix}$. Clearly Q is irreflexive.

$Q^2 = \begin{pmatrix} \langle 0.2, 0.5\rangle & \langle 0.2, 0.5\rangle & \langle 0.0, 1.0\rangle \\ \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle \\ \langle 0.4, 0.3\rangle & \langle 0.0, 1.0\rangle & \langle 0.2, 0.5\rangle \end{pmatrix} \Rightarrow Q^2 \nleq Q$. So Q is not transitive.

Now,

$$Q/Q = Q \overset{c}{\leftarrow} (Q \times Q),$$

$$Q/Q = \begin{pmatrix} \langle 0.0, 1.0\rangle & \langle 0.4, 0.3\rangle & \langle 0.2, 0.5\rangle \\ \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle \\ \langle 0.4, 0.3\rangle & \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle \end{pmatrix},$$

$$(Q/Q)^+ = (Q/Q) \vee (Q/Q)^2 \vee (Q/Q)^3,$$

$$(Q/Q)^+ = \begin{pmatrix} \langle 0.2, 0.5\rangle & \langle 0.4, 0.3\rangle & \langle 0.2, 0.5\rangle \\ \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle \\ \langle 0.4, 0.3\rangle & \langle 0.4, 0.3\rangle & \langle 0.2, 0.5\rangle \end{pmatrix}.$$

Therefore,

$$Q \neq (Q/Q)^+ = (Q/Q) \vee (Q/Q)^2 \vee (Q/Q)^3. \quad (3)$$

Theorem 3.3.

Let Q be an $n \times n$ irreflexive and transitive matrix. Then, the following conditions are equivalent:

(1) $Q/Q \leq S \leq Q$

(2) $S^+ = Q$ for any $n \times n$ IFM S.

Proof:

Let $S^k = (\langle s_{ij}^{(k)}, s_{ij}^{\prime(k)}\rangle)$ and $T = (\langle t_{ij}, t'_{ij}\rangle) = Q/Q$. That is,

$$\langle t_{ij}, t'_{ij}\rangle = \langle q_{ij}, q'_{ij}\rangle \overset{c}{\leftarrow} \langle \bigvee_{k=1}^{n} (q_{ik} \wedge q_{kj}), \bigwedge_{k=1}^{n} (q'_{ik} \vee q'_{kj})\rangle.$$

(1) Suppose that $Q/Q \leq S \leq Q$. Clearly from Theorem 3.1, $S^+ = Q$.

(2) Suppose that $S^+ = Q$, then we get $S \leq Q$.

(a) Let $n = 1$.

The only irreflexive matrix is $\langle 0, 1\rangle$. Thus,

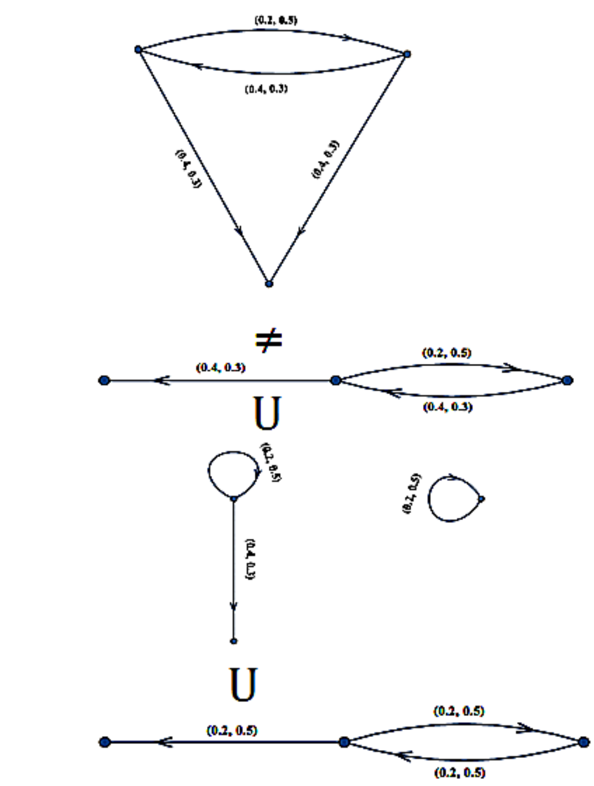

Figure 3: The graphical representation of equation (3)

$$S = Q \leq Q/Q.$$

(b) Let $n = 2$. Since S is nilpotent ($S^2 = \langle 0, 1 \rangle$), we get

$$Q/Q \leq Q = S^+ = S \vee S^2 = S.$$

(c) Let $n \leq 3$. Suppose that $\langle s_{ij}, s'_{ij} \rangle < \langle t_{ij}, t'_{ij} \rangle$, then

$\langle t_{ij}, t'_{ij} \rangle = \langle q_{ij}, q'_{ij} \rangle > \langle 0, 1 \rangle$ and $\langle \bigvee\limits_{k=1}^{n} (q_{ik} \wedge q_{kj}), \bigwedge\limits_{k=1}^{n} (q'_{ik} \vee q'_{kj}) \rangle < \langle q_{ij}, q'_{ij} \rangle$.

Now we get,

$\langle s_{il_1}, s'_{il_1} \rangle \wedge \langle s_{l_1 l_2}, s'_{l_1 l_2} \rangle \wedge ... \wedge \langle s_{l_h j}, s'_{l_h j} \rangle = \langle q_{ij}, q'_{ij} \rangle$ for suitable indices $l_1, l_2, ..., l_h (1 \leq h \leq n-2)$, so that,

$$\langle q_{il_1}, q'_{il_1} \rangle \wedge \langle q_{l_1 l_2}, q'_{l_1 l_2} \rangle \wedge ... \wedge \langle q_{l_h j}, q'_{l_h j} \rangle \leq \langle q_{ij}, q'_{ij} \rangle.$$

Thus, $\langle q_{il_1}, q'_{il_1} \rangle \geq \langle q_{ij}, q'_{ij} \rangle$ and $\langle q_{l_1 j}, q'_{l_1 j} \rangle \geq \langle q_{ij}, q'_{ij} \rangle$ for l_1. Then,

$$\langle \bigvee_{k=1}^{n} (q_{ik} \wedge q_{kj}), \bigwedge_{k=1}^{n} (q'_{ik} \vee q'_{kj}) \rangle \geq \langle q_{ij}, q'_{ij} \rangle.$$

$$\langle t_{ij}, t'_{ij}\rangle = \langle q_{ij}, q'_{ij}\rangle \leftharpoondown \langle \bigvee_{k=1}^{n}(r_{ik}\wedge r_{kj}), \bigwedge_{k=1}^{n}(r'_{ik}\vee r'_{kj})\rangle = \langle 0,1\rangle,$$

which contradicts the fact that $\langle t_{ij}, t'_{ij}\rangle > \langle 0,1\rangle$. Thus, $T \leq S$, so that $Q/Q \leq S \leq Q$.

By the properties of Theorem 3.3 above, Q/Q is minimal in the set of IFMs such that

$$S^{+} = Q.$$

Theorem 3.4.

Let Q be an $n \times n$ irreflexive and transitive IFM. Then, the following conditions are equivalent:

(1) $Q/Q \leq S \leq Q$.

(2) $Q/Q = S/Q$.

Proof:

Let $F = (\langle f_{ij}, f'_{ij}\rangle) = Q/Q$ and $G = (\langle g_{ij}, g'_{ij}\rangle) = S/Q$. Then,

$$\langle f_{ij}, f'_{ij}\rangle = \langle q_{ij}, q'_{ij}\rangle \leftharpoondown \langle \bigvee_{k=1}^{n}(q_{ik}\wedge q_{kj}), \bigwedge_{k=1}^{n}(q'_{ik}\vee q'_{kj})\rangle,$$
$$\langle g_{ij}, g'_{ij}\rangle = \langle s_{ij}, s'_{ij}\rangle \leftharpoondown \langle \bigvee_{k=1}^{n}(s_{ik}\wedge s_{kj}), \bigwedge_{k=1}^{n}(s'_{ik}\vee s'_{kj})\rangle.$$

(1) $\Rightarrow$ (2): Suppose that $Q/Q \leq S \leq Q$, so that $\langle f_{ij}, f'_{ij}\rangle \leq \langle s_{ij}, s'_{ij}\rangle \leq \langle r_{ij}, r'_{ij}\rangle$.

(a) First we show that, $\langle f_{ij}, f'_{ij}\rangle \leq \langle g_{ij}, g'_{ij}\rangle$. Let $\langle f_{ij}, f'_{ij}\rangle > \langle 0,1\rangle$.

Then, $\langle f_{ij}, f'_{ij}\rangle = \langle q_{ij}, q'_{ij}\rangle > \langle 0,1\rangle$, so that $\langle s_{ij}, s'_{ij}\rangle = \langle q_{ij}, q'_{ij}\rangle$ and

$$\langle \bigvee_{k=1}^{n}(q_{ik}\wedge q_{kj}), \bigwedge_{k=1}^{n}(q'_{ik}\vee q'_{kj})\rangle < \langle q_{ij}, q'_{ij}\rangle.$$

Since $\langle q_{ik}, q'_{ik}\rangle \geq \langle s_{ik}, s'_{ik}\rangle$, we have $\langle \bigvee_{k=1}^{n}(s_{ik}\wedge q_{kj}), \bigwedge_{k=1}^{n}(s'_{ik}\vee q'_{kj})\rangle < \langle q_{ij}, q'_{ij}\rangle$.

Thus,

$$\begin{aligned}\langle g_{ij}, g'_{ij}\rangle &= \langle s_{ij}, s'_{ij}\rangle \leftharpoondown \langle \bigvee_{k=1}^{n}(s_{ik}\wedge q_{kj}), \bigwedge_{k=1}^{n}(s'_{ik}\vee q'_{kj})\rangle \\ &= \langle q_{ij}, q'_{ij}\rangle \leftharpoondown \langle \bigvee_{k=1}^{n}(s_{ik}\wedge q_{kj}), \bigwedge_{k=1}^{n}(s'_{ik}\vee q'_{kj})\rangle\end{aligned}$$

so that $\langle f_{ij}, f'_{ij}\rangle \leq \langle g_{ij}, g'_{ij}\rangle$.

(b) Next we show that $\langle g_{ij}, g'_{ij}\rangle \leq \langle f_{ij}, f'_{ij}\rangle$.

Let $\langle g_{ij}, g'_{ij}\rangle > \langle 0, 1\rangle$, then

$$\langle g_{ij}, g'_{ij}\rangle = s_{ij}, s'_{ij} > \langle 0, 1\rangle, \text{ and hence,}$$

$$\langle \bigvee_{k=1}^{n}(s_{ik} \wedge q_{kj}), \bigwedge_{k=1}^{n}(s'_{ik} \vee q'_{kj})\rangle < \langle q_{ij}, q'_{ij}\rangle.$$

Recall that,

$$\langle f_{ij}, f'_{ij}\rangle = \langle q_{ij}, q'_{ij}\rangle \leftharpoondown \langle \bigvee_{k=1}^{n}(q_{ik} \wedge q_{kj}), \bigwedge_{k=1}^{n}(q'_{ik} \vee q'_{kj})\rangle \leq \langle s_{ij}, s'_{ij}\rangle \leq \langle q_{ij}, q'_{ij}\rangle.$$

Since $\langle s_{ij}, s'_{ij}\rangle > \langle 0, 1\rangle$, we have $\langle q_{ij}, q'_{ij}\rangle > \langle 0, 1\rangle$.

We shall show that if $\langle f_{ij}, f'_{ij}\rangle < \langle q_{ij}, q'_{ij}\rangle$, then there is a contradiction.

Suppose that $\left\langle f_{ij}, f'_{ij}\right\rangle < \left\langle q_{ij}, q'_{ij}\right\rangle$. Then,

$$\langle \bigvee_{k=1}^{n}(q_{ik} \wedge q_{kj}), \bigwedge_{k=1}^{n}(q'_{ik} \vee q'_{kj})\rangle \geq \langle q_{ij}, q'_{ij}\rangle \geq \langle 0, 1\rangle,$$

so that

$$\langle q_{ik(1)}, q'_{ik(1)}\rangle \geq \langle q_{ij}, q'_{ij}\rangle, \ \langle q_{k(1)j}, q'_{k(1)j}\rangle \geq \langle q_{ij}, q'_{ij}\rangle, \text{ and}$$

$$\langle q^{(2)}_{ik(1)}, q'^{(2)}_{ik(1)}\rangle = \langle q_{ij}, q'_{ij}\rangle > \langle q_{ij}, q'_{ij}\rangle, \text{ for some k(1).}$$

We have $\langle s_{ik(1)}, s'_{ik(1)}\rangle \leq \langle q_{ij}, q'_{ij}\rangle$, since $\langle q_{k(1)j}, q'_{k(1)j}\rangle \geq \langle q_{ij}, q'_{ij}\rangle$ and

$$\langle \bigvee_{k=1}^{n}(q_{ik} \wedge q_{kj}), \bigwedge_{k=1}^{n}(q'_{ik} \vee q'_{kj})\rangle < \langle s_{ij}, s'_{ij}\rangle \leq \langle q_{ij}, q'_{ij}\rangle.$$

Then, since $F \leq S$, we have $\langle f_{ik(1)}, f'_{ik(1)}\rangle < \langle q_{ij}, q'_{ij}\rangle$.

Furthermore, $\langle f_{ik(1)}, f'_{ik(1)}\rangle < \langle q_{ik(1)}, q'_{ik(1)}\rangle$, since $\langle q_{ik(1)}, q'_{ik(1)}\rangle \geq \langle q_{ij}, q'_{ij}\rangle$.

Thus,

$$\langle \bigvee_{k=1}^{n}(q_{ik} \wedge q_{kk(1)}), \bigwedge_{k=1}^{n}(q'_{ik} \vee q'_{kk(1)})\rangle \geq \langle s_{ik(1)}, s'_{ik(1)}\rangle \geq \langle q_{ij}, q'_{ij}\rangle.$$

Therefore,

$$\langle q_{ik(2)}, q'_{ik(2)}\rangle \geq \langle q_{ij}, q'_{ij}\rangle,\ \langle q_{k(2)k(1)}, q'_{k(2)k(1)}\rangle \geq \langle q_{ij}, q'_{ij}\rangle, \text{ and}$$

$$\langle q^{(3)}_{ij}, q'^{(3)}_{ij}\rangle = \langle q_{ij}, q'_{ij}\rangle > \langle 0, 1\rangle, \text{ for some k(2).}$$

Since $\langle q_{k(1)j}, q'_{k(1)j}\rangle \geq \langle q_{ij}, q'_{ij}\rangle$ and $\langle q_{k(2)k(1)}, q'_{k(2)k(1)}\rangle \geq \langle q_{ij}, q'_{ij}\rangle$,
we have $\langle q_{k(2)j}, q'_{k(2)j}\rangle \geq \langle q_{ij}, q'_{ij}\rangle$, so that $\langle s_{ik(2)}, s'_{ik(2)}\rangle < \langle q_{ij}, q'_{ij}\rangle$, since

$$\langle \bigvee_{k=1}^{n} (s_{ik} \wedge r_{kj}), \bigwedge_{k=1}^{n} (s'_{ik} \vee r'_{kj})\rangle < \langle s_{ij}, s'_{ij}\rangle \leq \langle q_{ij}, q'_{ij}\rangle.$$

Then, since $F \leq S$, we have $\langle f_{ik(2)}, f'_{ik(2)}\rangle < \langle q_{ij}, q'_{ij}\rangle$.
Moreover, $\langle f_{ik(2)}, f'_{ik(2)}\rangle < \langle q_{ik(2)}, q'_{ik(2)}\rangle$, since $\langle q_{ik(2)}, q'_{ik(2)}\rangle \geq \langle q_{ij}, q'_{ij}\rangle$.
Thus,

$$\langle \bigvee_{k=1}^{n} (q_{ik} \wedge q_{kk(2)}), \bigwedge_{k=1}^{n} (q'_{ik} \vee q'_{kk(2)})\rangle \geq \langle q_{ik(2)}, q'_{ik(2)}\rangle \geq \langle q_{ij}, q'_{ij}\rangle.$$

Therefore,

$$\langle q_{ik(3)}, q'_{ik(3)}\rangle \geq \langle q_{ij}, q'_{ij}\rangle,$$

$$\langle q_{k(3)k(2)}, q'_{k(3)k(2)}\rangle \geq \langle q_{ij}, q'_{ij}\rangle, \text{ and}$$

$$\langle q^{(4)}_{ij}, q'^{(4)}_{ij}\rangle = \langle q_{ij}, q'_{ij}\rangle > \langle 0, 1\rangle, \text{ for a few } k(3).$$

By continuing the same process, we get $\langle q^{(n)}_{ij}, q^{(n)}_{ij}\rangle > \langle 0, 1\rangle$. That is a contradiction, since Q is nilpotent.

Thus, $\langle f_{ij}, f'_{ij}\rangle \geq \langle q_{ij}, q'_{ij}\rangle$, so that $\langle g_{ij}, g'_{ij}\rangle = \langle s_{ij}, s'_{ij}\rangle \leq \langle q_{ij}, q'_{ij}\rangle \leq \langle f_{ij}, f'_{ij}\rangle$.

(2) $\Rightarrow$ (1) Suppose that $Q/Q = S/Q$.

(a) It is evident that $Q/Q = S/Q \leq S$.

(b) We prove that $S \leq Q$. Suppose that $\langle s_{ij}, s'_{ij}\rangle > \langle q_{ij}, q'_{ij}\rangle$. Since

$$\langle q_{ij}, q'_{ij}\rangle \leftharpoondown \langle \bigvee_{k=1}^{n} (q_{ik} \wedge q_{kj}), \bigwedge_{k=1}^{n} (q'_{ik} \vee q'_{kj})\rangle = \langle s_{ij}, s'_{ij}\rangle \leftharpoondown \langle \bigvee_{k=1}^{n} (s_{ik} \wedge q_{kj}), \bigwedge_{k=1}^{n} (s'_{ik} \vee q'_{kj})\rangle,$$

we have

$$\langle s_{ij}, s'_{ij}\rangle \leftharpoondown \langle \bigvee_{k=1}^{n}(q_{ik}\wedge q_{kj}), \bigwedge_{k=1}^{n}(s'_{ik}\vee s'_{kj})\rangle = \langle 0,1\rangle, \text{ so that}$$

$$\langle \bigvee_{k=1}^{n}(s_{ik}\wedge q_{kj}), \bigwedge_{k=1}^{n}(s'_{ik}\vee q'_{kj})\rangle \geq \langle s_{ij}, s'_{ij}\rangle > \langle 0,1\rangle.$$

Then,

$$\langle s_{ik(1)}, s'_{ik(1)}\rangle \geq \langle s_{ij}, s'_{ij}\rangle,$$

$$\langle q_{k(1)j}, q'_{k(1)j}\rangle \geq \langle s_{ij}, s'_{ij}\rangle, \text{ and}$$

$$\langle q^{(1)}_{k(1)j}, q'^{(1)}_{k(1)j}\rangle = \langle s_{ij}, s'_{ij}\rangle > \langle 0,1\rangle, \text{ for a few } k(1).$$

Since

$$\langle q_{ij}, q'_{ij}\rangle < \langle s_{ij}, s'_{ij}\rangle \text{ and } \langle q_{ij}, q'_{ij}\rangle \geq \langle q_{ik(1)}, q'_{ik(1)}\rangle \wedge \langle q_{k(1)j}, q'_{k(1)j}\rangle,$$

we have $\langle q_{ik(1)}, q'_{ik(1)}\rangle < \langle s_{ij}, s'_{ij}\rangle$, so that $\langle q_{ik(1)}, q'_{ik(1)}\rangle < \langle s_{ik(1)}, s'_{ik(1)}\rangle$. Since

$$\begin{aligned}\langle q_{ik(1)}, q'_{ik(1)}\rangle &\leftharpoondown \langle \bigvee_{k=1}^{n}(q_{ik}\wedge q_{kk(1)}), \bigwedge_{k=1}^{n}(q'_{ik}\vee q'_{kk(1)})\rangle\\ = \langle s_{ij}, s'_{ij}\rangle &\leftharpoondown \langle \bigvee_{k=1}^{n}(s_{ik}\wedge q_{kk(1)}), \bigwedge_{k=1}^{n}(s'_{ik}\vee q'_{kk(1)})\rangle\end{aligned}$$

we get

$$\langle \bigvee_{k=1}^{n}(s_{ik}\wedge q_{kk(1)}), \bigwedge_{k=1}^{n}(s'_{ik}\vee q'_{kk(1)})\rangle \geq \langle s_{ik(1)}, s'_{ik(1)}\rangle \geq \langle s_{ij}, s'_{ij}\rangle.$$

Then,

$$\langle s_{ik(2)}, s'_{ik(2)}\rangle \geq \langle s_{ij}, s'_{ij}\rangle,$$

$$\langle q_{k(2)k(1)}, q'_{k(2)k(1)}\rangle \geq \langle s_{ij}, s'_{ij}\rangle, \text{ and}$$

$$\langle q^{(2)}_{k(2)j}, q'^{(2)}_{k(2)j}\rangle \geq \langle s_{ij}, s'_{ij}\rangle > \langle 0,1\rangle, \text{ for a few } k(2).$$

Since $\langle q_{ik(1)}, q'_{ik(1)}\rangle < \langle s_{ij}, s'_{ij}\rangle$ and $\langle q_{ik(1)}, q'_{ik(1)}\rangle \geq \langle q_{ik(2)}, q'_{ik(2)}\rangle \wedge \langle q_{k(2)k(1)}, q'_{k(2)k(1)}\rangle$, we get

$$\langle q_{ik(2)}, q'_{ik(2)}\rangle < \langle s_{ij}, s'_{ij}\rangle, \text{ so that } \langle q_{ik(2)}, q'_{ik(2)}\rangle < \langle s_{ik(2)}, s'_{ik(2)}\rangle. \text{ Since}$$

$$\langle q_{ik(2)}, q'_{ik(2)}\rangle \leftharpoondown \langle \bigvee_{k=1}^{n}(q_{ik}\wedge q_{kk(2)}), \bigwedge_{k=1}^{n}(q'_{ik}\vee q'_{kk(2)})\rangle$$

$$= \langle s_{ik(2)}, s'_{ik(2)}\rangle \leftharpoondown \langle \bigvee_{k=1}^{n}(s_{ik}\wedge q_{kk(2)}), \bigwedge_{k=1}^{n}(s'_{ik}\vee q'_{kk(2)})\rangle$$

we get $\langle \bigvee_{k=1}^{n}(s_{ik}\wedge q_{kk(2)}), \bigwedge_{k=1}^{n}(s'_{ik}\vee q'_{kk(2)})\rangle \geq \langle s_{ik(2)}, s'_{ik(2)}\rangle \geq \langle s_{ij}, s'_{ij}\rangle$.

Then,

$$\langle s_{ik(3)}, s'_{ik(3)}\rangle \geq \langle s_{ij}, s'_{ij}\rangle,\ \langle q_{k(3)k(2)}, q'_{k(3)k(2)}\rangle \geq \langle s_{ij}, s'_{ij}\rangle, \text{ and}$$

$$\langle q^{(3)}_{k(3)j}, q'^{(3)}_{k(3)j}\rangle \geq \langle s_{ij}, s'_{ij}\rangle > \langle 0,1\rangle, \text{ for some } k(3).$$

By continuing the same argument,

$$\langle s_{ik(n)}, s'_{ik(n)}\rangle \geq \langle s_{ij}, s'_{ij}\rangle,\ \langle q_{k(n)k(n-1)}, q'_{k(n)k(n-1)}\rangle \geq \langle s_{ij}, s'_{ij}\rangle, \text{ and}$$

$$\langle q^{(n)}_{k(n)j}, q'^{(n)}_{k(n)j}\rangle \geq \langle s_{ij}, s'_{ij}\rangle > \langle 0,1\rangle,$$

which contradicts the fact Q is nilpotent. Hence $\langle s_{ij}, s'_{ij}\rangle < \langle q_{ij}, q'_{ij}\rangle$ for all i,j.

By Theorem 3.1, Theorem 3.3 and Theorem 3.4 we find the following Corollaries, which are handy for reduction of nilpotent IFMs or acyclic graphs. We should remember that if Q is nilpotent, then Q^+ is irreflexive and transitive.

Corollary 3.5.

If Q is nilpotent, then

(1) $Q^+/Q^+ \leq Q \leq Q^+$,

(2) $Q^+/Q^+ = Q/Q^+ = Q^+/Q$,

(3) $(Q^+/Q^+)^+ = Q^+$.

Proof:

(1) Is true by taking $Q = S$, by Theorem 3.3.

(2) By (1) and Theorem 3.4 we get

$$Q/Q^+ = Q^+/Q^+ = Q^+ \overset{c}{\leftharpoondown} (Q^2 \vee Q^3 \vee ... \vee Q^n - 1) = Q^+/Q.$$

(3) By Theorem 3.1.

Corollary 3.6.

Let Q be nilpotent. Then, the following statements are equivalent:

(1) $Q/Q^+ \leq S \leq Q^+$,

(2) $Q^+ = S^+$,

(3) $Q/Q^+ = S/Q^+$.

4. Conclusion

In this article some properties of irreflexive, transitive intuitionistic fuzzy matrices are explored. Further, reduction of nilpotent IFM are obtained by applying implication operator. The concept of reduction of nilpotent IFMs are discussed.

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