Applications and Applied Mathematics: An International Journal (AAM)

Volume 10 | Issue 1 | Article 27

6-2015

Solution of fractional Drinfeld-Sokolov-Wilson equation using Homotopy perturbation transform method

P. K. Singh
Indian Institute of Technology

K. Vishal
Indian Institute of Technology

T. Som
Indian Institute of Technology

Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 10, Issue 1 (June 2015), pp. 460 - 472

Solution of fractional Drinfeld-Sokolov-Wilson equation using Homotopy perturbation transform method

P.K. Singh, K. Vishal and T. Som
Department of Mathematical Sciences
Indian Institute of Technology (BHU),
Varanasi – 221 005, India
piushsingh82@gmail.com

Received: March 3, 2014; Accepted: February 26, 2015

Abstract

In this article, the approximate solutions of the non-linear Drinfeld-Sokolov-Wilson equation with fractional time derivative have been obtained. The fractional derivative is described in the Caputo sense. He's polynomial is used to tackle the nonlinearity which arise in our considered problems. A time fractional nonlinear partial differential equation has been computed numerically. The numerical procedures illustrate the effectiveness and reliability of the method. Effects of fractional order time derivatives on the solutions for different particular cases are presented through graphs.

Keywords: Drinfeld-Sokolov-Wilson equation; Caputo derivative; Homotopy; Perturbation Transform method

MSC 2010: 26A33, 35R11, 34A08, 35A20

1. Introduction

The Drinfeld-Sokolov-Wilson (DSW) equation

$$\begin{cases} u_t + av\,v_x = 0, \\ v_t + bv_{xxx} + cu\,v_x + d\,u_x v = 0, \end{cases} \tag{1}$$

where a, b, c and d are parameters, is one of the universal models proposed by Drinfeld and Sokolov (1981) and Drinfeld and Sokolov (1985). The DSW equation is a coupled nonlinear partial differential equations. Nonlinear partial differential equations (NPDEs) are widely used to describe complex phenomena in various sciences, especially in the physical sciences. Finding explicit and exact solutions, in particular, solitary wave solutions of nonlinear evolution equations in mathematical physics play an important role in nonlinear science. It is well known that there are infinite solutions for every NPDE, making the task of finding an exact solution, a difficult one.

Many phenomena in engineering and applied sciences can be described successfully by developing the models using fractional calculus, i.e., the theory of derivatives and integrals of non-integer order [Podlubny (1999), Gorenflo and Mainardi (1997), Luchko and Groreflo (1998), Bouagada and Dooren (2012)]. Thus, appearances of fractional order derivatives make the study more interesting and challenging. Fractional differential equations have garnered much attention since fractional order system response ultimately converges to the integer order system response. For high accuracy, fractional derivatives are used to describe the dynamics of some structures. An integer order differential operator is a local operator. Whereas the fractional order differential operator is non local in the sense that it takes into account the fact that the future state not only depends upon the present state but also upon the history of all of its previous states. For this realistic property, the fractional order systems are becoming popular. Another reason behind using fractional order derivatives is that these are naturally related to the systems with memory which prevails for most of the physical and scientific system models.

The importance of obtaining the exact or approximate solutions of nonlinear partial differential equations in physics and mathematics is still significant in the sense that it needs new methods to discover exact or approximate solutions. Most of the nonlinear equations do not have a precise analytic solution, so numerical methods have largely been used to handle these equations. Moreover, if the nonlinear systems are of fractional order then it becomes more complicated to solve. Due to its important applications in engineering and physics, the authors are motivated to solve the following DSW equation with fractional order time derivatives in the presence of dispersive term as

$$\begin{cases} \dfrac{\partial^{\alpha} u}{\partial t^{\alpha}} + av\, v_x = 0\,, \quad 0 < \alpha \leq 1, \\ \dfrac{\partial^{\beta} v}{\partial t^{\beta}} + bv_{xxx} + cu\, v_x + d\, u_x v = 0, \;\; 0 < \beta \leq 1\,. \end{cases} \tag{2}$$

Previously, modelling was mainly restricted to linear systems for which analytical treatment is tractable. But due to the advent of powerful computers and with improved computational techniques, nowadays it is possible to tackle even nonlinear problems to some extent. Nonlinearity is a phenomenon which is exhibited by most of the systems in nature and has gained increasing popularity during last few decades. Most of the nonlinear problems do not have a precise analytical solution; especially it is hard to obtain it for the fractional order nonlinear equations. So these types of equations are to be solved by any approximate methods or Numerical methods. Recently, many new approaches for the solution of nonlinear differential equations have been proposed, for example, the Tanh-function method by Fan (2000), Jacobian elliptic function method by Dai and Zhang (2006), the Variational iteration method by He (1999), Adomian decomposition method by Adomian (1994), Homotopy analysis method by Liao (1992), the Homotopy perturbation method by He (1999, 2000), etc. Integral Transform method is a very old and powerful technique for solving linear differential equation, but if nonlinearity occurs in the problem then one cannot apply it directly. So, there is a need of amalgamation of this method with other existing methods. Khan and Wu (2011) proposed a new method called homotopy perturbation transformation method (HPTM), which is a combination of the homotopy perturbation method, the Laplace Transform and He's polynomials [Ghorbani (2009)]. The advantage of this method is its capability of combining two powerful methods for obtaining even the exact solutions for some nonlinear equations. Like the homotopy perturbation method this method also does not require small parameters in the equation. Thus it overcomes the limitations of traditional perturbation techniques.

The DSW equation has been solved by Zha and Zhi (2008) using the Improved F-Expansion method and by Inc (2006) using the Adomian Decomposition method. Zhang (2011) has applied the variational approach to find the solitary solution for the DSW equation. But to the best of authors' knowledge the time fractional order DSW equation has not yet been studied by any researcher. In this article, the homotopy perturbation method in the framework of the Laplace transform, i.e., the HPTM is successfully applied to obtain the solution of the fractional order DSW equation.

2. Basic idea of the homotopy perturbation transform method (HPTM)

To illustrate the basic ideas of this method, we consider the following non-linear fractional differential equation

$$D_t^{\alpha} u(x,t) + Ru(x,t) + N u(x,t) = 0, \ \ 0 < \alpha \le 1, \tag{3}$$

with the initial condition

$$u(x,0) = f(x), \tag{4}$$

where R is the linear differential operator, N is the nonlinear differential operator and $D_t^{\alpha} u(x,t)$ is the Caputo fractional derivative of function $u(x,t)$ which is defined as

$$D_t^{\alpha} u = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{u^{(n)}(x,\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \qquad (n-1 < \alpha \le n, \ n \in N), \tag{5}$$

where $\Gamma(.)$ denotes the standard Gamma function.

One of the properties of Laplace transform for Caputo fractional derivative is

$$\mathrm{L}[D_t^{\alpha} u] = s^{\alpha}\mathrm{L}[u(x,t)] - \sum_{k=0}^{n-1} u^k(x,0^+)\, s^{\alpha-1-k}. \tag{6}$$

Taking the Laplace transform on both sides of equation (3), we get

$$\mathrm{L}[D_t^{\alpha} u] + \mathrm{L}[R u(x,t)] + \mathrm{L}[N u(x,t)] = 0. \tag{7}$$

In the view of equation (6), we have

$$\mathrm{L}[u(x,t)] = \frac{1}{s} u(x,0) - \frac{1}{s^{\alpha}} \mathrm{L}[R u(x,t)] - \frac{1}{s^{\alpha}} \mathrm{L}[N u(x,t)]. \tag{8}$$

Operating the inverse Laplace transform on both sides of equation (8), we get

$$u(x,t) = u(x,0) - \mathrm{L}^{-1}\left[\frac{1}{s^{\alpha}} \mathrm{L}[R u(x,t)] + \frac{1}{s^{\alpha}} \mathrm{L}[N u(x,t)]\right]. \tag{9}$$

Now we apply the homotopy perturbation method to the equation (9) using the expansion of $u(x,t)$ in power series in p as

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t), \tag{10}$$

and with the decomposition of the nonlinear term as

$$N u(x,t) = \sum_{n=0}^{\infty} p^n H_n(u), \tag{11}$$

where $p \in [0,1]$ is an embedding parameter and $H_n(u)$ is the He's polynomials [Ghorbani (2009)] defined as

$$H_n(u_0, u_1, u_2, \ldots, u_n) = \frac{1}{n!} \frac{\partial}{\partial p^n} [N(\sum_{i=0}^{\infty} p^i u_i)]_{p=0}, \quad n = 0,1,2,\ldots . \tag{12}$$

Substituting equations (10) and (11) into equation (9), we have

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = u(x,0) - p\left(\mathrm{L}^{-1}\left[\frac{1}{s^{\alpha}} \mathrm{L}[R \sum_{n=0}^{\infty} p^n u_n(x,t)] + \frac{1}{s^{\alpha}} \mathrm{L}[\sum_{n=0}^{\infty} p^n H_n(u)]\right]\right). \tag{13}$$

Equating the terms with identical powers in p, we obtain the following approximations as

$$p^0: \quad u_0(x,t) = u(x,0), \tag{14}$$

$$p^1: \quad u_1(x,t) = -\mathrm{L}^{-1}\left[\frac{1}{s^{\alpha}} \mathrm{L}[R u_0(x,t)] + \frac{1}{s^{\alpha}} \mathrm{L}[H_0(u)]\right], \tag{15}$$

$$p^2: \quad u_2(x,t) = -\mathrm{L}^{-1}\left[\frac{1}{s^\alpha}\mathrm{L}[Ru_1(x,t)] + \frac{1}{s^\alpha}\mathrm{L}[H_1(u)]\right], \tag{16}$$

$$p^3: \quad u_3(x,t) = -\mathrm{L}^{-1}\left[\frac{1}{s^\alpha}\mathrm{L}[Ru_2(x,t)] + \frac{1}{s^\alpha}\mathrm{L}[H_2(u)]\right]. \tag{17}$$

Proceeding in this manner, we get $u_n(x,t),\ n \geq 4$ and finally we approximate the analytical solution by the truncated series as

$$u(x,t) = \lim_{N\to\infty} \Phi_N(x,t), \tag{18}$$

where

$$\Phi_N(x,t) = \sum_{n=0}^{N-1} u_n(x,t).$$

3. Solution of the problem by HPTM

Consider the DSW equation with fractional order time derivative as

$$\begin{cases} D_t^\alpha u + 3v\, v_x = 0, & 0 < \alpha \leq 1 \\ D_t^\beta v + 2v_{xxx} + 2u\, v_x + u_x v = 0, & 0 < \beta \leq 1 \end{cases} \tag{19}$$

with initial conditions

$$\begin{cases} u(x,0) = 3\sec h^2(x) \\ v(x,0) = 2\sec h(x)\,. \end{cases} \tag{20}$$

Taking the Laplace transform of equation (19), we have

$$L[D_t^\alpha u] + 3L[v\, v_x] = 0$$

and

$$L[D_t^\beta v] + 2L[v_{\text{xxx}}] + 2L[u\, v_x] + L[u_x v] = 0.$$

Using equation (6) and then applying the inverse Laplace transform, we obtain

$$u(x,t) = L^{-1}\left[\frac{1}{s}u(x,0) - \frac{1}{s^\alpha}L[vv_x]\right] \tag{21}$$

and

$$\mathrm{v}(x,t) = L^{-1}\left[\frac{1}{s}v(x,0) - \frac{1}{s^\beta}\left\{2L[v_{\text{xxx}}] + 2L[u\, v_x] + L[u_x v]\right\}\right]. \tag{22}$$

For applying the homotopy perturbation method in equations (21) and (22), we express

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \text{ and } \mathrm{v}(x,t) = \sum_{n=0}^{\infty} p^n v_n(x,t).$$

The nonlinear terms can be decomposed as equation (11).

Equations (21) and (22) reduce to

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = u_0(x,t) - L^{-1}\left[\frac{1}{s^{\alpha}}\left\{2L[\sum_{n=0}^{\infty} p^n H_n(v)]\right\}\right]$$

$$\sum_{n=0}^{\infty} p^n v_n(x,t) = v_0(x,t)$$
$$-L^{-1}\left[\frac{1}{s^{\beta}}\{2L[(\sum_{n=0}^{\infty} p^n v_n)_{xxx}] + 2L[\sum_{n=0}^{\infty} p^n G_n(u,v)] + L[\sum_{n=0}^{\infty} p^n H_n^{'}(u,v)]\}\right],$$

where $H_n(v)$, $G_n(v)$ and $H_n^{'}(u,v)$ are He's polynomials [Khan and Wu (2011)] that represent nonlinear terms vv_x, uv_x and $u_x v$ respectively, which are given as

$$H_0(v) = v_0 v_{0x},$$
$$H_1(v) = v_1 v_{0x} + v_0 v_{1x},$$
$$H_2(v) = v_2 v_{0x} + v_1 v_{1x} + v_0 v_{2x},$$
$$G_0(u,v) = u_0 v_{0x},$$
$$G_1(u,v) = u_1 v_{0x} + u_0 v_{1x},$$
$$G_2(u,v) = u_2 v_{0x} + u_1 v_{1x} + u_0 v_{2x},$$
$$H_0^{'}(u,v) = u_{0x} v_0,$$
$$H_1^{'}(u,v) = u_{1x} v_0 + u_{0x} v_1,$$
$$H_1^{'}(u,v) = u_{2x} v_0 + u_{1x} v_1 + u_{0x} v_2 .$$

Comparing the coefficients of like powers of p, we obtain

$$u_0(x,t) = 3\sec h^2(x),$$
$$v_0(x,t) = 2\sec h(x),$$
$$u_1(x,t) = 4\sec h^2(x)\tanh(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)},$$
$$v_1(x,t) = 4\sec h(x)\tanh(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)},$$
$$u_2(x,t) = 8\sec h^2(x)\left(\cosh(2x)-2\right)\frac{t^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)},$$
$$v_2(x,t) = \sec h^5(x)\left(67+\cosh(4x)-52\cosh(2x)\right)\frac{t^{2\beta}}{\Gamma(1+2\beta)}$$
$$+8\sec h^5(x)\left(2\cosh(2x)-3\right)\frac{t^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)} .$$

Proceeding in a similar manner, the rest of the components of $u_n(x,t)$ and $v_n(x,t)$, $n \geq 3$ can be obtained and the series solutions can thus be entirely determined. Finally, we approximate the analytical solutions of $u(x,t)$ and $\mathrm{v}(x,t)$ by the truncated series as given in equation (18).

4. Numerical results and discussion

In this section, the numerical results of the field variables $u(x,t)$ and $v(x,t)$ are calculated for various fractional Brownian motions $\alpha = \beta = 0.5$ and also for the standard motion $\alpha = \beta = 1$ for various particular cases. The results are depicted through Figures (1-14). The variations of $u(x,t)$ and $v(x,t)$ with x and t for the standard DSW equation are shown in Figures 1 and 2 respectively, whereas those for $\alpha = \beta = 1/2$ are shown in Figures 3 & 4. The 2D figures of field variables depicted through Figures 5 & 6 and Figures 10 & 11 reveal the fact that the field variables decrease as the system approaches from fractional order to the standard order as time increases for the case $x = 1$. But the variations with x are different at $t = 1$ (Figures7 & 8 and Figures 12 & 13). The salient features of this section are the graphical presentations for the comparisons of approximate and exact solutions of $u(x,t)$ and $v(x,t)$ through Figures 9 and 14 for various particular cases, which clearly reveal the fact that our considered method HPTM is reliable and very much effective.

5. Conclusion

In this article the successful application of the Homotopy perturbation transformation method (HPTM) is used to demonstrate how to find the approximate solutions of the nonlinear Drinfeld-Sokolov-Wilson (DSW) equation with fractional order time derivative. A clear conclusion from the numerical results is that the method provides highly accurate numerical solutions even for the fractional order nonlinear partial differential equations.

REFERENCES

Adomian, G. (1994).Solving Frontier Problems of Physics: The Decomposition Method, Kluwer Acad. Publ., Boston.

Bouagada, D. and Dooren, P.V. (2012**).** State space solution of implicit fractional continuous time systems, Frac. Calc. Appl. Anal. 15, 356-361.

Dai, C.Q. and Zhang, J.F. (2006). Jacobian elliptic function method for nonlinear differential– difference equations, Chaos, Solitons & Fractals 27, 1042–1047.

Drinfeld, V. G. and Sokolov, V. V. (1981). Equations of Korteweg-de Vries type and simple Lie algebras, Sov. Math. Dokl. 23, 457-462.

Drinfeld, V. G. and Sokolov, V. V. (1985). Lie algebras and equations of Korteweg-de Vries type, J. Sov. Math. 30, 1975-2005.

Fan, E. (2000). Extended tanh-function method and its applications to nonlinear equation, Phys. Lett. A 277, 212–218.

Ghorbani, A. (2009). Beyond Adomian's polynomials: He polynomials, Chaos Solitons & Fractals 39, 1486–1492.

Gorenflo, R. and Mainardi, F. (1997). Fractional calculus: Integral and differential equations of fractional order, in: A. Carpinteri, F. Mainardi (Eds.), Fractals and Fractional Calculus,New York.

He, J.H. (1966). Variational iteration method—a kind of nonlinear analytical technique some examples, Int. J. Nonlin. Mech. 34, 699–708.

He, J.H. (1999). Homotopy perturbation technique, Comp. Methods in Appl. Mech. and Eng. 178, 257-262.

He, J.H. (2000). A coupling method of homotopy technique and perturbation technique for nonlinear problems, Int. J Nonlinear. Mech 35, 37–43.

He, J.H. (2000). Periodic solutions and bifurcations of delay-differential equations, Phys. Lett. A 347, 228-230.

Inc, M. (2006). On numerical doubly periodic wave solutions of the coupled Drinfeld–Sokolov–Wilson equation by the decomposition method, Applied Mathematics and Computation,172(1), 421-430.

Khan, Y. and Wu, Q. B. (2011). Homotopy perturbation transform method for nonlinear equations using He's polynomials, Comput. Math. Appl. 61, 1963-1967.

Liao, S.J. (1992). The Proposed homotopy analysis technique for the solution of nonlinear Problems, Ph.D. thesis, Shanghai Jio Tong University, Shanghai.

Luchko, A.Y. and Groreflo, R. (1998) The initial value problem for some fractional differential equations with the Caputo derivative, Preprint series, fachbreich mathematik und informatik, Freic Universitat, Berlin A08–98.

Podlubny, I. Fractional Differential Equations, (1999). Academic Press, New York.

Zha, X. Q. and Zhi, H. Y. (2008). An Improved F·Expansion Method and Its Application Coupled Drinfeld Sokolov Wilson Equation, Commun. Theor. Phys, 50, 309-314.

Zhang, W.M. (2011). Solitary Solutions and Singular Periodic Solutions of the Drinfeld-Sokolov-Wilson Equation by Variational Approach, Applied Mathematical Sciences, 5 (38), 1887-1894.

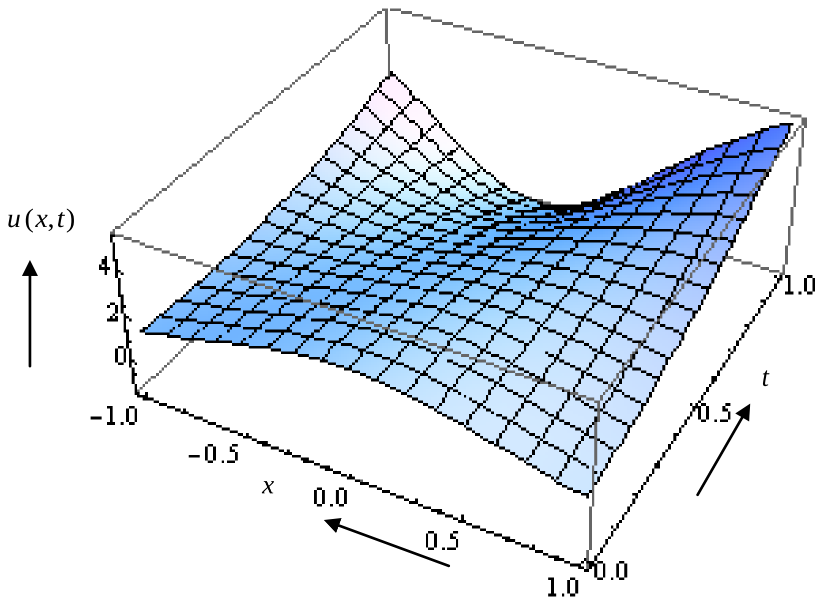

Figure1. Plot of $u(x,t)$ w.r.to x and t for $\alpha = 1$ and $\beta = 1$

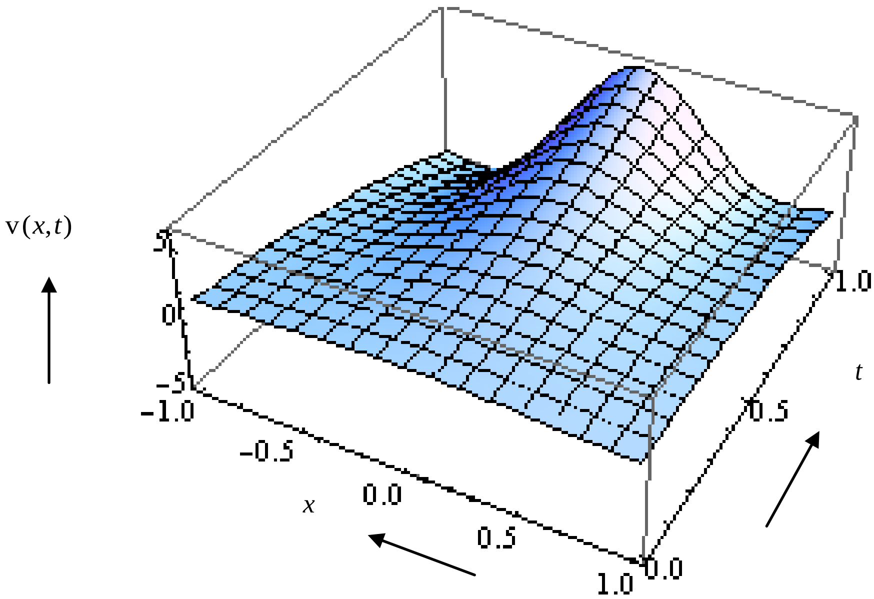

Figure 2. Plot of $v(x,t)$ w.r.to x and t for $\alpha = 1$ and $\beta = 1$

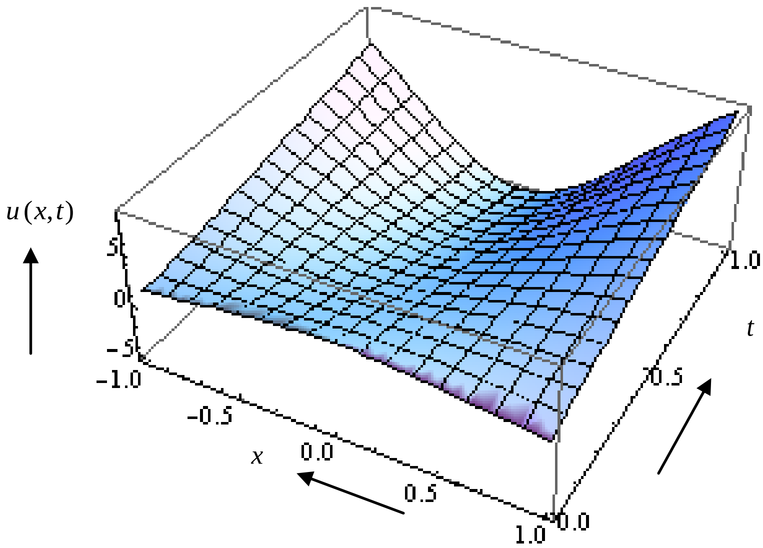

Figure 3. Plot of $u(x,t)$ w.r.t x and t for $\alpha = 0.5$ and $\beta = 0.5$

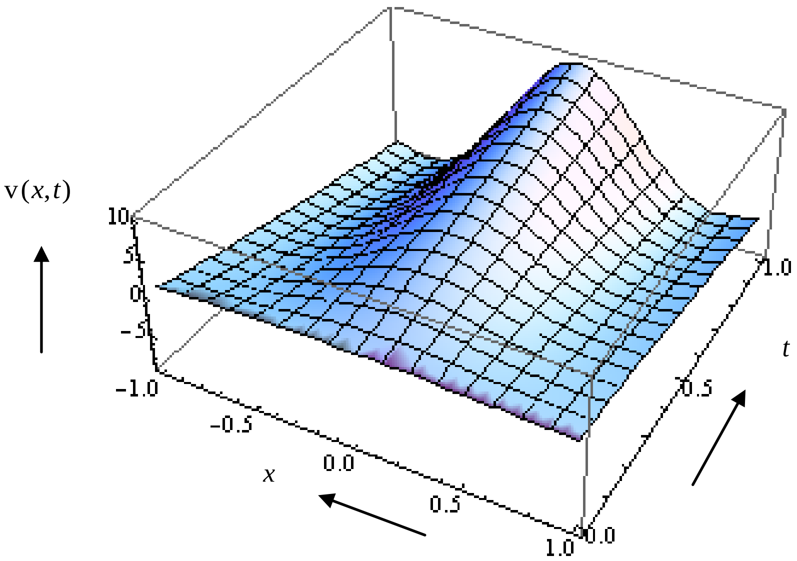

Figure 4. Plot of $v(x,t)$ w.r.t. x and t for $\alpha = 0.5$ and $\beta = 0.5$

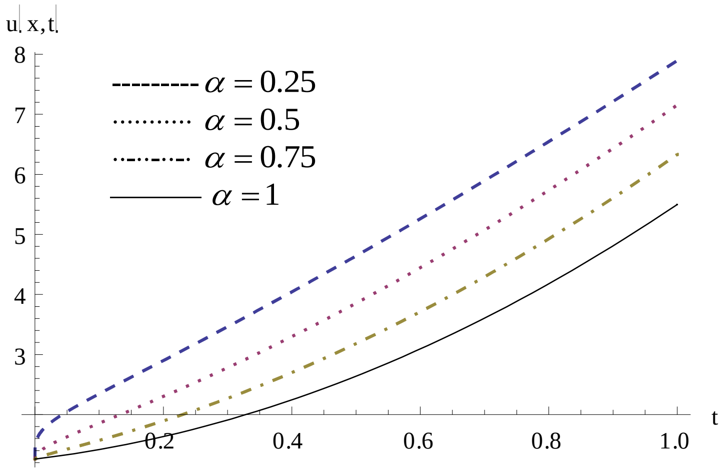

Figure 5. Plot of $u(x,t)$ vs t for different values of α at $\beta = 1$

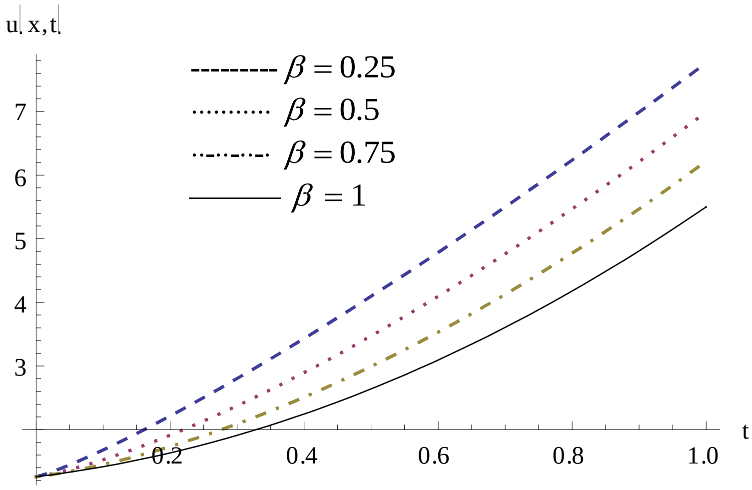

Figure 6. Plot of $u(x,t)$ vs t for different values of β at $\alpha = 1$

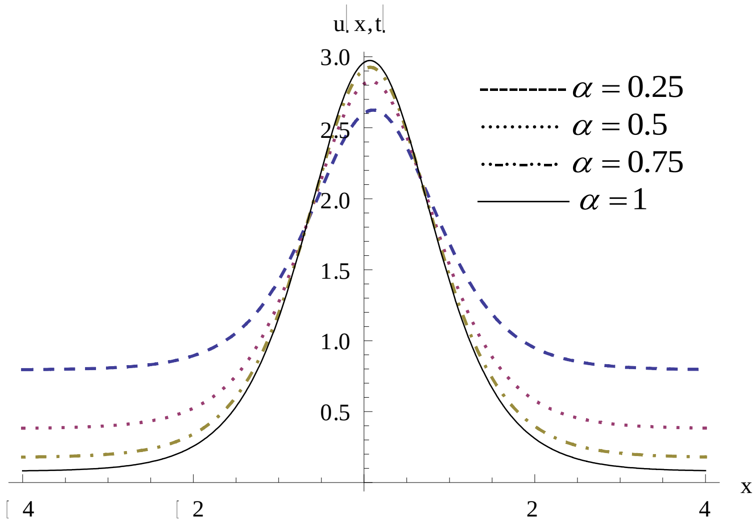

Figure 7. Plot of $u(x,t)$ vs x for different values of α at $\beta = 1$

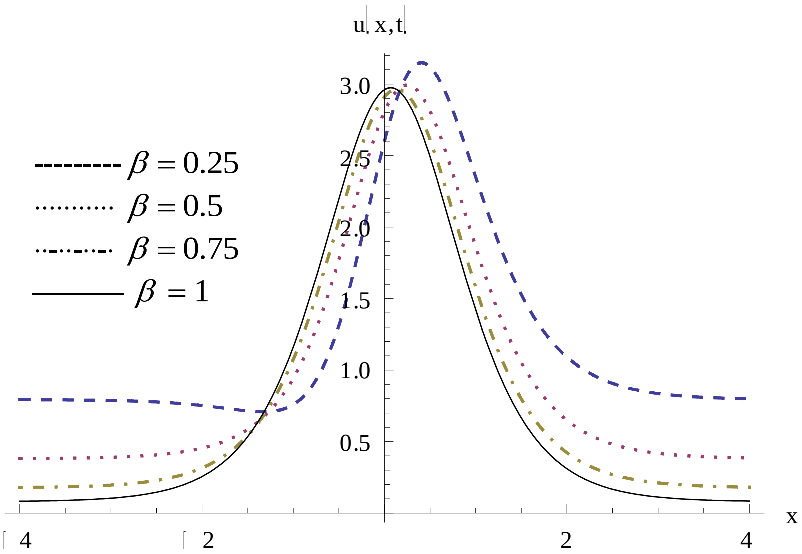

Figure 8. Plot of $u(x,t)$ vs t for different values of β at $\alpha = 1$

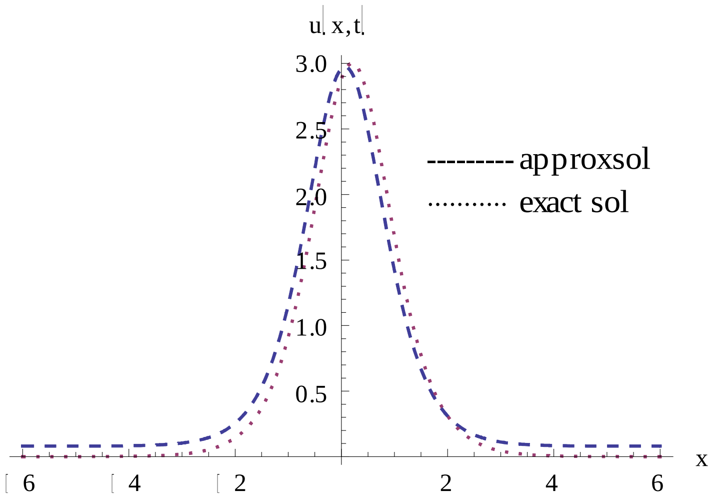

Figure 9. Comparison between approx. and exact values of $u(x,t)$ for $\alpha = \beta = 1$

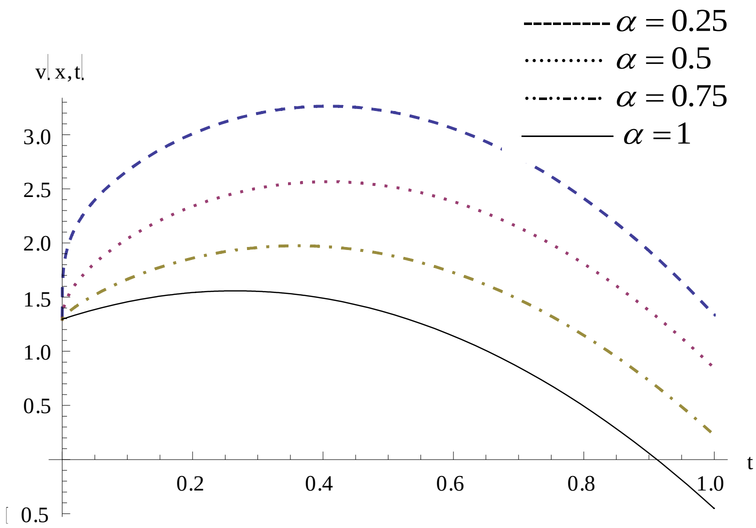

Figure 10. Plot of $v(x,t)$ vs t for different values of α at $\beta = 1$

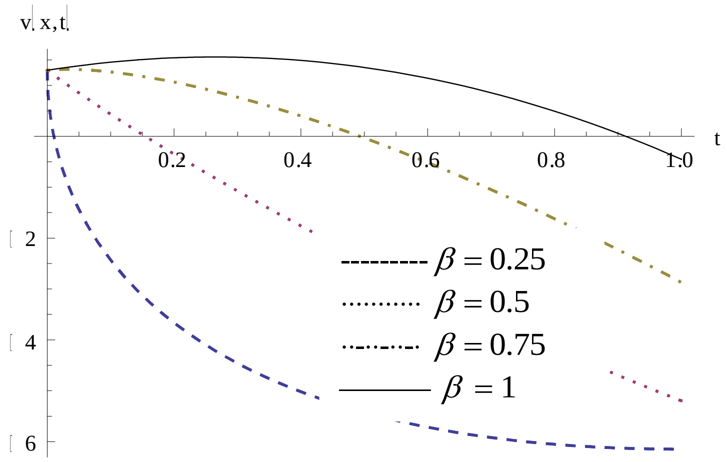

Figure 11. Plot of $v(x,t)$ vs t for different values of β at $\alpha = 1$

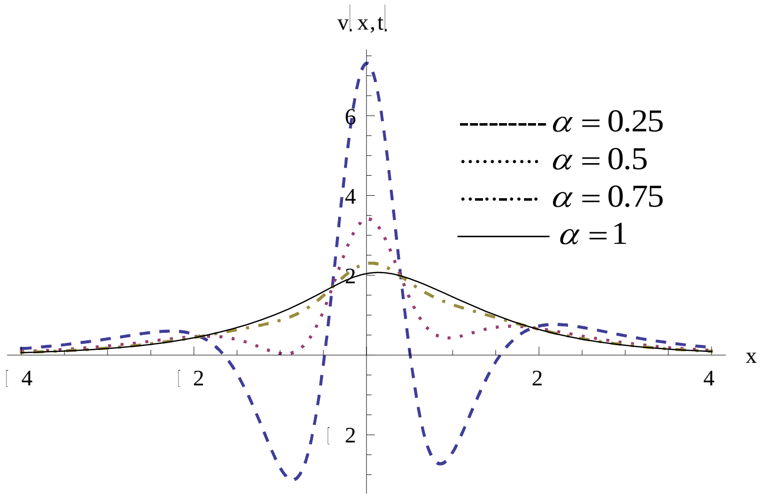

Figure 12. Plot of $v(x,t)$ vs x for different values of α at $\beta = 1$

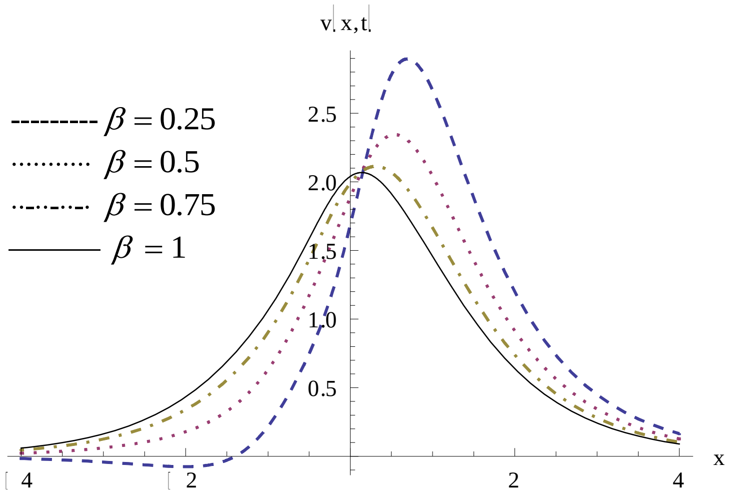

Figure 13. Plot of $v(x,t)$ vs t for different values of β at $\alpha = 1$

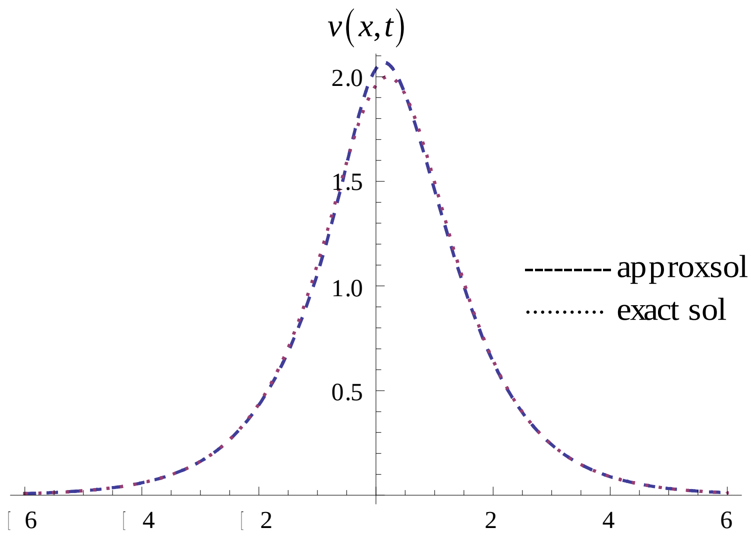

Figure 14. Comparison between approx. and exact values of $v(x,t)$ for $\alpha = \beta = 1$