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Abstract

In this discussion we consider an autonomous parabolic epidemic 2-dimensional system modelling the dynamics of transmission of immunizing diseases for a closed population into bounded regular domain. Our model takes into account diffusion of population with external influx as well as one class of infected individuals. We study the well-posedness two-component diffusion equations including external supplies with Neumann conditions using upper/lower solutions and analytic semigroups. In case of constant population or not, with non-oscillatory solution and constant diffusion, this problem admits travelling wave solutions whose minimum wave speed is surveyed here.

Keywords: Diffusion; modeling; well-posedness; upper/lower solutions; external influx; analytic semigroups; two-component diffusion equations; non-oscillatory solution

MSC 2010 No.: 35Q92, 92D30, 92D25

1. Introduction

Since the Fisher's Equation, introduced by Fisher (1937) (see Bubniakov (2007) and Murray (2001)) and works of Skellam (1951), the role of diffusion is acknowledged as important in the comprehension of the spatial dynamic of the disease (see Ducrot et al. (2011, 2014) and references therein). Another important parameter is the form of contact rate (see discussions in Thieme (2011), Carrero et al. (2003), and Calsina et al. (2014)). We work with basic diffusion as the Laplace operator and contacts between susceptible and infected individuals under mass action law. In fact according to Li et al. (2009), a more accurate force of infection under general dispersal is $\lambda_m(t,x,i,e) = \int_\Omega p_1(t,x,y)(i(t,y))^m dy$ whose second-order approximation leads to simple diffusion, for example through the kernel $p_1(t,x,y) \equiv k(x-y)$. Combining these features in one model is sometimes difficult since we need to discuss the well-posedness of the model. We present a result of well-posedness for two-component diffusion equations with Neumann conditions using upper and lower solutions and semigroup theory on a regular domain Ω. We study, under some suitable assumptions, characterizations of steady states for our 2-dimensional system. The note is organized as follow. We present the model in the next section and justify well-posedness of the solution and possible existence of steady states in the section before the conclusion.

2. Preliminaries

In the model we analyse, the spatial variable x belongs to a bounded set $\Omega \subset \mathbb{R}^n$ (with $n \in \mathbb{N} - \{0\}$) with a smooth boundary $\partial\Omega$, and a density dependant "immediate" contact rate is defined as $\lambda_m(t,x)S(t,x)$ with $m \geq 0$, $\lambda_m(t,x) = \beta_i i^m(t,x)$ and

$$\begin{cases} \partial_t S(t,x) - d_S \Delta S(t,x) = b - (\lambda_1(t,x) + \mu)\, S, \\ \partial_t i(t,x) - d_i \Delta i(t,x) = a - (d_1 + \mu)\, i(t,x) + \lambda_1(t,x) S, \\ \partial_\nu S(t,x') = 0 \text{ with } t \geq 0 \text{ with no flux and } x' \in \partial\Omega, \\ \partial_\nu i(t,x') = 0 \text{ with } t \geq 0 \text{ with no flux and } x' \in \partial\Omega, \\ S(0,x) = S_0(x) \geq 0 \text{ with } x \in \Omega, \\ i(0,x) = i_0(x) \geq 0 \text{ with } x \in \Omega. \end{cases} \tag{1}$$

We denote $\Delta := \Delta_x$. a, b are respectively the strictly positive influx of infected and susceptible individuals. Here, $S(t,x)$ denotes the space-specific density of susceptible, $i(t,x)$ respectively denotes the space-specific density of infected individuals (that can be symptomatic or asymptomatic) while $r(t,x)$ denotes the density of recovered and definitively immune individuals from infection. Since we consider that the $r-$component of the system decouples from the other, it has no impact upon the long time behavior of the system. It will then be omitted in the sequel. Moreover to perform such an analysis we shall assume that the contacts between individuals are homogeneous among the different cohorts. We assume that d_i, d_S, β_i, μ and d_1 are positive and bounded below by $0 < \kappa < 1$. Our model incorporates influx a and b which traduce the entries of new individuals in the bounded domain. These new individuals could be susceptible for b or infected for a.

The problem in Equation (1) is equivalent (after introducing a space scaling $\tilde{x} = x/\sqrt{d_i}$, $d = d_S/d_i$, $d_{x_s} = \mu$, $d_{x_i} = d_1 + d_{x_s}$) to the following autonomous semi-linear problem $(E_{a,b,d,\beta_i,d_{x_i},d_{x_s}})$

named (2):

$$\begin{cases} \partial_t i(t,x) - \Delta i(t,x) = f(i;S) = a - d_{x_i} i(t,x) + \lambda_1(t,x)S, \\ \partial_t S(t,x) - d\Delta S(t,x) = g(i;S) = b - (\lambda_1(t,x) + d_{x_s})\, S, \\ \partial_\nu i(t,x') = 0 \text{ with } t \geq 0 \text{ and } x' \in \partial\Omega, \\ \partial_\nu S(t,x') = 0 \text{ with } t \geq 0 \text{ and } x' \in \partial\Omega, \\ i(0,x) = i_0(x) \geq 0 \text{ with } x \in \Omega, \\ S(0,x) = S_0(x) \geq 0 \text{ with } x \in \Omega. \end{cases} \tag{2}$$

3. Upper, lower solutions and analytic semigroups

A problem is well-posed in the sense of Hadamard if the following conditions are satisfied:

- (W1) Existence and uniqueness;
- (W2) Existence for all times;
- (W3) Continuously dependency on initial conditions;
- (W4) The solution is non-negative for non-negative initial data;
- (W5) The solution is bounded for all bounded initial data.

The method of Pao (1992) proves (W1-W4-W5) for the system (2) while (W2-W3) require the use of semigroup theory presented in Kristiansen (2008). But for our model (2), global existence is shown using methods of Pao (1992). We characterize also steady-states with the method of Pao (1992).

3.1. Method of Pao (1992) for existence and uniqueness of solution.

3.1.1. Method of Pao (1992): evolution problem:

The method of Pao (1992) assumes that f and g are assumed to be continuously differentiable and mixed quasimonotone functions: $\partial_S f(i;S) \geq 0$ and $\partial_i g(i;S) \leq 0$ with $i_0, S_0 \in C^0_{L^\infty} \equiv C([0,T]; L^\infty(\Omega))$ for some $T > 0$. The use of upper and lower solutions follows from the following theorem proved by induction (Pao (1992), Theorem 8.3.3).

Theorem 1. [See Pao (1992, Theorem 8.3.3)]

Let $\underline{U} = (\underline{u}, \underline{v})$ and $\bar{U} = (\bar{u}, \bar{v})$ be a pair of ordered upper and lower solutions and let f and g be mixed quasimonotone. Then there exists a unique solution $U = (u,v)$, and it is in the sector

$$< \underline{U}, \bar{U} >= \left\{ Z \in \left(C^{2,1}_{L^\infty}(\Omega \times [0;T]) \right)^2 |, \underline{U} \leq Z \leq \bar{U} \right\}.$$

The following result relies on a partial application of the above Theorem 1.

Theorem 2.

Under positivity of coefficients d_i, d_S, β_i, μ, d_1, initial continuous functions i_0, S_0 and boundary conditions, system (2) has a unique nonnegative global solution localized between a lower solution and an upper solution.

Proof:

In system (2), f and g are obviously continuously differentiable and mixed quasimonotone functions.

In order to use Theorem 1 in our system (2), we need to find upper ($\underline{U}$) and lower solutions ($\bar{U}$) which satisfies for $\underline{U} = (\underline{i}; \underline{S})$ and $\bar{U} = (\bar{i}; \bar{S})$:

$$\begin{cases} \underline{U} \leq \bar{U}, \\ \partial_t \bar{i}(t,x) - \Delta \bar{i}(t,x) \geq f(\bar{i}; \bar{S}), \\ \partial_t \bar{S}(t,x) - d\Delta \bar{S}(t,x) \geq g(\underline{i}; \bar{S}), \\ \partial_t \underline{i}(t,x) - \Delta \underline{i}(t,x) \leq f(\underline{i}; \underline{S}), \\ \partial_t \underline{S}(t,x) - d\Delta \underline{S}(t,x) \leq g(\bar{i}; \underline{S}), \\ \partial_\nu \bar{i}(t,x') \geq 0 \geq \partial_\nu \underline{i}(t,x') \text{ with } t \geq 0 \text{ and } x' \in \partial\Omega, \\ \partial_\nu \bar{S}(t,x') \geq 0 \geq \partial_\nu \underline{S}(t,x') \text{ with } t \geq 0 \text{ and } x' \in \partial\Omega, \\ \bar{i}(0,x) \geq 0 \geq \underline{i}(0,x) \text{ with } x \in \Omega, \\ \bar{S}(0,x) \geq 0 \geq \underline{S}(0,x) \text{ with } x \in \Omega. \end{cases} \tag{3}$$

We start by setting $\underline{S} = 0$. Then, one has:

$$\begin{cases} \partial_t \underline{i}(t,x) - \Delta \underline{i}(t,x) \leq a - d_{x_i} \underline{i}, \\ \partial_\nu \underline{i}(t,x') \leq 0 \text{ with } t \geq 0 \text{ and } x' \in \partial\Omega, \\ \underline{i}(0,x) \leq i_0(x) \text{ with } x \in \Omega. \end{cases} \tag{4}$$

Naturally, $\underline{i} = m = min(a; inf_\Omega i_0)$ is solution of the system (4). For upper solutions, let start with $\bar{S}$ which satisfies

$$\begin{cases} \partial_t \bar{S}(t,x) - d\Delta \bar{S}(t,x) \geq b - (d_{x_s} + \beta_i m)\bar{S}, \\ \partial_\nu \bar{S}(t,x') \geq 0 \text{ with } t \geq 0 \text{ and } x' \in \partial\Omega, \\ \bar{S}(0,x) \geq S_0(x) \text{ with } x \in \Omega. \end{cases} \tag{5}$$

We look for solutions S^* independent of the spatial variable and satisfying

$$\begin{cases} \partial_t S^*(t,x) = b - (d_{x_s} + \beta_i m) S^*, \\ S^*(0) \geq sup_{x\in\Omega} S_0(x). \end{cases} \tag{6}$$

such that

$$S^*(t) = \frac{b}{(d_{x_s} + \beta_i m)} + \left(\sup_{x\in\Omega} S_0(x) - \frac{b}{(d_{x_s} + \beta_i m)} \right) e^{-(d_{x_s} + \beta_i m)t}.$$

Then, by setting

$$M = \max\left(\sup_{x\in\Omega} S_0(x); \frac{b}{(d_{x_s} + \beta_i m)} \right),$$

it is obvious to consider as upper solution $\bar{S} = M$.

Now $\bar{i}$ verifies

$$\begin{cases} \partial_t \bar{i}(t,x) - \Delta \bar{i}(t,x) \geq a + (\beta_i \bar{S} - d_{x_i})\bar{i}, \\ \partial_\nu \bar{i}(t,x') \geq 0 \text{ with } t \geq 0 \text{ and } x' \in \partial\Omega, \\ \bar{i}(0,x) \geq S_0(x) \text{ with } x \in \Omega. \end{cases} \tag{7}$$

We look for solutions i^* independent of the spatial variable and satisfying

$$\begin{cases} \partial_t i^*(t) - \Delta \bar{i}(t,x) \geq a + (\beta_i M - d_{x_i})i^*, \\ i^*(0) \geq \sup_{x \in \Omega} S_0(x). \end{cases} \tag{8}$$

- If $M \neq \frac{d_{x_i}}{\beta_i}$, then $i^*(t) = \frac{a}{(\beta_i M - d_{x_i})} + \left(\sup_{x\in\Omega} S_0(x) - \frac{a}{(\beta_i M - d_{x_i})}\right) e^{(\beta_i M - d_{x_i})t}$;
- If $M = \frac{d_{x_i}}{\beta_i}$, then $i^*(t) = at + \sup_{x \in \Omega}$.

Obviously we take $\bar{i} \equiv i^*$.

Positivity of solution [W4] is proved by methods of Pao (1992, Theorem 2.1.4) through essentially the maximum principle result of Kristiansen (2008, Prop. 1, p. 16). Properties (W2-W3) are proved by application of results from Kristiansen (2008, Theorem 3, p. 45) and Kristiansen (2008, Theorem 2, p. 32) using semigroup theory recalled in subsection 3.2. □

3.1.2. Method of Pao (1992): stationary case:

The stationary model:

$$\begin{cases} -\Delta i(x) = f(i;S) = a - d_{x_i} i(x) + \lambda_1(x) S, \\ -d\Delta S(x) = g(i;S) = b - (\lambda_1(x) + d_{x_s}) S, \\ \partial_\nu i(x') = 0 \text{ with } t \geq 0 \text{ and } x' \in \partial\Omega, \\ \partial_\nu S(x') = 0 \text{ with } t \geq 0 \text{ and } x' \in \partial\Omega, \\ \lambda_1(x) = \beta_i i(x). \end{cases} \tag{9}$$

could be analysed with the spirit of Theorem 1 of Pao (1992, Theorem 5.2): there exists a unique solution $U = (u,v)$ in the sector $< \underline{U}, \bar{U} >$ but there may exist a solution outside this sector.

3.2. Semigroup theory for completeness

3.2.1. Semigroup and evolution problem:

Definition 3 [Semigroup, see Henry (1981, Definition 1.3.3)].

An analytic semigroup is a family of bounded, linear operators on a Banach space X, $\{G(t)\}_{t\geq 0}$, satisfying:

- $G(0) = I$, $G(t)G(s) = G(t+s)$ for $t, s \geq 0$

- $G(t)x \to x$ as $t \to 0+$, for each $x \in X$
- $t \to G(t)x$ is real analytic on $0 < t < +\infty$ for each $x \in X$

Definition 4. [Sectorial operator, Ducrot et al. (2010, Def 3.1, p. 269)]

Let $L : D(L) \subset X \to X$ be a linear operator on a Banach space X. L is sectorial if there exists two constants $\omega \in \mathbb{R}$, $\theta \in (\frac{\pi}{2}; \pi)$, and $M > 0$ such that

(i) $\rho(L) \supset S_{\theta,\omega} = \{\lambda \in \mathbb{C} : \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\}$,

(ii) $(\lambda I - L)^{-1} \leq \frac{M}{|\lambda - \omega|}, \forall \lambda \in S_{\theta,\omega}$.

The Neumann realisation A of -Laplacian $(-\Delta)$ generates an analytic semigroup on $H^2(\Omega)$. Moreover Kristiansen (2008, Assertion 2 and 3, p. 109-110) shows that A is closed by application of Rellich-Kondrachov compactness theorem and has a countable set of real, non-negative eigenvalues that can be arranged in a non-decreasing sequence converging to 0. Moreover A is sectorial according to Kristiansen (2008, Assertion 4, p. 111) in L^p. By working on domains of fractional powers of A, Kristiansen (2008) shows through a variation of constants formula for space dimensions 1, 2 and 3, that:

- (W3): The solution depends continuously on the initial data following Kristiansen (2008, Theorem 3, p. 45);
- (W5): Global existence and boundedness of solution is retrieved by Kristiansen (2008, Theorem 2, p. 32).

3.2.2. Diffusion driven instabilities:

For system (9) it is easy to find uniform steady-state $(i^0; S^0)$ independent of the spatial variable. It turns out that:

$$\begin{cases} d_{x_s}(d_{x_i} - \beta_i)(S^0)^2 + (a+b)\beta_i S^0 + [a - (a+b)d_{x_i}] = 0, \\ \dfrac{a + b - d_{x_s} S^0}{d_{x_i}} = i^0. \end{cases} \tag{10}$$

As Kristiansen (2008, Theorem 4, p. 50), by Taylor expansion, we can consider the Jacobian matrix $\mathcal{A}$ at $(i^0; S^0)$ for $(f; g)$: a linear analysis of the linearised system of (2) around $(i^0; S^0)$ is useful to study the diffusion-driven instability of Turing (1952). We recall that $(i^0; S^0)$ satisfies the system (10).

Definition 5.

The autonomous diffusion model has a diffusion-driven instability at (i^0, S^0) if the following two conditions are satisfied:

- (i^0, S^0) is asymptotically stable in the absence of diffusion;
- (i^0, S^0) is unstable in the presence of diffusion.

Kristiansen (2008, Theorem 5, p. 60) gives sufficient conditions to have a diffusion-driven instability for an autonomous diffusion model by playing on an uniform spatial and temporal scaling.

Remark 6.

In case of constant population N, with non-oscillatory solution and constant diffusion rate $D := d_{x_i} = d_{x_s}$, the problem (in one dimensional space with no influx: $a = b = 0$) through $i(t,x) = I(x - ct) = I(z) = exp[(\frac{-c+/-\sqrt{c^2-4D\beta_i N}}{2D})z]$ tends to the traveling wave with minimum wave speed: $|c| \geq c^* = \sqrt{4D\beta_i N}$ (see Bubniakov (2007) and Murray (2001)). This result is similar to those of (Fisher, 1937). Ducrot et al. (2011, p.2895) observe that this kind of result implies the existence of planar travelling wave solutions for the n-dimensional case in space by considering: $S(t,x) = \hat{S}(xe - ct)$ and $i(t,x) = \hat{i}(xe - ct)$ with $e \in \mathbb{S}^{n-1}$, (the unit sphere of $\mathbb{R}^n$), $n \in \mathbb{N}^*$. Without external influx, a more general model deriving results with monotone functions and reaction-diffusion terms is studied by Huang (2006) without assuming constant population and considering $d \neq 1$ and $d = 1$.

4. Conclusion

It is obvious that combined methods of upper/lower solutions and semigroup theory give an interesting strategy to study the dynamic of a diffusion model. We consider an "immediate" contact rate and many conditions above involving β_i underline its importance – e.g. the minimal speed is reduced accordingly to the square roots of the size N of the population, the diffusion rate D (see Remark 6) and the form of the force of infection trough contact rate β_i, the diffusion rate D and the transmission rate β_i. Considering external influx makes our model more realistic. It is important to see how it is helpful to combine lower-upper methods with semigroup theory; the first "localizes" the solution while the second method gives an interesting variation constant formula in dynamical systems and further studies on properties of the solution in complex Sobolev spaces. When the contacts between S and i are more general with generalizations as saturations similar to Holling, Beddington-DeAngelis or Crowley-Martin-type functional responses, our model could be applied in ecology with S as preys and i as predators (see Shi and Ruan (2015)).

This study contributes to the study of epidemic model with diffusion factors not necessarily the same for susceptible (S) and infected individuals (i). A perspective could be to introduce an age structure, vaccination strategy, variable exponents $(i^m(t,x).S^\alpha(t,x))$ in the nonlinear contact between S and i and some pulse strategies in movements or displacements of infected or susceptible individuals in time or space. But the main problem that could arise then in mathematical analysis will be the control of the parameters in order to avoid the blow-up of solution either into bounded or unbounded space domain (disconnected or not). Other useful works of Calsina et al. (2014) solve models with 2-dimensional but non-monotone nonlinearities.

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