[Applications and Applied Mathematics: An International Journal (AAM)](https://digitalcommons.pvamu.edu/aam)

Volume 9 | Issue 2 Article 20

12-2014

Modern Approach for Designing and Solving Interval Estimated Linear Fractional Programming Models

S. Ananthalakshmi
Manonmaniam Sundaranar University

C. Vijayalakshmi
VIT University, Chennai Campus

V. Ganesan
VIT University, Chennai Campus

Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 9, Issue 2 (December 2014), pp. 795-810

Modern Approach for Designing and Solving Interval Estimated Linear Fractional Programming Models

S. Ananthalakshmi
Department of Statistics
Manonmaniam Sundaranar University
Tirunelveli 627 012
Tamilnadu, India
ananthisvasan@gmail.com

C. Vijayalakshmi and V. Ganesan
Department of Mathematics
VIT University, Chennai Campus
Chennai-600 127
Tamilnadu, India
India.vijusesha2002@yahoo.co.in

Received: March 23, 2013; Accepted: June 26, 2013

Abstract

Optimization methods have been widely applied in statistics. In mathematical programming, the coefficients of the models are always categorized as deterministic values. However uncertainty always exists in realistic problems. Therefore, interval-estimated optimization models may provide an alternative choice for considering the uncertainty into the optimization models. In this aspect, this paper concentrates, the lower and upper values of interval estimated linear fractional programming model (IELFPM) are obtained by using generalized confidence interval estimation method. An IELFPM is a LFP with interval form of the coefficients in the objective function and all requirements. The solution of the IELFPM is also analyzed.

Keywords: Interval estimated linear fractional programming model (IELFPM); linear fractional programming (LFP); interval valued function (IVF); optimum solution; confidence interval

MSC 2010 No.: 90C05, 90C32, 90C30, 65K99

1. Introduction

The optimization models have widely applied to many research fields. In mathematical programming, the coefficients of the models are always categorized as deterministic values. However uncertainty always exists in realistic problems. Fuzzy optimization and stochastic approaches are commonly used to describe the uncertain elements present in a decision model. In fuzzy optimization, fuzzy parameters are assumed to be with known membership functions and in stochastic programming, the uncertain coefficients are regarded as random variables and their probability distributions are assumed to be known. However, in reality, it is not always easy to specify the membership function or probability distribution in an inexact environment. Therefore, interval-estimated optimization models may provide an alternative choice for considering the uncertainty into IELFP Models. The generalized confidence intervals have established to be useful tools for making inferences in many practical uncertain IELFP models. That is, an objective function in general, is formed as the ratio of two interval estimated linear functions and all requirements are interval form and the coefficients in the IELFP Models are assumed as closed intervals. The bounds of uncertain data (i.e., determining the closed intervals to bind the possible observed data) are easier to be finding the generalized confidence intervals. Therefore, we interest to study the generalized confidence intervals on IELFPM. The applications of IELFP are production planning, financial and corporate planning, health care and hospital planning.

In this paper, first section describes the introduction of IELFPM. Second section deals with literature review and third section discusses some preliminaries on interval arithmetic. In fourth section deals, how to find interval values through confidence interval is discussed. The solving procedure is presented in fifth section.

2. Literature survey

Charnes and Cooper (1962) have proposed their method depends on transforming LFP to an equivalent linear program. LFP Models have been discussed by several contributors, namely, Schaible (1981), Schaible and Ibaraki (1983), and Suresh Chandra et al. (2011). Interval analysis was introduced by Moore (1966, 1979). Interval analysis has been studied by several researchers, such as Alefeld and Herzberger (1983) Atanu Sengupta and Tapan Kumar Pal (2000), etc. Charnes et al. (1977) have developed mathematical programming methodology in which coefficients can be expressed as interval form. LP models with interval coefficients have been studied by several researchers, such as Atanu Sengupta et al. (2001), Chinneck and Ramadan (2000), Dantzig (1955), Herry Suprajitno and Ismail bin Mohd (2010), Kuchta (2008). Hladik (2007) computes exact range of the optimal value for LPM in which input data can vary in some given real compact intervals, and he able to characterize the primal and dual solution sets, the bounds of the objective function resulted from two nonlinear programming models.

Effati and Pakdaman (2012) discussed solving procedure of interval valued LFPM. Hsien-Chung Wu (2007, 2008) proved and derived the Karush-Kuhn-Tucker (KKT) optimality conditions for an optimization model with interval valued objective function. Sengupta et al. (2000, 2001) have reduced the interval number LPM into a bi-objective classical LPM and then obtained an optimal solution. Suprajitno and Mohd (2008) and Suprajitno et al. (2009) presented some interval linear programming models, where the coefficients and variables are in the form of intervals.

Krishnamoorthy and Mathew (2004) discussed on one sided tolerance limits in balanced and unbalanced one-way random effects ANOVA model. Weerahandi (2004) has introduced the concept of a generalized pivotal quantity (GPQ) for a scalar parameter μ and using that parameter, one can construct an interval estimator for μ in situations where standard pivotal quantity based approaches may not be applicable. He referred to such intervals as generalized confidence intervals (GCI).

3. Preliminaries

This section is to present some notations, which are useful in our further consideration.

Let us denote by I the class of all closed and bounded intervals in $\mathcal{R}$. If $[a]$ and $[b]$ are closed and bounded intervals, we also adopt the notation $[a]=[\underline{a}, \overline{a}]$ and $[b]=[\underline{b}, \overline{b}]$, where $\underline{a}$, $\underline{b}$ and $\overline{a}, \overline{b}$ mean the lower and upper bounds of $[a]$ and $[b]$. Let $[a]=[\underline{a}, \overline{a}]$ and $[b]=[\underline{b}, \overline{b}]$ be in I. Then, by definition,

(i) $[a]+[b]=\left[\underline{a}+\underline{b},\ \overline{a}+\overline{b}\right]\in I.$

(ii) $[a]-[b]=\left[\underline{a}-\overline{b},\ \overline{a}-\underline{b}\right]\in I.$

(iii) $-[a]=\left[-\overline{a}, -\underline{a}\right]\in I.$

(iv) $$x\left[\underline{a}, \overline{a}\right]=\begin{cases}\left[x\underline{a}, x\overline{a}\right], & \text{if } x\geq 0,\\ \left[x\overline{a}, x\underline{a}\right], & \text{if } x\leq 0,\end{cases}$$
where x is a real number.

(v) An interval $[a]$ is said to be positive, if $\underline{a}>0$ and negative, if $\overline{a}<0$.

(vi) If $[a]=[\underline{a}, \overline{a}]$ and $[b]=[\underline{b}, \overline{b}]$ are bounded and real intervals, we define the multiplication of two intervals as follows:

$$[a][b]=\left[\min\left\{\underline{ab}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{ab}\right\},\ \max\left\{\underline{ab}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{ab}\right\}\right],$$

1) If $0\leq\underline{a}\leq\overline{a}$ and $0\leq\underline{b}\leq\overline{b}$, then we have

$$[a][b]=\left[\underline{ab}, \overline{ab}\right]. \tag{3.1}$$

2) If $0 \le \underline{a} \le \overline{a}$ and $\underline{b} < 0 < \overline{b}$, then we have

$$[a][b] = [\overline{a}\underline{b},\ \overline{ab}]. \tag{3.2}$$

(vii) There are several approaches to define interval division. We define the quotient of two intervals as follows:

Let $[a] = [\underline{a},\ \overline{a}]$ and also $[b] = [\underline{b},\ \overline{b}]$ be two nonempty bounded real intervals. Then, if $0 \notin [\underline{b}, \overline{b}]$, we have

$$[a]/[b] = [\underline{a},\ \overline{a}]\left[\frac{1}{\underline{b}},\ \frac{1}{\overline{b}}\right]. \tag{3.3}$$

(viii) Power of interval for $n \in Z$ is given as:

When n is positive and odd or $[a]$ is positive, then $[a]^n = [\underline{a}^n,\ \overline{a}^{\,n}]$.

When n is positive and even, then

$$[a]^n = \begin{cases} [\underline{a}^n,\ \overline{a}^{\,n}], & \text{if } \underline{a} \ge 0, \\ [\overline{a}^{\,n},\ \underline{a}^n], & \text{if } \overline{a} < 0, \\ [0, \max\{(\underline{a})^n,\ (\overline{a})^n\}], & \text{otherwise.} \end{cases}$$

When n is negative and odd or even, then

$$[a]^n = \frac{1}{[a]^n}.$$

(ix) For an interval $[a]$ such that $\underline{a} \ge 0$, define the square root of $[a]$ denoted by $\sqrt{[a]}$ as: $\sqrt{[a]} = \{\sqrt{b} : \underline{a} \le b \le \overline{a}\}$.

(x) Mid-point of an interval $[a]$ is defined as $m([a]) = \frac{1}{2}\left(\underline{a} + \overline{a}\right)$.

(xi) Width of an interval $[a]$ is defined as $w([a]) = \overline{a} - \underline{a}$.

(xii) Half-width of an interval $[a]$ is defined as $hw([a]) = \frac{1}{2}(\overline{a} - \underline{a})$.

Remark:

Note that every real number $a \in \mathcal{R}$ can be considered as an interval $[a, a] \in I$.

Definition 3.1.

The function F: $\mathcal{R}^n \rightarrow$I defined on the Euclidean space $\mathcal{R}^n$ called an Interval Valued Function (IVF) i.e., $F(x) = F(x_1, x_2, \ldots, x_n)$ is a closed interval in $\mathcal{R}$. The IVF F can also be written as $F(x) = [\underline{F}(x), \overline{F}(x)]$, where $\underline{F}(x)$ and $\overline{F}(x)$ are real-valued functions defined on $\mathcal{R}^n$ and satisfy $\underline{F}(x) \leq \overline{F}(x)$ for every $x \in \mathcal{R}^n$. We say that the IVF F is differentiable at $x_0 \in \mathcal{R}^n$ if and only if the real–valued functions $\underline{F}(x)$ and $\overline{F}(x)$ are differentiable at x_0. For more details on the topic of interval analysis, we refer to Moore (1966, 1979) and Alefeld and Herzberger (1983).

4. Description of Confidence Interval

The usual LFPM requires the parameters to be known as constants. In practical point of view, however, the values are seldom known exactly and have to be estimated. Therefore, we interest to study interval LFP where it's the coefficients and variables are in the form of interval. We use the method of estimation and obtain fiducial limits for the interval coefficients.

In practical studies, the data on virtually the same object of interest are made by fixed (k) number of experimental entities. The i^{th} entity repeats its data n_i times, for large n_i. The entities may exhibit different within entity variances (heteroscedasticity). Here we will assume that the data follow normal distribution. We consider the following fixed effects model

$$Y_{ij} = \mu_i + \xi_{ij}, \tag{4.1}$$

with mutually independent errors, assumed to normally distributed with mean zero and (unknown) variance σ_i^2, $i = 1, 2, \ldots, k$.

The task is make inference about the common mean μ, especially confidence intervals for μ, so we need an estimator of μ. Consider an unbiased estimator $\hat{\mu}$ of the common mean μ with variance $Var\ (\hat{\mu}) = \sum_{i=1}^{k} \lambda_i \sigma_i^2$, where $\lambda_i > 0$. If the variance components σ_i^2 are known then the pivot

$$Z = \frac{\hat{\mu} - \mu}{\sqrt{Var\ (\hat{\mu})}} \approx N(0, 1). \tag{4.2}$$

The (1 - α) 100% confidence interval is

$$\hat{\mu} - \mu(1-\alpha/2)\sqrt{Var(\hat{\mu})} \leq \mu \leq \hat{\mu} + \mu(1-\alpha/2)\sqrt{Var(\hat{\mu})}, \tag{4.3}$$

where μ(.) is quantile function of normal distribution. If the variance components σ_i^2 are unknown then we find the exact distribution of Z.

So we want to compare some approximate confidence intervals for common mean derived from the simple t-statistic, the t-statistic with Satterthwaite's degrees of freedom, the t-statistic derived from Kenward- Roger method and by Welch's quantile approximation.

Interval derived from simple *t*-statistic

The simple t-statistic T is given by

$$T = \frac{\overline{Y}_n - \mu}{\sqrt{Var(\overline{Y}_n)}}, \tag{4.4}$$

where

$$Var(\overline{Y}_n) = S^2 / N,$$

$$N = \sum_{i=1}^{k} n_i, \quad \overline{Y}_{i.} = n_i^{-1}\sum_{j=1}^{n_i} Y_{ij}, \quad \overline{Y}_n = N^{-1}\sum_{i=1}^{k} n_i \overline{Y}_{i.},$$

$$S_i^2 = (n_i - 1)^{-1}\sum_{j=1}^{n_i}(Y_{ij} - \overline{Y}_{i.})^2, \quad S^2 = (N-k)^{-1}\sum_{i=1}^{k}(n_i - 1)S_i^2.$$

This statistic was derived under the assumption of the variance homogeneity and has a t-distribution with $N - k$ degrees of freedom.

The (1 - α) 100% confidence interval is

$$\overline{Y}_n - t_{N-k(1-\alpha/2)}\sqrt{Var(\overline{Y}_n)} \leq \mu \leq \overline{Y}_n + t_{N-k(1-\alpha/2)}\sqrt{Var(\overline{Y}_n)}, \tag{4.5}$$

where $t_{df}(.)$ is quantile function of Student's t-distribution with df degrees of freedom.

Interval derived from *t*-statistic with Satterthwaite's degrees of freedom

The t-test, Ts, is given by

$$T_S = \frac{\overline{Y}_n - \mu}{\sqrt{Var(\overline{Y}_n)}}, \tag{4.6}$$

where

$$Var\ (\bar{Y}_n) = N^{-1}\sum_{i=1}^{k} n_i S_i^2 \ .$$

In Satterthwaite approximated, the sum of χ^2 random variables to derive the null distribution of the statistic T_S as a t random variable with approximately v degrees of freedom:

$$\hat{v} = (\sum_{i=1}^{k} n_i\, S_i^2\,)^2 \,/\, (\sum_{i=1}^{k} (n_i - 1)^{-1}\, n_i^2\, S_i^2\,)\,.$$

The (1 - α) 100% confidence interval is

$$\bar{Y}_n - t_{\hat{v}(1-\alpha/2)}\sqrt{Var\ (\bar{Y}_n)} \le \mu \le \bar{Y}_n + t_{\hat{v}(1-\alpha/2)}\sqrt{Var\ (\bar{Y}_n)}\ . \tag{4.7}$$

Welch's Quantile Approximation

Consider this probability equation

$$\Pr\left\lfloor \bar{Y}_n - \mu\ < u(\xi)\sqrt{Var\ (\bar{Y}_n)}\ \right\rfloor = \xi\,. \tag{4.8}$$

If the variance components σ_i^2 are known then equation (4.8) holds true. If the variance components are unknown we have to estimate S_i^2 . ξ is specified probability. Welch's approach was to approximate the distribution, i.e. to find such a quantile function h

$$\Pr\left[\bar{Y}_n - \mu < h\ (S_1^2,\ldots,S_k^2,\ \xi)\right] = \xi\,, \tag{4.9}$$

that the equation (4.9) holds true.

The (1 - α) 100% confidence interval is

$$\bar{Y}_n - h\ (S_1^2,\ldots,S_k^2,\ 1-\alpha/2) \le \mu \le \bar{Y}_n + h\ (S_1^2,\ldots,S_k^2,\ 1-\alpha/2)\,, \tag{4.10}$$

where the appropriated function h is

$$h\ (S_1^2,\ldots,S_k^2,\ \xi) = u_{\xi}\sqrt{\sum_{i=1}^{k}\lambda_i S_i^2}\ \left[\ 1 + \ell + \wp\ \right], \tag{4.11}$$

$$\ell = \frac{1+u_{\xi}^{2}\sum_{i=1}^{k}\lambda_i^2 S_i^4 / f_i}{4\left(\sum_{i=1}^{k}\lambda_i S_i^2\right)^2} - \frac{1+u_{\xi}^{2}\sum_{i=1}^{k}\lambda_i^2 S_i^4 / f_i^2}{2\left(\sum_{i=1}^{k}\lambda_i S_i^2\right)^2},$$

$$\wp = \frac{3+5u_{\xi}^{2}+u_{\xi}^{4}\sum_{i=1}^{k}\lambda_i^3 S_i^6 / f_i^2}{3\left(\sum_{i=1}^{k}\lambda_i S_i^2\right)^3} - \frac{15+32u_{\xi}^{2}+9u_{\xi}^{4}u_{\xi}^{4}(\sum_{i=1}^{k}\lambda_i^2 S_i^4 / f_i)^2}{32\left(\sum_{i=1}^{k}\lambda_i S_i^2\right)^4},$$

$$f_i = n_i - 1,\ \lambda_i = \frac{n_i}{N^2},\ \text{for } i = 1, 2, \ldots, k.$$

Interval Derived by Kenward Roger Method

Kenward and Roger derived the method to estimate the variance of the generalized least square estimator (GLSE) and derived a test statistic about expected values.

$$T_{KR} = \frac{\bar{Y}_{\hat{\omega}} - \mu}{\sqrt{Var\ (\bar{Y}_{\hat{\omega}})}}\ , \tag{4.12}$$

where

$$Var\ (\bar{Y}\hat{\omega}) = \left(\sum_{i=1}^{k}\hat{\omega}_i\right)^{-1} + 2\hat{\Lambda},$$

$$\hat{\omega}_i = n_i / S_i^2,\ \ \bar{Y}\hat{\omega} = \left(\sum_{i=1}^{k}\hat{\omega}_i\right)^{-1}\sum_{i=1}^{k}\hat{\omega}_i\bar{Y}i\ ,$$

and $\hat{\Lambda}$ is penalty derived from Kenward and Roger method. The statistic T_{KR} has a *t*-distribution with approximately $\hat{m}$ degrees of freedom, where degrees of freedom $\hat{m}$ are derived by Satterthwaite's method.

The (1 - α) 100% confidence interval is

$$\bar{Y}_{\hat{\omega}} - t_{\ddot{m}(1-\alpha/2)}\sqrt{Var\ (\bar{Y}_{\hat{\omega}})} \le \mu \le \bar{Y}_{\hat{\omega}} + t_{\hat{m}(1-\alpha/2)}\sqrt{Var\ (\bar{Y}_{\hat{\omega}})}\,. \tag{4.13}$$

5. General Model of Interval Estimated Linear Fractional Programming Model (IELFPM)

Consider the following LFPM

$$\left.\begin{array}{ll} \text{Minimize} & Z = \dfrac{cx+\alpha}{dx+\beta}, \\ \text{Subject to} & Ax = b, \\ & x \geq 0, \\ \multicolumn{2}{l}{\text{where } c, d \in R^n, A \in R^{mxn}, b \in R^m, \ \alpha, \beta \in R.} \end{array}\right\} \tag{5.1}$$

The feasible solution set $S^* = \{x \in R^n : Ax \leq b \text{ and } x \geq 0\}$ is assumed to be nonempty and bounded. Assume that $dx + \beta \neq 0$.

The coefficients of LFPM (5.1) are fixed values. That is always not possible in real life models. Therefore, as was described in the previous section, by using the confidence interval technique obtain the intervals of LFP models. Thus the model (5.1) can be rewrite as follows:

$$\left.\begin{array}{ll} \text{Minimize} & f(x) = \dfrac{\sum_{j=1}^{n}\left[\underline{c}_j x_j + \underline{\alpha}_j,\ \overline{c}_j x_j + \overline{\alpha}_j\right]}{\sum_{j=1}^{n}\left[\underline{d}_j x_j + \underline{\beta}_j,\ \overline{d}_j x_j + \overline{\beta}_j\right]} \\ \text{Subject to} & \sum_{j=1}^{n}\left[\underline{a}_{ij},\ \overline{a}_{ij}\right] x_j \leq = \geq \left[\underline{b}_i,\ \overline{b}_i\right] \\ x_j \geq 0 & \end{array}\right\} \tag{5.2}$$

Then, we say that $x = (x_1, x_2, \ldots, x_n)$ is a feasible solution of model (5.2) if and only if $x_1 a_{i1} + \ldots + x_j a_{ij} + \ldots + x_n a_{in} \in \left[\underline{b}_i, \overline{b}_i\right]$, for all possible $a_{ij} \in \left[\underline{a}_{ij}, \overline{a}_{ij}\right]$, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. In other words, $x = (x_1, x_2, \ldots, x_n)$ is a feasible solution of model (5.2) if and only if $\underline{b}_i \leq \sum_{j=i}^{n} a_{ij} x_j \leq \overline{b}_i$ for all possible $a_{ij} \in \left[\underline{a}_{ij}, \overline{a}_{ij}\right]$, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. We adopt the notations $\underline{b}_i = \left(\underline{b}_1, \underline{b}_2, \ldots, \underline{b}_m\right)$ and $\overline{b}_i = \left(\overline{b}_1, \overline{b}_2, \ldots, \overline{b}_m\right)$. Also the feasible solution set $S^* = \{ x_j \in R^n$: $\sum_{j=1}^{n}\left[\underline{a}_{ij}, \overline{a}_{ij}\right] x_j \leq = \geq \left[\underline{b}_i, \overline{b}_i\right]$ and $x_j \geq 0$ } is assumed to be nonempty and bounded. Assume that $\sum_{j=1}^{n}\left[\underline{d}_j x_j + \underline{\beta}_j,\ \overline{d}_j x_j + \overline{\beta}_j\right] \neq 0$.

Let

$$p(x) = \sum_{j=1}^{n} \left[\underline{c}_j x_j + \underline{\alpha}_j,\ \overline{c}_j x_j + \overline{\alpha}_j \right], \tag{5.3}$$

$$q(x) = \sum_{j=1}^{n} \left[\underline{d}_j x_j + \underline{\beta}_j,\ \overline{d}_j x_j + \overline{\beta}_j \right]. \tag{5.4}$$

From (5.3) and (5.4) we consider

$$\underline{p}(x) = \sum_{j=1}^{n} (\underline{c}_j x_j + \underline{\alpha}_j),\quad \overline{p}(x) = \sum_{j=1}^{n} (\overline{c}_j x_j + \overline{\alpha}_j),$$

$$\overline{q}(x) = \sum_{j=1}^{n} (\overline{d}_j x_j + \overline{\beta}_j),\quad \underline{q}(x) = \sum_{j=1}^{n} (\underline{d}_j x_j + \underline{\beta}_j).$$

We suppose that $0 \notin q(x)$ for each feasible solution x, so we should have

$$0 < \underline{q}(x) \le \overline{q}(x) \quad \text{or} \quad \underline{q}(x) \le \overline{q}(x) < 0. \tag{5.5}$$

Using preliminaries (vii) and equation (3.3) the objective function of IVLFP of the model (5.2) can be rewrite into the following form

$$f(x) = \sum_{j=1}^{n} \left[\underline{c}_j x_j + \underline{\alpha}_j,\ \overline{c}_j x_j + \overline{\alpha}_j \right] \left[\frac{1}{\overline{d}_j x_j + \overline{\beta}_j}, \frac{1}{\underline{d}_j x_j + \underline{\beta}_j} \right]. \tag{5.6}$$

Now we can consider two possible cases:

Case (1) When $0 < \underline{q}(x) \le \overline{q}(x)$, we have two possibilities

i) If $0 \le \underline{p}(x) \le \overline{p}(x)$, using preliminaries (vi) and equation (3.1) we have

$$f(x) = \sum_{j=1}^{n} \left[\frac{\underline{c}_j x_j + \underline{\alpha}_j}{\overline{d}_j x_j + \overline{\beta}_j}, \frac{\overline{c}_j x_j + \overline{\alpha}_j}{\underline{d}_j x_j + \underline{\beta}_j} \right]. \tag{5.7}$$

ii) If $\underline{p}(x) < \overline{p}(x) < 0$, using preliminaries (vi) and equation (3.2) we have

$$f(x) = \sum_{j=1}^{n}\left[\frac{\underline{c}_j x_j + \underline{\alpha}_j}{\underline{d}_j x_j + \underline{\beta}_j}, \frac{\overline{c}_j x_j + \overline{\alpha}_j}{\underline{d}_j x_j + \underline{\beta}_j}\right]. \tag{5.8}$$

Case (2) When $\underline{q}(x) \leq \overline{q}(x) < 0$, we have two possibilities:

(i) If $0 \leq \underline{p}(x) \leq \overline{p}(x)$, using preliminaries (vi) and equation (3.1) we have

$$f(x) = \sum_{j=1}^{n}\left[\frac{\overline{c}_j x_j + \overline{\alpha}_j}{\overline{d}_j x_j + \overline{\beta}_j}, \frac{\underline{c}_j x_j + \underline{\alpha}_j}{\underline{d}_j x_j + \underline{\beta}_j}\right]. \tag{5.9}$$

(ii) If $\underline{p}(x) < \overline{p}(x) < 0$, using preliminaries (vi) and equation (3.2) we have

$$f(x) = \sum_{j=1}^{n}\left[\frac{\overline{c}_j x_j + \overline{\alpha}_j}{\overline{d}_j x_j + \overline{\beta}_j}, \frac{\underline{c}_j x_j + \underline{\alpha}_j}{\overline{d}_j x_j + \overline{\beta}_j}\right]. \tag{5.10}$$

We see that the interval-valued optimization models of (5.6) to (5.10) have the common form as shown below:

$$\left.\begin{aligned}
&\text{Minimize} \quad F(x) \\
&\quad \text{Subject to} \quad g_i\,(x) = \left(\sum_{j=1}^{n} \underline{a}_{ij} x_j = \underline{b}_i\right) \leq 0, \; i = 1, 2, \ldots, m, \\
&\qquad\qquad\qquad hi\,(x\,) = \left(\sum_{j=1}^{n} \overline{a}_{ij}\; x_j = \overline{b}_i\right) \leq 0\,,\; i = 1, 2, \ldots, m, \\
&\qquad\qquad\qquad x_i \geq 0.
\end{aligned}\right\} \tag{5.11}$$

where $F : R^n \rightarrow I$ is an interval-valued function, and $g_i : R^n \rightarrow R$ and $h_i : R^n \rightarrow R$, $i = 1, 2, \ldots, m$, are real-valued functions.

Definition 5.1.

To interpret the meaning of optimization of IVF, we introduce a partial ordering $\preceq$ over I. Let $A = [\underline{a}, \overline{a}]$, $B = [\underline{b}, \overline{b}]$ be two closed, bounded, real intervals, $(A, B \in I)$, then we say that $A \preceq B$, if and only if $\underline{a} \leq \underline{b}$ *and* $\overline{a} \leq \overline{b}$. Also we write $A \prec B$, if and only if A $\preceq$ B and A $\neq$ B. In the other words, we say $A \prec B$, if and only if

$$\begin{matrix} \underline{a} < \underline{b} \\ \overline{a} \leq \overline{b} \end{matrix} \quad or \quad \begin{matrix} \underline{a} \leq \underline{b} \\ \overline{a} < \overline{b} \end{matrix} \quad or \quad \begin{matrix} \underline{a} < \underline{b} \\ \overline{a} < \overline{b}. \end{matrix} \tag{5.12}$$

Definition 5.2.

Let x^* be a feasible solution of model (5.11).We say that x^* is a nondominated solution of model (5.11), if there exists no feasible solution x such that $f(x) \prec f(x^*)$. In this case we say that $f(x^*)$ is the nondominated objective value of f.

6. Karush–Kuhn–Tucker (KKT) Optimality Conditions for Interval-estimated Optimization Models

Now we consider the following optimization model,

$$\left.\begin{aligned} &\text{Minimize} && F(x) = \underline{F}(x) + \overline{F}(X) \\ &\text{Subject to} && g_i(x) = \left(\sum_{j=1}^{n} \underline{a_{ij}} x_j = \underline{b_i} \right) \leq 0,\ i = 1, 2, \ldots, m \\ & && hi(x) = \left(\sum_{j=1}^{n} \overline{a_{ij}}\ x_j = \overline{b_i} \right) \leq 0,\ i = 1, 2, \ldots, \mathrm{m} \\ & && x_i \geq 0, \end{aligned}\right\} \tag{6.1}$$

where $F : R^n \to I$ is an interval-valued function, and $g_i : R^n \to R$ and $h_i : R^n \to R$, $i = 1, 2, \ldots, m$ are real-valued functions. Then we have the following observation.

Proposition 6.1.

If x^* is an optimal solution of model (5.11), then x^* is a nondominated solution of model (5.2).

Proof:

We see that model (5.11) and (5.2) have the identical feasible sets. Suppose that x^* is not a nondominated solution. Then there exists a feasible solution x such that $F(x) \prec F(x^*)$. From (5.12) it means that

$$\begin{matrix} \underline{F}(x) < \underline{F}(x^*), \\ \overline{F}(x) \leq \overline{F}(x^*) \end{matrix} \quad or \quad \begin{matrix} \underline{F}(x) \leq \underline{F}(x^*) \\ \overline{F}(x) < \overline{F}(x^*) \end{matrix} \quad or \quad \begin{matrix} \underline{F}(x) < \underline{F}(x^*) \\ \overline{F}(x) < \overline{F}(x^*). \end{matrix}$$

It also shows that $F(x) < F(x^*)$, which contradicts the fact that x^*, is an optimal solution of model (5.11). We complete the proof.

Theorem 6.1. (KKT Conditions)

Suppose that x^* is an optimal solution of model (6.1) and F, g_i, and h_i, $i = 1, 2, \ldots, m$ are differential at x^*. We also assume that the constraint functions g_i, and h_i, $i = 1, 2, \ldots, m$ satisfy the Kuhn- Tucker constraint at x^*. Then there exists KKT multipliers μ_i, $\lambda_i \in \mathcal{R}$ for $i = 1, 2, \ldots, m$ such that

1. $\nabla \underline{F}(x^*) + \nabla \overline{F}(x^*) \sum_{i=1}^{m} \mu_i . \nabla g_i(x^*) + \sum_{i=1}^{m} \lambda_i . \nabla h_i(x^*) = 0$,
2. $\mu_i . g_i(x^*) = 0 = \lambda_i \, h_i(x^*)$ for all i = 1, 2, …, m,
3. $g_i(x^*), h_i(x^*) \leq 0$ for all i = 1, 2, …, m, and
4. $\mu_i, \lambda_i \geq 0$.

7. Numerical Example

We consider multiple period productions – smoothing model with shipping costs and preferring routes, crisp supplies and demands. Here, there is an example of using data obtained from confidence interval technique. Thus, the given IELFPM can be written as the following

$$\left.\begin{array}{ll}
\text{Minimize} & f(x) = \dfrac{[898.48,\ 970.52]\, x_1 + [620.20,\ 686.68]\, x_2 + [770.05,\ 890.20]}{[311.64,\ 343.20]\, x_1 + [800.50,\ 812.50]\, x_2 + [550.56,\ 670.90]} \\
\text{Subject to} & [15.04, 22.01]\, x_1 + [22.90, 34.56]\, x_2 \leq [504.78,\ 888.35] \\
 & [16.22, 35.60]\, x_1 + [37.40,\ 47.20]\, x_2 \leq [544.10,\ 846.33] \\
 & x_i \geq 0, \qquad \forall\, i = 1, 2.
\end{array}\right\} \quad (7.1)$$

We have $0 < \underline{q}(x) \leq \overline{q}(x)$ and also $0 < \underline{p}(x) \leq \overline{p}(x)$, so we should apply in section 5.0, case (1)

(i) Finally we will have the following optimization model

$$\left.\begin{aligned}
&\text{Minimize} \quad f(x) = \left[\frac{898.48x_1 + 620.20x_2 + 770.05}{311.64x_1 + 800.50x_2 + 550.56}, \frac{970.52x_1 + 686.68x_2 + 890.20}{343.20x_1 + 812.50x_2 + 670.90}\right] \\
&\text{Subject to} \quad g_1(x_1, x_2) = 15.04\, x_1 + 22.90\, x_2 = 504.78 \\
&\qquad h_1(x_1, x_2) = 22.01\, x_1 + 34.56\, x_2 = 888.35 \\
&\qquad g_2(x_1, x_2) = 16.22\, x_1 + 37.40\, x_2 = 544.10 \\
&\qquad h_2(x_1, x_2) = 35.60\, x_1 + 47.20\, x_2 = 846.33 \\
&\qquad x_1,\ x_2 \geq 0.
\end{aligned}\right\} \tag{7.2}$$

Now to obtain a nondominated solution for (7.2), we use proposition (6.1) and solve the following optimization model

$$\left.\begin{aligned}
&\text{Minimize} \quad f(x) = \left[\frac{898.48x_1 + 620.20x_2 + 770.05}{311.64x_1 + 800.50x_2 + 550.56} + \frac{970.52x_1 + 686.68x_2 + 890.20}{343.20x_1 + 812.5x_2 + 670.90}\right] \\
&\text{Subject to} \quad 15.04\, x_1 + 22.90\, x_2 = 504.78 \\
&\qquad 22.01\, x_1 + 34.56\, x_2 = 888.35 \\
&\qquad 16.22\, x_1 + 37.40\, x_2 = 544.10 \\
&\qquad 35.60\, x_1 + 47.20\, x_2 = 846.33 \\
&\qquad x_1,\ x_2 \geq 0.
\end{aligned}\right\} \tag{7.3}$$

By using Excel Solver, the optimal solution is $x_1^* = 10.55255$, $x_2^* = 9.971596$ with optimal value $(x^*) = 2.840901$

8. Conclusion

In this paper, first we introduce a LFPM with interval valued parameters. Then we have suggested using confidence intervals for estimating interval values to IELFPM. In practical point of view, confidence intervals based on t- statistic and Welch's method has very good reporting properties for almost all cases. The method based on Satterthwaite's degrees of freedom has good reporting properties whenever the number of observations in one experimental unit is sufficiently large or number of experimental units is increasing. The method based on Kenward and Roger does not have good properties for this model with small number of observations in one experimental unit. By using Karush–Kuhn–Tucker optimality conditions, it is proved that we can convert the model of the IELFPM to the nonlinear fractional programming model and obtained an optimal solution. The study of very complicated system can be done with the help of this model and can be adapted to adjust the variation in the uncertain environments of real situations. Work is in progress to apply and check the approach for solving optimization problem under interval data environment.

REFERENCES

Alefeld, G. and Herzberger, J. (1983). Introduction to interval computations, *Academic Press*, New York.

Atanu Sengupta, and Tapan Kumar Pal (2000). Theory and Methodology: On comparing interval numbers, *European Journal of Operational Research*, 27: 28 - 43.

Atanu Sengupta, Tapan Kumar Pal and Debjani Chakraborty (2001). Interpretation of inequality constraints involving interval coefficients and a solution to interval linear programming-Fuzzy Sets and Systems, *European Journal of Operational Research*, 119:129-138.

Charnes, A. and Cooper, W. W. (1962). Naval Research Log. Quart., 9:181-186.

Charnes, A., Granot, F. and Phillips, E. (1977). An algorithm for solving interval linear programming problems, *Operations Research* 25: 688-695.

Chinneck, J.W. and Ramadan, K. (2000). Linear programming with interval coefficients, *JORS*, 51: 209-220.

Dantzig, G. B. (1955). Linear programming under uncertainty, *Management Science*, 1: 197-206.

Herry Suprajitno and Ismail bin Mohd (2010). Linear Programming with Interval Arithmetic, *Int. J. Contemp. Math. Sciences*, 5(7): 323 – 332.

Hladik M. (2007). Optimal value range in interval linear programming, Faculty of Mathematics and Physics, Charles University, Prague.

Krishnamoorthy, K. and Mathew, T. (2004). One-sided tolerance limits in balanced and unbalanced one-way random models based on generalized confidence limits, *Technometrics*, 46: 44–52.

Kuchta, D. (2008). A modification of a solution concept of the linear programming problem with interval coefficients in the constraints, *CEJOR*, 16: 307-316.

Moore, R. E. (1966). Interval analysis. Englewood Cliffs, NJ: Prentice-Hall.

Moore, R. E. (1979). Method and Application of Interval Analysis, Philadelphia, SIAM.

Schaible, S. (1981). Fractional programming: applications and algorithms, *European Journal of Operational Research*, 7(2): 111-120.

Schaible, S. and Ibaraki, T. (1983). Fractional programming, *European Journal of Operational Research*, 12(3): 325-338.

Sohrab Effati and Morteza Pakdaman (2012). Solving the Interval-Valued Linear Fractional Programming Problem, American Journal of Computational Mathematics, 2:51-55

Suprajitno H. and Mohd I.B. (2008). Interval Linear Programming, presented in ICoMS-3, Bogor, Indonesia.

Suprajitno H., Mohd I.B., Mamat M. and Ismail F. (2009). Linear Programming with Interval Variables, presented in 2nd ICOWOBAS, Johor Bahru, Malaysia.

Suresh Chandra, Jayadeva and Aparna Mehra (2011). Numerical Optimization with Applications, *Narosa Publishing House*, New Delhi, 442-475.

Weerahandi S. (2004). Generalized Inference in Repeated Measures, New York Wiley.

Wu H.-C. (2007). The Karush-Kuhn-Tucker Optimality Conditions in an Optimization Problem with Interval- valued Objective Function, *European Journal of Operational Research*, 176(1): 46-59.

Wu H.-C. (2008). On Interval-Valued Nonlinear Programming Problems, *Journal of Mathematical Analysis and Applications*, 338(1): 299-316.

Authors' Bios

Dr. S. Ananthalakshmi is currently working as Assistant Professor in Department of Mathematics, Annai Hajira Women's, College, Malapalayam, Tirunelveli Tamilnadu (India). She is a Visiting Professor in Department of Statistics, Manonmaniam Sundaranar University, Tirunelveli, Tamil Nadu Her research interests concern applied mathematical methods for advanced linear programming model, risk analysis, uncertainty analysis, inventory management, sensitivity analysis and decision analysis. She is a life member of the Indian Society for Probability and Statistics.

Dr. C. Vijayalakshmi is currently working as Associate Professor in SAS,Mathematics Division,VIT University, Chennai, Tamil Nadu. She has more than 15 years of teaching experience at Graduate and Post Graduate level. She has published more than Twenty research papers in International and National Journals and authored three books for Engineering colleges, Anna University. She has received Bharat Jyothi Award for outstanding services, achievements and contributions, Best Teachers award for academic excellence. Her area of specialization is Stochastic Processes and their applications. Her other research interests include Optimization Techniques, Data mining and Bio-Informatics.

Dr. V. Ganesan is a Visiting Professor in Department of Statistics, Manonmaniam Sundaranar University, Tirunelveli, Tamil Nadu. He has more than 30 years of teaching experience at Graduate and Post Graduate level in Government colleges. He has published Twenty Three research papers in International and National Journals and authored one book for quality services. His area of specialization is Queueing Processes and their applications. His other research interests include Optimization Techniques and Stochastic Processes.