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Amina A. Bouchentouf
Djillali Liabes University of Sidi Bel Abbas

Abbes Rabhi
Djillali Liabes University of Sidi Bel Abbas

Lahcene Yahiaoui
Dr. Moulay Taher University

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Stability Condition of a Retrial Queueing System with Abandoned and Feedback Customers

Amina Angelika Bouchentouf¹, Abbas Rabhi¹ and Lahcene Yahiaoui²

¹Department of Mathematics, Djillali Liabes University of Sidi Bel Abbas
B. P. 89, Sidi Bel Abbas 22000, Algeria
bouchentouf_amina@yahoo.fr, rabhi_abbes@yahoo.fr

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²Department of Mathematics, Dr. Moulay Taher University
Saida 20000, Algeria
lahceneya8@gmail.com

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Abstract

This paper deals with the stability of a retrial queueing system with two orbits, abandoned and feedback customers. Two independent Poisson streams of customers arrive to the system, and flow into a single-server service system. An arriving one of type i , $i = 1, 2$, is handled by the server if it is free; otherwise, it is blocked and routed to a separate type- i retrial (orbit) queue that attempts to re-dispatch its jobs at its specific Poisson rate. The customer in the orbit either attempts service again after a random time or gives up receiving service and leaves the system after a random time. After the customer is served completely, the customer will decide either to join the retrial group again for another service or leave the system forever with some probability.

Keywords: Queueing system, call center, retrial queue, abandonment, feedback

MSC 2010 No.: 60K25; 68M20; 90B22

1. Introduction

In classical Queueing theory, it is assumed that any customer who cannot get service immediately

upon arrival either joins a waiting line or leaves the system forever. But there are real situations where the blocked customers leave the service area temporarily but returns to repeat their demand after some random time. This Queueing behavior is referred to as retrial queues (Parveen and Begum (2014)).

Retrial queues are characterized by the feature that if a customer finds the server busy or down or on vacation, he/she may decide to join a group of blocked customers (called orbit) for repeating their demand, or request after some random amount of time, or leave the system immediately.

Retrial queues have wide applications. In the case of many real life systems, these later can be applied in the performance modelling, for instance, in modelling magnetic disk memory system, cellular mobile networks, computer networks, and local area networks with non-persistent CSMA/CD protocols, with star topology, with random access protocols, and with multiple access protocols.

The study of retrial queues in queueing theory has been a focus by many authors because of its wide applicability in web access, telephone switching systems, telecommunication networks and computer networks, and many daily life situations. Extensive survey articles in retrial queues are due to Yang and Templeton (1987) and Falin (1990). For an excellent scenario of retrial queues, a monograph on this topic is given by Falin and Templeton (1997).

Feedback in queueing literature represents customer dissatisfaction because of inappropriate quality of service. In case of feedback, after getting partial or incomplete service, the customer retries for service. In computer communication, the transmission of a protocol data unit is sometimes repeated due to occurrence of an error. This usually happens because of non-satisfactory quality of service. Rework in industrial operations is also an example of a queue with feedback (Sharma and Kumar (2014)). Takacs (1963) was the first to study feedback queueing models. The author studied queues with feedback to determine the stationary process for the queue size and the first two moments of the distribution function of the total time spent in the system by a customer. Studies on queue length, the total sojourn time and the waiting time for an $M/G/1$ queue with Bernoulli feedback were provided by Vanden Berg and Boxma (1991). Choi et al. (1998) studied an $M/M/c$ retrial queueing model with geometric loss and feedback. Santhakumaran and Thangaraj (2000) considered a single server feedback queue with impatient and feedback customers. Choudhury and Paul (2005) derived the queue size distribution at random epoch, and at a service completion epoch for $M/G/1$ queue with two phases of heterogeneous services and Bernoulli feedback system. Krishna Kumar et al. (2006) considered a generalized $M/G/1$ feedback queue in which customers are either “positive” or “negative”. Thangaraj and Vanitha (2009) obtained transient solution of $M/M/1$ feedback queue with catastrophes using continued fractions, the steady-state solution, moments under steady state and busy period analysis are calculated. Ayyapan et al. (2010) studied $M/M/1$ retrial queueing systems with loss and feedback, under non preemptive priority service, by matrix geometric method. Kumar and Sharma (2012) analyzed a single server queueing system with retention of reneged customers. Arivudainambi and Godhandaraman (2012) considered a batch arrival queueing system with two phases of service, feedback and K optional vacations under a classical retrial policy. Bouchentouf et al. (2014) analyzed a queueing model with two heterogeneous servers balking, reneging and feedback.

A queueing system with two orbits and two exogenous streams of different types serves as a model for two competing job streams in a carrier sensing multiple access system, where the jobs, after a failed attempt to network access, wait in an orbit queue (Nain (1985); Szpankowski (1994)). The retrial queueing systems with a constant retrial rate and a single type of jobs were considered in Fayolle (1986), Choi et al. (1993a), Choi et al. (1993b), Artalejo et al. (2001), and Avrachenkov and Yechiali (2010). A two-class retrial system with a single server, no waiting room, batch arrivals and classical retrial scheme was introduced and analyzed in Kulkarni (1986). Then, in Falin (1988), the author extended the analysis of the model in Kulkarni (1986) to the multi-class setting with arbitrary number of classes. In Grishechkin (1992) the author established equivalence between the multi-class batch arrival retrial queues with classical retrial policy and branching processes with immigration. In Moutzoukis and Langaris (1996) a non-preemptive priority mechanism was added to the model of Falin (1988) and Kulkarni (1986). In Langaris and Dimitriou (2010), the authors considered a multi-class retrial system where retrial classes are associated with different phases of service. Retrial queueing model $M/MAP/M_2/1$ with two orbits was studied in Avrachenkov et al. (2010), where the authors considered a retrial single-server queueing model with two types of customers. In case of the server occupancy at the arrival epoch, the customer moves to the orbit depending on the type of the customer, where one orbit is infinite while the second one is finite. Joint distribution of the number of customers in the orbits and some performance measures are computed. In Bouchentouf and Belarbi (2013), the authors considered two retrial queueing system with balking and feedback, the joint generating function of the number of busy server, and the queue length was found by solving Kummer differential equation, and by the method of series solution. In Avrachenkov et al. (2014), the authors analyzed a retrial model with two input streams and two orbit queues. Bouchentouf et al. (2014) gave a note on fluid approximation of retrial queueing system with two orbits, abandonment and feedback. And in Bouchentouf and Sakhi (2015), the authors presented a note on an $M/M/s$ queueing system with two reconnect and two redial orbits.

Motivated by the need to analyze retrial queueing networks, and by the need to develop analytical tools that support performance analysis of many large telecommunication systems (call centers), where abandonments, retrial, and feedback arise naturally and are prevalent, we study in the present paper a retrial queueing model with abandonment and feedback customers, system with two orbits, constant retrials, abandoned and feedback customers is carried out.

The layout of the paper is given as follows. After the introduction, the retrial queueing model is described in Section 2. In Section 3, we give the main result; we formulate our retrial system as a three-dimensional Markovian queueing network, then we derive balance equations and generating functions, and finally we give its necessary stability condition.

2. The Model

The Markovian retrial queueing system with two classes of jobs and constant retrial, abandonment and feedback customers, is considered (Figure 1).

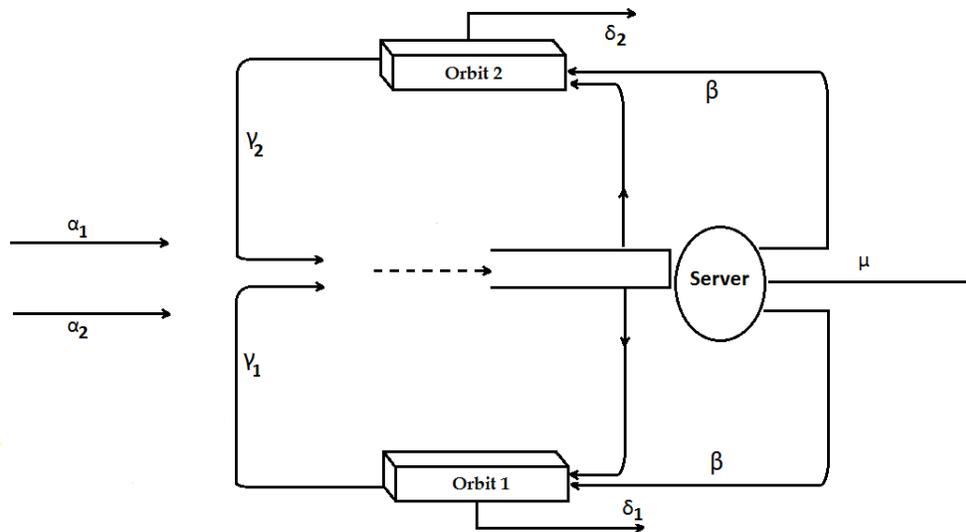


Figure 1: A queueing model with two orbits, abandonment and feedback

Two independent Poisson streams of jobs, S_1 and S_2 , flow into a single-server service system. The service system can hold at most one job. The arrival rate of stream S_i is α_i , $i = 1, 2$, with $\alpha_1 + \alpha_2 = \alpha$. The required service time of each job is independent of its type and is exponentially distributed with mean $1/\mu$. If an arriving type- i job finds the (main) server busy, it is routed to a dedicated retrial (orbit) queue from which jobs are re-transmitted at an exponential rate. The rates of retransmissions may be different from the rates of the original input streams. So the blocked jobs of type i form a type- i single-server orbit queue that attempts to retransmit jobs (if any) to the main service system at a Poisson rate of γ_i , $i = 1, 2$. This creates a system with three dependent queues. The customer in the orbit either attempts service again after a random time or gives up receiving service and leaves the system after a random time at rate δ_i , $i = 1, 2$. After the customer is served completely, it will decide either to join the retrial group again for another service with probability β or to leave the system forever with probability $\bar{\beta} = 1 - \beta$.

Let $C(t)$ denotes the number of jobs in the main queue. $C(t)$ takes the values of 0 or 1. Let $N_i(t)$ be the number of jobs in orbit queue i , $i = 1, 2$. The Markov process $\{(N_1(t), N_2(t), C(t)) : t \in [0, +\infty)\}$ is irreducible on the state-space $\{0, 1, \dots\} \times \{0, 1, \dots\} \times \{0, 1\}$.

Such a network can serve as a model for two competing job streams in a carrier sensing multiple access system "CSMA". A Local Area Computer Network (LAN) can be an example of CSMA.

3. Main Result: Necessary stability condition

The main objective of this work is to give the necessary stability condition of a retrial queueing system with two orbits, constant retrials, abandoned and feedback customers. The main result is given in the following proposition.

Proposition 1.

The following condition

$$\frac{\alpha(\gamma_1 + \delta_1)(\gamma_2 + \delta_2)}{[\alpha + (\beta + 1)\mu](\gamma_1 + \delta_1)(\gamma_2 + \delta_2) - \alpha\gamma_1\gamma_2 - \alpha_1\delta_1\gamma_2 - \alpha_2\delta_2\gamma_1} \left(1 + \frac{\alpha_i}{\gamma_i + \delta_i}\right) < 1, \quad (1)$$

for $i = 1, 2$ and

$$[\alpha + (\beta + 1)\mu](\gamma_1 + \delta_1)(\gamma_2 + \delta_2) - \alpha\gamma_1\gamma_2 - \alpha_1\delta_1\gamma_2 - \alpha_2\delta_2\gamma_1 \neq 0 \quad (2)$$

is necessary for the stability of the system.

Proof:

We will prove that our Markovian retrial queueing system with two classes of jobs and constant retrial rates, abandonment and feedback customers, is stable.

First we have to show that

$$\frac{\alpha(\gamma_1 + \delta_1)(\gamma_2 + \delta_2)}{[\alpha + (\beta + 1)\mu](\gamma_1 + \delta_1)(\gamma_2 + \delta_2) - \alpha\gamma_1\gamma_2 - \alpha_1\delta_1\gamma_2 - \alpha_2\delta_2\gamma_1} \leq 1 \quad (3)$$

and for $i = 1, 2$

$$\frac{\alpha(\gamma_1 + \delta_1)(\gamma_2 + \delta_2)}{[\alpha + (\beta + 1)\mu](\gamma_1 + \delta_1)(\gamma_2 + \delta_2) - \alpha\gamma_1\gamma_2 - \alpha_1\delta_1\gamma_2 - \alpha_2\delta_2\gamma_1} \left(1 + \frac{\alpha_i}{\gamma_i + \delta_i}\right) \leq 1 \quad (4)$$

are necessary conditions for the existence of a steady-state.

◆ Let us present the balance equations and generating functions. Consider the system in steady-state, where we define by (N_1, N_2, C) the stationary version of the Markov chain $\{(N_1(t), N_2(t), C(t)) : t \in [0, \infty)\}$. Define the set of stationary probabilities $P_{n_1 n_2}(c)$ as follows:

$$\begin{aligned} P_{n_1 n_2}(c) &= \lim_{t \rightarrow \infty} P(N_1(t) = n_1, N_2(t) = n_2, L(t) = c) \\ &= P(N_1 = n_1, N_2 = n_2, C = c), \end{aligned}$$

for $n_1, n_2 = 0, 1, \dots$, and $c = 0, 1$, when these limits exist. Define the marginal probabilities

$$P_{n_1 \cdot}(c) = \sum_{n_2=0}^{\infty} P_{n_1 n_2}(c) = P(N_1 = n_1, C = c), n_1 = 0, 1, 2, \dots, c = 0, 1$$

and

$$P_{\cdot n_2}(c) = \sum_{n_1=0}^{\infty} P_{n_1 n_2}(c) = P(N_2 = n_2, C = c), n_2 = 0, 1, 2, \dots, c = 0, 1.$$

Now, let us write the balance equations

1. $N_2 = n_2 = 0$

1.1. $N_1 = n_1 = 0, c = 0$

$$\alpha P_{00}(0) = \bar{\beta}\mu P_{00}(1) + \delta_1 P_{10}(0) + \delta_2 P_{01}(0). \quad (5)$$

1.2. $N_1 = n_1 \geq 1, c = 0$

$$(\alpha + \gamma_1 + \delta_1)P_{n_1 0}(0) = \bar{\beta}\mu P_{n_1 0}(1) + \beta\mu P_{n_1-1 0}(1) + \delta_1 P_{n_1+1 0}(0) + \delta_2 P_{n_1 1}(0). \quad (6)$$

1.3. $N_1 = 0, c = 1$

$$(\alpha + \mu)P_{0 0}(1) = \alpha P_{0 0}(0) + \gamma_1 P_{1 0}(0) + \gamma_2 P_{0 1}(0) + \delta_1 P_{1 0}(1) + \delta_2 P_{0 1}(1). \quad (7)$$

1.4. $N_1 = n_1 \geq 1, c = 1$

$$\begin{aligned} (\alpha + \mu + \delta_1)P_{n_1 0}(1) &= \alpha P_{n_1 0}(0) + \gamma_1 P_{n_1+1 0}(0) + \gamma_2 P_{n_1 1}(0) \\ &+ \delta_1 P_{n_1+1 0}(1) + \delta_2 P_{n_1 1}(1) + \alpha_1 P_{n_1-1 0}(1). \end{aligned} \quad (8)$$

2. $N_2 = n_2 \geq 1$

2.1. $N_1 = 0, c = 0$

$$(\alpha + \gamma_2 + \delta_2)P_{0 n_2}(0) = \bar{\beta}\mu P_{0 n_2}(1) + \beta\mu P_{0 n_2-1}(1) + \delta_1 P_{1 n_2}(1) + \delta_2 P_{0 n_2+1}(0). \quad (9)$$

2.2. $N_1 = n_1 \geq 1, c = 0$

$$\begin{aligned} (\alpha + \gamma_1 + \gamma_2 + \delta_1 + \delta_2)P_{n_1 n_2}(0) &= \beta\mu P_{n_1-1 n_2}(1) + \beta\mu P_{n_1 n_2-1}(1) \\ &+ \bar{\beta}\mu P_{n_1 n_2}(1) + \delta_1 P_{n_1+1 n_2}(0) + \delta_2 P_{n_1 n_2+1}(0). \end{aligned} \quad (10)$$

2.3. $N_1 = 0, c = 1$

$$\begin{aligned} (\alpha + \mu + \delta_2)P_{0 n_2}(1) &= \alpha P_{0 n_2}(0) + \gamma_1 P_{1 n_2}(0) + \gamma_2 P_{0 n_2+1}(0) + \delta_1 P_{1 n_2}(1) \\ &+ \delta_2 P_{0 n_2+1}(1) + \alpha_2 P_{0 n_2-1}(1). \end{aligned} \quad (11)$$

2.4. $N_1 = n_1 \geq 1, c = 1$

$$\begin{aligned} (\alpha + \mu + \delta_1 + \delta_2)P_{n_1 n_2}(1) &= \alpha P_{n_1 n_2}(0) + \gamma_1 P_{n_1+1 n_2}(0) + \gamma_2 P_{n_1 n_2+1}(0) \\ &+ \delta_1 P_{n_1+1 n_2}(1) + \delta_2 P_{n_1 n_2+1}(1) + \alpha_1 P_{n_1-1 n_2}(1) + \alpha_2 P_{n_1 n_2-1}(1). \end{aligned} \quad (12)$$

The probability generating function of the stationary version of the Markov process $\{(N_1(t), N_2(t), C(t)) : t \in [0, +\infty)\}$ is given by

$$F(z_1, z_2, z) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{c=0}^1 P_{n_1 n_2}(c) z_1^{n_1} z_2^{n_2} z^c. \quad (13)$$

Let us also define the following (partial) probability generating function

$$R_{n_2}^{(c)}(z_1) = \sum_{n_1=0}^{\infty} P_{n_1 n_2}(c) z_1^{n_1}, \quad c = 0, 1, n_2 = 0, 1, \dots,$$

and

$$F^{(c)}(z_1, z_2) = \sum_{n_2=0}^{\infty} \sum_{n_1=0}^{\infty} P_{n_1 n_2}(c) z_1^{n_1} z_2^{n_2} = \sum_{n_2=0}^{\infty} R_{n_2}^{(c)}(z_1) z_2^{n_2}, \quad c = 0, 1. \quad (14)$$

$$F(z_1, z_2, z) = F^{(0)}(z_1, z_2) + zF^{(1)}(z_1, z_2), \quad |z_1| \leq 1, \quad |z_2| \leq 1.$$

For $n_2 = 0$ and $c = 0$ we multiply (5) and (6) by $z_1^{n_1}$, we get

$$\begin{aligned} & ((\alpha + \gamma_1 + \delta_1)z_1 - \delta_1) R_0^{(0)}(z_1) - ((\gamma_1 + \delta_1)z_1 - \delta_1) P_{00}(0) \\ &= (\bar{\beta}\mu z_1 + \beta\mu z_1^2) R_0^{(1)}(z_1) + \delta_2 z_1 R_1^{(0)}(z_1). \end{aligned} \tag{15}$$

For $n_2 = 0$ and $c = 1$ we multiply (7) and (8) by $z_1^{n_1}$, we get

$$\begin{aligned} & ((\alpha + \mu + \delta_1)z_1 - \delta_1 - \alpha_1 z_1^2) R_0^{(1)}(z_1) - (\delta_1 z_1 - \delta_1) P_{00}(1) \\ &= (\alpha z_1 + \gamma_1) R_0^{(0)}(z_1) - \gamma_1 P_{00}(0) + \delta_2 z_1 R_1^{(1)} + \gamma_2 z_1 R_1^{(0)}. \end{aligned} \tag{16}$$

For $n_2 \geq 1$ and $c = 0$ we multiply (9) and (10) by $z_1^{n_1}$, we get

$$\begin{aligned} & ((\alpha + \gamma_1 + \gamma_2 + \delta_1 + \delta_2)z_1 - \delta_1) R_{n_2}^{(0)}(z_1) - ((\gamma_1 + \delta_1)z_1 - \delta_1) P_{0n_2}(0) \\ &= (\bar{\beta}\mu z_1 + \beta\mu z_1^2) R_{n_2}^{(1)}(z_1) + \beta\mu z_1^2 R_{n_2-1}^{(1)}(z_1) + \delta_2 z_1 R_{n_2+1}^{(1)}(z_1). \end{aligned} \tag{17}$$

For $n_2 \geq 1$ and $c = 1$ we multiply (11) and (15) by $z_1^{n_1}$, we get

$$\begin{aligned} & ((\alpha + \mu + \delta_1 + \delta_2)z_1 - \delta_1 - \alpha_1 z_1^2) R_{n_2}^{(1)}(z_1) - (\delta_1 z_1 - \delta_1) P_{0n_2}(1) \\ &= (\alpha z_1 + \gamma_1) R_{n_2}^{(0)} z_1 + \gamma_2 z_1 R_{n_2+1}^{(0)}(z_1) + \alpha_2 z_1 R_{n_2-1}^{(1)}(z_1) \\ &+ \delta_2 z_1 R_{n_2+1}^{(1)}(z_1) - \gamma_1 P_{0n_2}(0). \end{aligned} \tag{18}$$

Using Equations (15) and (17) then multiplying by $z_2^{n_2}$, we get

$$\begin{aligned} & (z_1(\alpha + \gamma_1 + \gamma_2 + \delta_1 + \delta_2)z_2 - \delta_1 z_2 - \delta_2 z_1) F^{(0)}(z_1, z_2) \\ & - (z_1(\gamma_2 + \delta_2)z_2 - \delta_2 z_1) F^{(0)}(z_1, 0) - (z_1(\gamma_1 + \delta_1)z_2 - \delta_1) z_2 F^{(0)}(0, z_2) \\ &= (\bar{\beta}\mu z_1 z_2 + \beta\mu z_1^2 z_2 + \beta\mu z_1^2 z_2^2) F^{(1)}(z_1, z_2). \end{aligned} \tag{19}$$

We do the same with Equations (16) and (18)

$$\begin{aligned} & (z_1(\alpha + \mu + \delta_1 + \delta_2)z_2 - \alpha_1 z_1^2 z_2 - \delta_1 z_2 - \delta_2 z_1 - \alpha_2 z_1 z_2^2) F^{(1)}(z_1, z_2) \\ & - (z_1 z_2 \delta_2 - \delta_2 z_1) F^{(1)}(z_1, 0) + \gamma_2 z_1 F^{(0)}(z_1, 0) \\ & - z_2(\delta_1 z_1 - \delta_1) F^{(1)}(0, z_2) + \gamma_1 F^{(0)}(0, z_2) \\ &= ((\alpha z_1 + \gamma_1)z_2 + \gamma_2 z_1) F^{(0)}(z_1, z_2). \end{aligned} \tag{20}$$

Let

$$\alpha_1 P_{n_1 \cdot}(1) = \gamma_1 P_{n_1+1 \cdot}(0) + \delta_1 P_{n_0+1 \cdot}(1). \tag{21}$$

Summing over n_1 , we get

$$1 - F^{(0)}(0, 1) = \left(1 + \frac{\alpha_1}{\gamma_1 + \delta_1}\right) F^{(1)}(1, 1). \quad (22)$$

Then by symmetry

$$1 - F^{(0)}(1, 0) = \left(1 + \frac{\alpha_2}{\gamma_2 + \delta_2}\right) F^{(1)}(1, 1). \quad (23)$$

Then

$$F^{(1)}(1, 1) = \frac{\alpha(\gamma_1 + \delta_1)(\gamma_2 + \delta_2)}{[\alpha + (\beta + 1)\mu](\gamma_1 + \delta_1)(\gamma_2 + \delta_2) - \alpha\gamma_1\gamma_2 - \alpha_1\delta_1\gamma_2 - \alpha_2\delta_2\gamma_1}, \quad (24)$$

with $[\alpha + (\beta + 1)\mu](\gamma_1 + \delta_1)(\gamma_2 + \delta_2) - \alpha\gamma_1\gamma_2 - \alpha_1\delta_1\gamma_2 - \alpha_2\delta_2\gamma_1 \neq 0$.

Second, we have to prove that for $i = 1, 2$, if

$$\frac{\alpha(\gamma_1 + \delta_1)(\gamma_2 + \delta_2)}{[\alpha + (\beta + 1)\mu](\gamma_1 + \delta_1)(\gamma_2 + \delta_2) - \alpha\gamma_1\gamma_2 - \alpha_1\delta_1\gamma_2 - \alpha_2\delta_2\gamma_1} \left(1 + \frac{\alpha_i}{\gamma_i + \delta_i}\right) = 1, \quad (25)$$

then both queues N_1 and N_2 are unbounded with probability one.

✂ Assume, for instance, that

$$\frac{\alpha(\gamma_1 + \delta_1)(\gamma_2 + \delta_2)}{[\alpha + (\beta + 1)\mu](\gamma_1 + \delta_1)(\gamma_2 + \delta_2) - \alpha\gamma_1\gamma_2 - \alpha_1\delta_1\gamma_2 - \alpha_2\delta_2\gamma_1} \left(1 + \frac{\alpha_i}{\gamma_i + \delta_i}\right) = 1,$$

so that $F^{(0)}(1, 0) = 0$ from (23). Since $F^{(0)}(1, 0) = \sum_{n_1=0} P_{n_1,0}(0)$ (see (14)), the condition

$F^{(0)}(1, 0) = 0$ implies that

$$P_{n_1,0}(0) = 0 \text{ for } n_1 = 0, 1, \dots, \quad (26)$$

so that from (5) to (6)

$$P_{n_1,0}(1) = 0 \text{ for } n_1 = 0, 1, \dots \quad (27)$$

We now use an induction argument to prove that

$$P_{n_1,n_2}(0) = 0 \text{ for } n_1, n_2 = 0, 1, \dots \quad (28)$$

We have already shown in (26) that (28) is true for $n_2 = 0$. Assume that (28) is true for $n_2 = 0, 1, \dots, k$ and let us show that it is still true for $n_2 = k + 1$.

From (10) and the induction hypothesis we get that $P_{n_1,k}(0) = P_{n_1,k}(1) = 0$ for $n_1 = 1, 2, \dots$. The latter equality implies, using (15), that $P_{n_1,k+1}(0) = 0$. This shows that (28) holds for $n_1 = 0, 1, \dots$, and $n_2 = k + 1$, and completes the induction argument, proving that (28) is true. We have therefore proved that $P_{n_1,n_2}(0) = 0$ for all $n_1, n_2 = 0, 1, \dots$

Let us prove that $P_{n_1,n_2}(1) = 0$ for all $n_1, n_2 = 0, 1, \dots$. The latter is true for $n_1, n_2 = 1, 2, \dots$, (10). It is also true for $n_2 = 0, n_1 = 0, 1, \dots$, from (27).

It remains to investigate the case where $n_1 = 0$ and $n_2 = 0, 1, \dots$

By (9) and (28) we get that $P_{0,n_2}(1) = 0$ for $n_2 = 1, 2, \dots$, whereas we have already noticed that $P_{0,0}(1) = 0$. In summary, $P_{n_1,n_2}(0) = P_{n_1,n_2}(1) = 0$ for all $n_1, n_2 = 0, 1, \dots$, so that $P(N_1 = n_1, N_2 = n_2) = P_{n_1,n_2}(0) + P_{n_1,n_2}(1)$ for all $n_1, n_2 = 0, 1, \dots$, which completes the proof.

4. Conclusion

In this article, a Markovian retrial queueing system with two classes of jobs and constant retrial, abandonment and feedback customers is studied. A necessary condition for the stability of this system is derived. For further work, it will be interesting to analyze the sufficient condition for the stability of the system. To this end we have to obtain the generating functions for this system via the solution of a Riemannian Hilbert boundary value technique.

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