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Abstract

In this paper, a numerical method for solving the systems of variable-coefficient coupled Burgers' equation is proposed. The method is based on two-dimensional Legendre wavelets. Two-dimensional operational matrices of integration are introduced and then employed to find a solution to the systems of variable-coefficient coupled Burgers' equation. Two examples are presented to illustrate the capability of the method. It is shown that the numerical results are in good agreement with the exact solutions for each problem.

Keywords: variable-coefficient coupled Burgers' equation; two-dimensional Legendre wavelets; operational matrix integration

MSC (2010) No.: 65T60, 35A35, 42C40

1. Introduction

The Burgers' equation retains the nonlinear aspects of the governing equations in many applications, such as the mathematical model of turbulence, heat conduction, and the approximate theory of flow through a shock wave traveling in a viscous fluid [Burger (1948); Cole (1951); Rashidi and Erfani (2009)]. The study to coupled Burgers equations is very significant in that the system is a simple model of sedimentation or evolution of scaled volume concentrations of two kinds of particles in fluid suspensions or colloids, under the effect of gravity [Nee and Duan (1998)]. It has been studied by many authors using different methods [Esipov (1995); Biazar and Aminikhah (2009); Abbasbandy and Darvishi (2005); Na (2010) and Aminikhah (2013)].

In the present work, a numerical algorithm based on the two-dimensional Legendre wavelets is proposed, and then applied to the nonlinear systems of variable coefficient coupled Burgers' equation that can be written in the following basic form [Na (2010) and Aminikhah (2013)]

$$\begin{aligned} u_t + r_1(t)\,u_{xx} + s_1(t)\,uu_x + p_1(t)(uv)_x &= 0, \\ v_t + r_2(t)\,v_{xx} + s_2(t)\,vv_x + p_2(t)(uv)_x &= 0, \end{aligned} \tag{1}$$

subject to the initial conditions:

$$u(x,0) = f(x), v(x,0) = g(x),$$

and the boundary conditions:

$$\begin{aligned} u(0,t) &= f_1(t), u_x(0,t) = f_2(t), \\ v(0,t) &= g_1(t), v_x(0,t) = g_2(t), \end{aligned}$$

where the subscripts $r_1(t), r_2(t), s_1(t), s_2(t), p_1(t)$ and $p_2(t)$ are arbitrary smooth functions of t.

Wavelet theory is a relatively new and an emerging area in mathematical research. Wavelets analysis possesses several useful properties, such as orthogonality, compact support, exact representation of polynomials to a certain degree, and multiresolution (MRA) [Yousefi (2011)]. Moreover wavelets establish a connection with fast numerical algorithms [Beilkin et al. (1991)]. Therefore the wavelet is successfully used in many fields. The fundamental idea of the Legendre wavelet method is, using the operational matrices, the nonlinear system of variable-coefficient coupled Burgers' equation which satisfies the boundary and initial conditions that can be converted into a set of algebraic equations.

The article is summarized as follows. In the section 2 we introduce the two-dimensional Legendre wavelets and we introduced operational matrices of integration in section 3. Section 4 is devoted to the solutions of (1) that utilize the aforementioned matrices and the 2-D Legendre wavelets. In Section 5, by considering numerical examples reported in our work, the accuracy of the proposed scheme is demonstrated.

2. Two-Dimensional Legendre wavelets

Two–dimensional Legendre wavelets in $L^2(\mathbb{R})$ over interval $[0,1]\times[0,1]$ defined as [Parsian (2005)]

$$\psi_{n,m,n',m'}(x,y)=\begin{cases}\sqrt{\left(m+\frac{1}{2}\right)\left(m'+\frac{1}{2}\right)}2^{\frac{k+k'}{2}} & \\ P_m\left(2^k x-2n+1\right)P_{m'}\left(2^{k'}y-2n'+1\right), & \frac{n-1}{2^{k-1}}\le x\le\frac{n}{2^{k-1}}, \\ & \frac{n'-1}{2^{k'-1}}\le y\le\frac{n'}{2^{k'-1}}; \\ 0 & otherwise\end{cases} \tag{2}$$

and

$$n=1,2,\ldots,2^{k-1}, n'=1,2,\ldots 2^{k'-1},$$
$$m=0,1,2,\ldots,M-1, m'=0,1,2,\ldots M'-1.$$

The coefficient $\sqrt{\left(m+\frac{1}{2}\right)\left(m'+\frac{1}{2}\right)}$ is for orthonormality. Here, $P_m(x)$ are the well-known Legendre polynomials of order m which are defined on the interval $[-1,1]$, and can be determined with the aid of the following recurrence formulae:

$$P_0(x)=1, P_1(x)=x,$$
$$P_{m+1}(x)=\left(\frac{2m+1}{m+1}\right)x\,P_m(x)-\left(\frac{m}{m+1}\right)P_{m-1}(x),\ m=1,\ 2,\ldots,$$

where the two-dimensional Legendre wavelets are an orthonormal set over $[0,1]\times[0,1]$,

$$\int_0^1\int_0^1 \psi_{n,m,n',m'}(x,y)\,\psi_{n_1,m_1,n'_1,m'_1}(x,y)\,dx\,dy=\delta_{n,n_1}\delta_{m,m_1}\delta_{n',n'_1}\delta_{m',m'_1},\ \delta_{i,j}=\begin{cases}1 & i=j\\ 0 & i\neq j\end{cases} \tag{3}$$

The function $u(x,y)\in L^2(\mathbb{R})$ defined over $[0,1]\times[0,1]$ may be expanded as

$$u(x,y)=X(x)Y(y)\cong\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n'=1}^{\infty}\sum_{m'=0}^{\infty}c_{n,m,n',m'}\,\psi_{n,m,n',m'}(x,y). \tag{4}$$

If the infinite series (4) is truncated, then can be written as

$$u(x,y)=X(x)Y(y)\cong\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\sum_{n'=1}^{2^{k'-1}}\sum_{m'=0}^{M'-1}c_{n,m,n',m'}\,\psi_{n,m,n',m'}(x,y), \tag{5}$$

where

$$c_{n,m,n',m'} = \int_0^1 \int_0^1 X(x)\, Y(y)\, \psi_{n,m,n',m'}(x,y)\, dx\, dy. \tag{6}$$

The equation (5) may be expressed in the form

$$u(x,y) = C^T\, \Psi(x,y), \tag{7}$$

where C and $\Psi(x,y)$ are the coefficient matrix and the wavelet vector respectively. The dimensions of those are $2^{k-1}2^{k'-1}MM' \times 1$ and given by [Beilkin, Coifman and Rokhlin (1991)] in the form

$$\begin{aligned} C = [&c_{1,0,1,0},\dots c_{1,0,1,M'-1}, c_{1,0,2,0}\dots, c_{1,0,2,M'-1},\dots, c_{1,0,2^{k'-1},0}\dots, c_{1,0,2^{k'-1},M'-1},\\ &c_{1,1,1,0},\dots c_{1,1,1,M'-1}, c_{1,1,2,0},\dots, c_{1,1,2,M'-1},\dots c_{1,1,2^{k'-1},0},\dots, c_{1,1,2^{k'-1},M'-1},\dots,\\ &c_{1,M-1,1,0},\dots, c_{1,M-1,1,M'-1}, c_{1,M-1,2,0},\dots, c_{1,M-1,2,M'-1},\dots, c_{1,M-1,2^{k'-1},0},\dots,\\ &c_{1,M-1,2^{k'-1},M'-1}, c_{2,0,1,0},\dots, c_{2,0,1,M'-1}, c_{2,0,2,0},\dots, c_{2,0,2,M'-1},\dots,\\ &c_{2,0,2^{k'-1},0},\dots, c_{2,0,2^{k'-1},M'-1},\dots, c_{2,M-1,1,0},\dots, c_{2,M-1,1,M'-1}, c_{2,M-1,2,0},\dots,\\ &c_{2,M-1,2,M'-1},\dots, c_{2,M-1,2^{k'-1},0},\dots, c_{2,M-1,2^{k'-1},M'-1},\dots, c_{2^{k-1},0,1,0},\dots,\\ &c_{2^{k-1},0,1,M'-1}, c_{2^{k-1},0,2,0},\dots, c_{2^{k-1},0,2,M'-1},\dots, c_{2^{k-1},0,2^{k'-1},0},\dots\\ &c_{2^{k-1},0,2^{k'-1},M'-1},\dots, c_{2^{k-1},M-1,1,0},\dots, c_{2^{k-1},M-1,1,M'-1}, c_{2^{k-1},M-1,2,0},\dots,\\ &c_{2^{k-1},M-1,2,M'-1},\dots, c_{2^{k-1},M-1,2^{k'-1},0},\dots, c_{2^{k-1},M-1,2^{k'-1},M'-1}]^T, \end{aligned} \tag{8}$$

and

$$\begin{aligned} \Psi(x,y) = [&\psi_{1,0,1,0},\dots, \psi_{1,0,1,M'-1}, \psi_{1,0,2,0},\dots, \psi_{1,0,2,M'-1},\dots, \psi_{1,0,2^{k'-1},0},\dots\psi_{1,0,2^{k'-1},M'-1},\\ &\psi_{1,1,1,0},\dots, \psi_{1,1,1,M'-1}, \psi_{1,1,2,0},\dots, \psi_{1,1,2,M'-1},\dots, \psi_{1,1,2^{k'-1},0},\dots,\\ &\psi_{1,1,2^{k'-1},M'-1},\dots, \psi_{1,M-1,1,0},\dots, \psi_{1,M-1,1,M'-1}, \psi_{1,M-1,2,0},\dots,\\ &\psi_{1,M-1,2,M'-1},\dots, \psi_{1,M-1,2^{k'-1},0},\dots, \psi_{1,M-1,2^{k'-1},M'-1}, \psi_{2,0,1,0},\dots,\\ &\psi_{2,0,1,M'-1}, \psi_{2,0,2,0},\dots, \psi_{2,0,2,M'-1},\dots, \psi_{2,0,2^{k'-1},0},\dots, \psi_{2,0,2^{k'-1},M'-1},\dots,\\ &\psi_{2,M-1,1,0},\dots, \psi_{2,M-1,1,M'-1}, \psi_{2,M-1,2,0},\dots, \psi_{2,M-1,2,M'-1},\dots,\\ &\psi_{2,M-1,2^{k'-1},0},\dots, \psi_{2,M-1,2^{k'-1},M'-1},\dots, \psi_{2^{k-1},0,1,0},\dots, \psi_{2^{k-1},0,1,M'-1},\\ &\psi_{2^{k-1},0,2,0},\dots, \psi_{2^{k-1},0,2,M'-1},\dots, \psi_{2^{k-1},0,2^{k'-1},0},\dots, \psi_{2^{k-1},0,2^{k'-1},M'-1},\dots,\\ &\psi_{2^{k-1},M-1,1,0},\dots, \psi_{2^{k-1},M-1,1,M'-1}, \psi_{2^{k-1},M-1,2,0},\dots, \psi_{2^{k-1},M-1,2,M'-1},\dots,\\ &\psi_{2^{k-1},M-1,2^{k'-1},0},\dots, \psi_{2^{k-1},M-1,2^{k'-1},M'-1}]^T \end{aligned} \tag{9}$$

The integration of the product of the two Legendre wavelet function vectors is obtained as

$$\int_0^1 \int_0^1 \Psi(x,y)\, \Psi^T(x,y)\, dx\, dy = I, \tag{10}$$

where I is identity matrix.

3. Two-dimensional operational matrix of integration

3.1. Operational matrix of integration for the *x* variable

The integration matrix for the x variable defined by

$$\int_0^x \Psi(x',y)\,dx' = P_x \Psi(x,y), \tag{11}$$

where P_x is the $2^{k-1}2^{k'-1}MM' \times 2^{k-1}2^{k'-1}MM'$ operational matrix for integration given by [Beilkin et al. (1991)] as

$$P_x = \frac{1}{M'2^{k+k'-1}}\begin{bmatrix} L & F & F & F & \cdots & F \\ O & L & F & F & \cdots & F \\ O & O & L & F & \cdots & F \\ O & O & O & L & \cdots & F \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & O & \cdots & L \end{bmatrix}, \tag{12}$$

and F, L and O are $2^{k-1}MM' \times 2^{k-1}MM'$ matrices that defined as below:

$$F = \begin{bmatrix} 2D & O' & O' & \cdots & O' \\ O' & O' & O' & \cdots & O' \\ O' & O' & O' & \cdots & O' \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O' & O' & O' & \cdots & O' \end{bmatrix},$$

$$L = \begin{bmatrix} D & \frac{1}{\sqrt{3}}D & O' & \cdots & O' \\ -\frac{1}{\sqrt{3}}D & O' & \frac{\sqrt{3}}{3\sqrt{5}}D & \cdots & O' \\ O' & \frac{-\sqrt{5}}{5\sqrt{3}}D & O' & \cdots & O' \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O' & O' & O' & \cdots & O' \end{bmatrix},$$

and

$$O=\begin{bmatrix} O' & O' & O' & \cdots & O' \\ O' & O' & O' & \cdots & O' \\ O' & O' & O' & \cdots & O' \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O' & O' & O' & \cdots & O' \end{bmatrix},$$

where D is the $2^{k'-1}M' \times 2^{k'-1}M'$ matrix defined below as:

$$D=\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix},$$

O' is $2^{k'-1}M' \times 2^{k'-1}M'$ zero matrix.

3.2. Operational matrix of integration for y variable

The integration matrix for the y variable defined as:

$$\int_0^y \Psi\left(x,y'\right)dy' = P_y \Psi\left(x,y\right). \tag{13}$$

Here,

$$P_y = \frac{1}{M2^{k-1}}\begin{bmatrix} P & P & P & P & \cdots & P \\ P & P & P & P & \cdots & P \\ P & P & P & P & \cdots & P \\ P & P & P & P & \cdots & P \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ P & P & P & P & \cdots & P \end{bmatrix}, \tag{14}$$

P_y is $2^{k-1}2^{k'-1}MM' \times 2^{k-1}2^{k'-1}MM'$ matrix, P is $2^{k'-1}M' \times 2^{k'-1}M'$ matrix and defined as:

$$P=\frac{1}{2^{k'}}\begin{bmatrix} L & F & F & F & \cdots & F \\ O & L & F & F & \cdots & F \\ O & O & L & F & \cdots & F \\ O & O & O & L & \cdots & F \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & O & O & L \end{bmatrix},$$

where O, L and F are $M' \times M'$ matrices. O is the zero matrix and L, F are defined as:

$$F=\begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

and

$$L=\begin{bmatrix} 1 & \frac{1}{\sqrt{3}} & 0 & \cdots & 0 \\ -\frac{1}{\sqrt{3}} & 0 & \frac{\sqrt{3}}{3\sqrt{5}} & \cdots & 0 \\ 0 & \frac{-\sqrt{5}}{5\sqrt{3}} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

4. Two-dimensional Legendre wavelets applied to the systems of variable-coefficient coupled Burgers' Equation

Consider the nonlinear systems of variable-coefficient coupled Burgers' Equation (1). Let "$\cdot$" and "$'$" denote differentiation with respect to x and t respectively.
In order to using Legendre wavelets to approximate $\ddot{u}'(x,t)$ and $\ddot{v}'(x,t)$ we have

$$\ddot{u}'(x,t) \approx {C_1}^T \Psi(x,t), \tag{15}$$

$$\ddot{v}'(x,t) \approx {C_2}^T \Psi(x,t). \tag{16}$$

Integrating Equation (15) with respect to t once from 0 to t and with respect to x twice from 0 to x, we obtain

$$\begin{aligned} \ddot{u}(x,t) &\approx C_1^T P_t \Psi(x,t) + \ddot{u}(x,0) \\ &= C_1^T P_t \Psi(x,t) + U_0^T \Psi(x,t), \end{aligned} \tag{17}$$

$$\begin{aligned} \dot{u}(x,t) &\approx C_1^T P_t P_x \Psi(x,t) + \dot{u}(x,0) - \dot{u}(0,0) + \dot{u}(0,t) \\ &= C_1^T P_t P_x \Psi(x,t) + U_1^T \Psi(x,t), \end{aligned} \tag{18}$$

$$\begin{aligned} u(x,t) &\approx C_1^T P_t P_x^2 \Psi(x,t) + u(x,0) - u(0,0) - x\dot{u}(0,0) + x\dot{u}(0,t) + u(0,t) \\ &= C_1^T P_t P_x^2 \Psi(x,t) + U_2^T \Psi(x,t). \end{aligned} \tag{19}$$

Integrating Equation (15) with respect to x twice from 0 to x, we obtain

$$\dot{u}'(x,t) \approx C_1^T P_x \Psi(x,t) + \dot{u}'(0,t), \tag{20}$$

$$\begin{aligned} u'(x,t) &\approx C_1^T P_x^2 \Psi(x,t) + x\dot{u}'(0,t) + u'(0,t) \\ &= C_1^T P_x^2 \Psi(x,t) + U_3^T \Psi(x,t). \end{aligned} \tag{21}$$

Similarly, integrating Equation (16) with respect to t once from 0 to t and with respect to x twice from 0 to x, we obtain

$$\begin{aligned} \ddot{v}(x,t) &\approx C_2^T P_t \Psi(x,t) + \ddot{v}(x,0) \\ &= C_2^T P_t \Psi(x,t) + V_0^T \Psi(x,t), \end{aligned} \tag{22}$$

$$\begin{aligned} \dot{v}(x,t) &\approx C_2^T P_t P_x \Psi(x,t) + \dot{v}(x,0) - \dot{v}(0,0) + \dot{v}(0,t) \\ &= C_2^T P_t P_x \Psi(x,t) + V_1^T \Psi(x,t), \end{aligned} \tag{23}$$

$$\begin{aligned} v(x,t) &\approx C_2^T P_t P_x^2 \Psi(x,t) + v(x,0) - v(0,0) - x\dot{v}(0,0) + x\dot{v}(0,t) + v(0,t) \\ &= C_2^T P_t P_x^2 \Psi(x,t) + V_2^T \Psi(x,t). \end{aligned} \tag{24}$$

Also, integrating Equation (16) with respect to x twice from 0 to x, we obtain

$$\dot{v}'(x,t) \approx C_2^T P_x \Psi(x,t) + \dot{v}'(0,t), \tag{25}$$

$$\begin{aligned} v'(x,t) &\approx C_2^T P_x^2 \Psi(x,t) + x\dot{v}'(0,t) + v'(0,t) \\ &= C_2^T P_x^2 \Psi(x,t) + V_3^T \Psi(x,t), \end{aligned} \tag{26}$$

where the coefficients U_0, U_1, U_2, U_3 and V_0, V_1, V_2, V_3 are known and obtained from the initial and boundary conditions. P_x and P_t are defined similarly in Equations (12) and (14).

Now consider the following approximations

$$
\begin{aligned}
&r_1(t)u_{xx} \approx Y_1^T\Psi(x,t),\\
&s_1(t)uu_x \approx Y_2^T\Psi(x,t),\\
&p_1(t)(uv)_x = p_1(t)(u_xv+v_xu) \approx Y_3^T\Psi(x,t),\\
&r_2(t)v_{xx} \approx Y_4^T\Psi(x,t),\\
&s_2(t)vv_x \approx Y_5^T\Psi(x,t),\\
&p_2(t)(uv)_x = p_2(t)(u_xv+v_xu) \approx Y_6^T\Psi(x,t),
\end{aligned}
\tag{27}
$$

where Y_1, Y_2, Y_3, Y_4, Y_5 and Y_6 are column vectors with the entries of the vectors C_1 and C_2.

Substitution of approximations (21), (26) and (27) in to the systems (1) results in

$$
\begin{aligned}
&C_1^T\, P_x^2\,\Psi(x,t) + U_3^T\Psi(x,t) + Y_1^T\Psi(x,t) + Y_2^T\Psi(x,t) + Y_3^T\Psi(x,t) = 0,\\
&C_2^T\, P_x^2\,\Psi(x,t) + V_3^T\Psi(x,t) + Y_4^T\Psi(x,t) + Y_5^T\Psi(x,t) + Y_6^T\Psi(x,t) = 0.
\end{aligned}
\tag{28}
$$

From the simplified system (28) the nonlinear system of the entries of C_1, C_2 is obtained,

$$
\begin{aligned}
&C_1^T\, P_x^2 + U_3^T + Y_1^T + Y_2^T + Y_3^T = 0,\\
&C_2^T\, P_x^2 + V_3^T + Y_4^T + Y_5^T + Y_6^T = 0,
\end{aligned}
\tag{29}
$$

The elements of the vector functions C_1 and C_2 can be computed by solving systems (29).

5. Numerical examples

In this section, two examples of systems of the variable-coefficient coupled Burgers’ equation are considered and will be solved by the method proposed.

Example1.

Consider the following variable coefficient coupled Burgers’ equation

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= -\frac{\partial^2 u}{\partial x^2} + 2e^{2t}\sin(2t)\,u\frac{\partial u}{\partial x} - \sin(2t)\frac{\partial (uv)}{\partial x},\\
\frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2} - 2e^{-2t}\cos(2t)\,v\frac{\partial v}{\partial x} + \cos(2t)\frac{\partial (uv)}{\partial x},
\end{aligned}
\tag{30}
$$

subject to the initial conditions:

$$u(x,0) = v(x,0) = e^x,$$

and the boundary conditions:

$$u(0,t) = e^{-t}, \frac{\partial u}{\partial x}(0,t) = e^{-t}.$$

$$v(0,t) = e^{t}, \frac{\partial v}{\partial x}(0,t) = e^{t}.$$

The exact solution of the equation is $u(x,t) = e^{x-t}$ and $v(x,t) = e^{x+t}$.

We solve the system (30) by introduced method in the paper with $k = k' = 1$ and $M = M' = 5$.

Let's consider the following approximations:

$$\ddot{u}'(x,t) \approx {C_1}^T \Psi(x,t),$$

$$\ddot{v}'(x,t) \approx {C_2}^T \Psi(x,t),$$

$$\begin{aligned}\ddot{u}(x,t) &\approx C_1^T P_t \Psi(x,t) + \ddot{u}(x,0)\\ &= C_1^T P_t \Psi(x,t) + U_0^T \Psi(x,t),\end{aligned}$$

$$\begin{aligned}\dot{u}(x,t) &\approx C_1^T P_t P_x \Psi(x,t) + \dot{u}(x,0) - \dot{u}(0,0) + \dot{u}(0,t)\\ &= C_1^T P_t P_x \Psi(x,t) + U_1^T \Psi(x,t),\end{aligned}$$

$$\begin{aligned}u(x,t) &\approx C_1^T P_t P_x^2 \Psi(x,t) + u(x,0) - u(0,0) - x\dot{u}(0,0) + x\dot{u}(0,t) + u(0,t)\\ &= C_1^T P_t P_x^2 \Psi(x,t) + U_2^T \Psi(x,t),\end{aligned}$$

$$\begin{aligned}u'(x,t) &\approx C_1^T P_x^2 \Psi(x,t) + x\dot{u}'(0,t) + u'(0,t)\\ &= C_1^T P_x^2 \Psi(x,t) + U_3^T \Psi(x,t),\end{aligned}$$

$$\begin{aligned}\ddot{v}(x,t) &\approx C_2^T P_t \Psi(x,t) + \ddot{v}(x,0)\\ &= C_2^T P_t \Psi(x,t) + V_0^T \Psi(x,t),\end{aligned}$$

$$\begin{aligned}\dot{v}(x,t) &\approx C_2^T P_t P_x \Psi(x,t) + \dot{v}(x,0) - \dot{v}(0,0) + \dot{v}(0,t)\\ &= C_2^T P_t P_x \Psi(x,t) + {V_1}^T \Psi(x,t),\end{aligned}$$

$$\begin{aligned}v(x,t) &\approx C_2^T P_t P_x^2 \Psi(x,t) + v(x,0) - v(0,0) - x\dot{v}(0,0) + x\dot{v}(0,t) + v(0,t)\\ &= C_2^T P_t P_x^2 \Psi(x,t) + V_2^T \Psi(x,t),\end{aligned}$$

$$
\begin{aligned}
v'(x,t) &\approx C_2^T\,P_x^2\,\Psi(x,t) + x\dot{v}'(0,t) + v'(0,t)\\
&= C_2^T\,P_x^2\,\Psi(x,t) + V_3^T\Psi(x,t),\\
e^{2t}\sin(2t)\,u\frac{\partial u}{\partial x} &\approx Y_1^T\Psi(x,t),\\
\sin(2t)\frac{\partial(uv)}{\partial x} &= \sin(2t)\left(v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x}\right) \approx Y_2^T\Psi(x,t),\\
e^{-2t}\cos(2t)\,v\frac{\partial v}{\partial x} &\approx Y_3^T\Psi(x,t),
\end{aligned}
$$

and

$$
\cos(2t)\frac{\partial(uv)}{\partial x} = \cos(2t)\left(v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x}\right) \approx Y_4^T\Psi(x,t).
$$

Substitution into the systems (30) and simplifying, we obtain:

$$
\begin{aligned}
C_1^T P_x^2 + U_3^T &= -\left(C_1^T P_t + U_0^T\right) + 2Y_1^T - Y_2^T,\\
C_2^T P_x^2 + V_3^T &= C_2^T\,P_t + V_0^T + 2Y_3^T + Y_4^T.
\end{aligned} \tag{31}
$$

Solving system (31) elements of the vector functions C_1 and C_2 can be obtained via the Maple package as follows:

$$
\begin{aligned}
C_1 = [&-1.087493405, 0.03061682727, -0.042153684020,\\
&0.001449648352, -0.0004844560546, -0.3101041425,\\
&0.08472002985, -0.01460270085, -0.001655686190,\\
&-0.0008626378535, -0.04056061053, 0.009419167299,\\
&-0.003631522310, -0.001627775833, -0.0006359870059,\\
&-0.003772120368, 0.0001359764102, -0.001136925203,\\
&-0.0008129285993, -0.0003001325444, -0.0003817937971,\\
&-0.0002461953633, -0.0003850232469, -0.0003292268874,\\
&-0.0001066526579]^T,
\end{aligned}
$$

and

$$
\begin{aligned}
C_2 = [&2.952130086, 0.8378741410, 0.1069559032, 0.008624692470,\\
&-0.0005488962243, 0.8377935328, 0.2370796653, 0.02950114344,
\end{aligned}
$$

$$0.001664032482, -0.001348812292, 0.1068907691,$$
$$0.02958926421, 0.002954714985, -0.0005214776846,$$
$$-0.001072652369, 0.008713733128, 0.002039310536, -0.0002143372932,$$
$$-0.0004460174704, -0.0005347054352, 0.0004232992796,$$
$$-0.00008642252597, -0.0002332484704, -0.000198172675, -0.0002003594432]^T.$$

Resulting in, the following solutions will result:

$$\begin{aligned} u(x,t) &\approx \left(C_1^T P_t P_x^2 + U_2^T\right)\Psi(x,t) \\ &= (-0.02045780897\,t^4 + 0.02626512245\,t^3 + 0.01602087677\,t^2 \\ &\quad - 0.06665092588\,t + 0.06898619322)\,x^4 + (0.03458774114\,t^4 \\ &\quad - 0.07351816784\,t^3 + 0.09541143195\,t^2 - 0.01435494014\,t + \\ &\quad 0.1406033728)\,x^3 + (-0.000655580043\,t^4 - 0.05609814020\,t^3 \\ &\quad + 0.2422704207\,t^2 - 0.5088769723\,t + 0.5100091115)\,x^2 \\ &\quad + (0.02812378780\,t^4 - 0.1576936525\,t^3 + 0.4966734337\,t^2 \\ &\quad - 0.9980851067t + 0.9985530837)\,x + 0.02537646739\,t^4 \\ &\quad - 0.1532932779\,t^3 + 0.4951680489\,t^2 - 0.9993694794\,t + 1.000023933, \end{aligned}$$

and

$$\begin{aligned} v(x,t) &\approx \left(C_2^T P_t P_x^2 + V_2^T\right)\Psi(x,t) \\ &= (-0.005521140823\,t^4 + 0.03001500346\,t^3 + 0.02234065672\,t^2 \\ &\quad + 0.07266448724\,t + 0.06891807128)\,x^4 + (0.02722651489\,t^4 \\ &\quad - 0.01411439480\,t^3 + 0.09231748308\,t^2 + 0.1343044398\,t \\ &\quad + 0.1407159291)\,x^3 + (0.02769224341\,t^4 + 0.08627879549\,t^3 \\ &\quad + 0.2513012432\,t^2 + 0.5122036141\,t + 0.5099738628)\,x^2 \\ &\quad + (0.07125706429\,t^4 + 0.1358251233\,t^3 + 0.5120322391\,t^2 \\ &\quad + 0.9963394641\,t + 0.9986422408)\,x + 0.06791994170\,t^4 \\ &\quad + 0.14279334337\,t^3 + 0.5086018908\,t^2 + 0.9989496681\,t + 1.000077107. \end{aligned}$$

Figure 1 and Figure 2 show the numerical solution for Equation (30) obtained by the two-dimensional Legendre wavelets method, for $x, t \in [0,1]$.

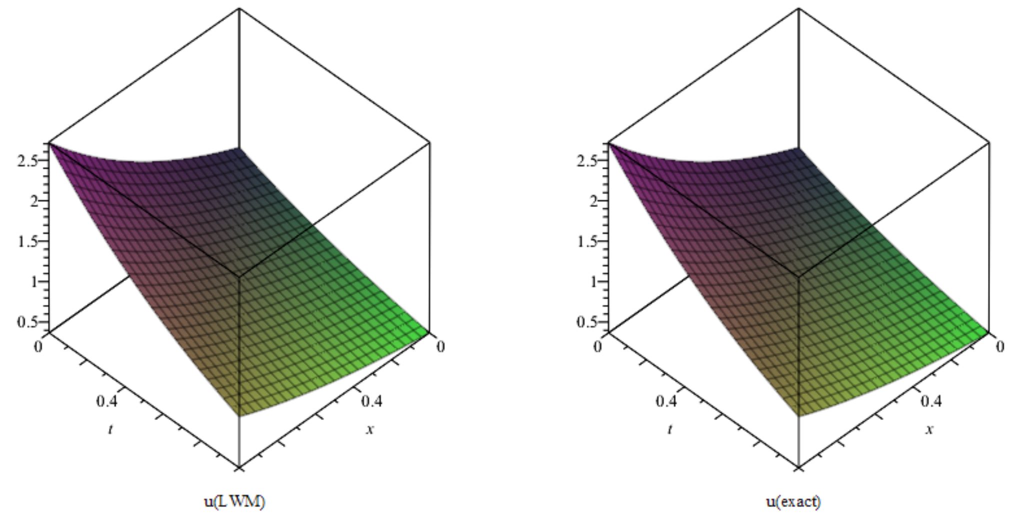

Figure 1. The exact and LWM solution $u(x,t)$ of example1

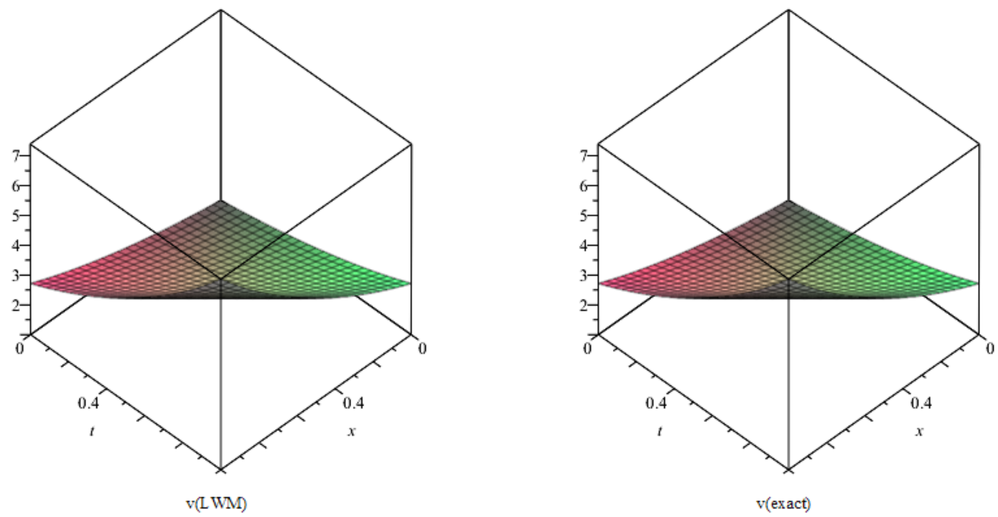

Figure 2: The exact and LWM solution $v(x,t)$ of Example 1

In Tables 1.1-1.5, we show the Comparisons between numerical and analytical solutions of Equation (30) in $t=0, t=0.25, t=0.5, t=0.75$ and $t=1$, for various values of x.

Table 1.1. Numerical results of example1 for $t=0$

x	u_{exact}	u_{LWM}	$\lvert u_{exact}-u_{LWM}\rvert$	v_{exact}	v_{LWM}	$\lvert v_{exact}-v_{LWM}\rvert$
0.0	1.0000000000	1.0000239330	0.0000239330	1.0000000000	1.0000771070	0.0000771070
0.1	1.1051709180	1.1051268340	0.0000440840	1.1051709180	1.1051886770	0.0000177590
0.2	1.2214027580	1.2213701190	0.0000326390	1.2214027580	1.2214405060	0.0000377480
0.3	1.3498588080	1.3498457570	0.0000130510	1.3498588080	1.3499249930	0.0000661850
0.4	1.4918246980	1.4918112870	0.0000134110	1.4918246980	1.4918999430	0.0000752450
0.5	1.6487212710	1.6486898110	0.0000314600	1.6487212710	1.6487885640	0.0000672930
0.6	1.8221188000	1.8220700020	0.0000487980	1.8221188000	1.8221794650	0.0000606650
0.7	2.0137527070	2.0137060980	0.0000466090	2.0137527070	2.0138266610	0.0000739540
0.8	2.2255409280	2.2255179030	0.0000230250	2.2255409280	2.2256495690	0.0001086410
0.9	2.4596031110	2.4595907890	0.0000123220	2.4596031110	2.4597330110	0.0001299000
1.0	2.7182818280	2.7181756940	0.0001061340	2.7182818280	2.7182818280	0.0000453830

Table 1.2. Numerical results of example1 for $t = 0.25$

x	u_{exact}	u_{LWM}	$\lvert u_{exact} - u_{LWM} \rvert$	v_{exact}	v_{LWM}	$\lvert v_{exact} - v_{LWM} \rvert$
0.0	0.7788007831	0.7788334856	0.0000327025	1.2840254170	1.2840986000	0.0000731830
0.1	0.8607079764	0.8606910215	0.0000169549	1.4190675490	1.4190522880	0.0000152610
0.2	0.9512294245	0.9512227240	0.0000067005	1.5683121850	1.5683139090	0.0000017240
0.3	1.0512710960	1.0512797430	0.0000086470	1.7332530180	1.7332832890	0.0000302710
0.4	1.1618342430	1.1618420050	0.0000077620	1.9155408290	1.9155736820	0.0000328530
0.5	1.2840254170	1.2840182060	0.0000072110	2.1170000170	2.1170117690	0.0000117520
0.6	1.4190675490	1.4190458180	0.0000217310	2.3396376560	2.3396376560	0.0000091960
0.7	1.5683121850	1.5682910840	0.0000211010	2.5857048790	2.5857048790	0.0000047800
0.8	1.7332530180	1.7332490200	0.0000039980	2.8576803980	2.8576803980	0.0000292800
0.9	1.9155408290	1.9155434120	0.0000025830	3.1582446040	3.1582446040	0.0000516940
1.0	2.1170000170	2.1169268230	0.0000731940	3.4902913100	3.4902913100	0.0000516470

Table 1.3. Numerical results of example1 for $t = 0.5$

x	u_{exact}	u_{LWM}	$\lvert u_{exact} - u_{LWM} \rvert$	v_{exact}	v_{LWM}	$\lvert v_{exact} - v_{LWM} \rvert$
0.0	0.6065306597	0.6065555750	0.0000249153	1.6487212710	1.6487965780	0.0000753070
0.1	0.6703200460	0.6703087370	0.0000113090	1.8221188000	1.8220782260	0.0000405740
0.2	0.7408182207	0.7408158592	0.0000023615	2.0137525070	2.0137334570	0.0000192500
0.3	0.8187307531	0.8187408750	0.0000101219	2.2255409280	2.2255588470	0.0000179190
0.4	0.9048374180	0.9048477290	0.0000103110	2.4596031110	2.4596251630	0.0000220520
0.5	1.0000000000	1.0000003730	0.0000003730	2.7182818280	2.7182773500	0.0000044780
0.6	1.1051709180	1.1051627700	0.0000081480	3.0041660240	3.0041345300	0.0000314940
0.7	1.2214027580	1.2213988930	0.0000038650	3.3201169230	3.3200900130	0.0000269100
0.8	1.3498588080	1.3498727190	0.0000139110	3.6692966680	3.6693112870	0.0000146190
0.9	1.4918246980	1.4918482390	0.0000235410	4.0551999670	4.0552400230	0.0000400560
1.0	1.6487212710	1.6486894540	0.0000318170	4.4816890700	4.4815920730	0.0000969970

Table 1.4. Numerical results of example1 for $t = 0.75$

x	u_{exact}	u_{LWM}	$\lvert u_{exact} - u_{LWM} \rvert$	v_{exact}	v_{LWM}	$\lvert v_{exact} - v_{LWM} \rvert$
0.0	0.4723665527	0.4723875222	0.0000209695	2.1170000170	2.1171091580	0.0001091410
0.1	0.5220457768	0.5220388282	0.0000069486	2.3396468520	2.3396025610	0.0000442910
0.2	0.5769498104	0.5769501236	0.0000003132	2.5857096590	2.5856907110	0.0000189480
0.3	0.6376281516	0.6376380607	0.0000099091	2.8576511180	2.8576786350	0.0000275170
0.4	0.7046880897	0.7046975765	0.0000094868	3.1581929100	3.1582239270	0.0000310170
0.5	0.7788007831	0.7788018871	0.0000011040	3.4903429570	3.4903367360	0.0000062210
0.6	0.8607079764	0.8607024925	0.0000054839	3.8574255310	3.8573797600	0.0000457710
0.7	0.9512294245	0.9512291741	0.0000002504	4.2631145150	4.2630682630	0.0000462520
0.8	1.0512710960	1.0512899900	0.0000188940	4.7114701830	4.7114700610	0.0000001220
0.9	1.1618342430	1.1618712860	0.0000370430	5.2069798270	5.2070055330	0.0000257060
1.0	1.2840254170	1.2840376900	0.0000122730	5.7546026760	5.7644476080	0.0001550680

Table 1.5. Numerical results of example1 for $t = 1$

x	u_{exact}	u_{LWM}	$\lvert u_{exact} - u_{LWM}\rvert$	v_{exact}	v_{LWM}	$\lvert v_{exact} - v_{LWM}\rvert$
0.0	0.3678794412	0.3679056920	0.0000262508	2.7182818280	2.7183419520	0.0000601240
0.1	0.4065696597	0.4065852860	0.0000156263	3.0041660240	3.0040253530	0.0001406710
0.2	0.4493289641	0.4493528960	0.0000239319	3.3201169230	3.3200042360	0.0001126870
0.3	0.4965853038	0.4966167190	0.0000314152	3.6692966680	3.6692395960	0.0000570720
0.4	0.5488116361	0.5488429470	0.0000313109	4.0551999670	4.0551446400	0.0000553270
0.5	0.6065306597	0.6065557630	0.0000251033	4.4816890700	4.4815847720	0.0001042980
0.6	0.6703200460	0.6703373400	0.0000172940	4.9530324240	4.9528775910	0.0001548330
0.7	0.7408182207	0.7408278480	0.0000096273	5.4739473920	5.4737949060	0.0001544860
0.8	0.8187307531	0.8187254460	0.0000053071	6.0496474640	6.0495527220	0.0000947420
0.9	0.9048374180	0.9047862850	0.0000511330	6.6858944420	6.6858312490	0.0000631930
1.0	1.0000000000	0.9998245120	0.0001754880	7.3890560990	7.3887548920	0.0003012070

Example2.

Consider the following coupled Burgers' Equation

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + 2\,u\frac{\partial u}{\partial x} - \frac{\partial (uv)}{\partial x},\\ \frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2} + 2\,v\frac{\partial v}{\partial x} - \frac{\partial (uv)}{\partial x},\end{aligned} \tag{32}$$

subject to the initial conditions:

$$u(x,0) = v(x,0) = \sin(x),$$

and the boundary conditions:

$$\begin{aligned}&u(0,t) = v(0,t) = 0,\\ &\frac{\partial u}{\partial x}(0,t) = \frac{\partial v}{\partial x}(0,t) = e^{-t}.\end{aligned}$$

The exact solution of the equation is $u(x,t) = v(x,t) = e^{-t}\sin x$. We solve the system (32) by the proposed method with $k = k' = 1$ and $M = M' = 5$. The vectors C_1 and C_2 are computed by solving the system of nonlinear equations, via the Maple package, as follows:

$$\begin{aligned}C_1 = [&0.2905703766, -0.08254037472, 0.01055648322, -0.0009064946308,\\ &0.00004868812793, 0.1561591494, -0.04436673862, 0.005668369680,\\ &-0.0004888574038, 0.00002822902943, -0.01109904289,\\ &0.003142878545, -0.000410514098, 0.000030038563324,\\ &-0.00000304325305, -0.001724343752, 0.0004878243884,\end{aligned}$$

$$
\begin{aligned}
&-0.00006409071239, 0.000004956342306, -4.358683355\times 10^{-7},\\
&\ 0.00005940327606, -0.00001722231480, 0.000002690225550,\\
&-3.214003183\times 10^{-7}, 0.000001050671712]^T,
\end{aligned}
$$

and

$$
\begin{aligned}
C_2 = [&0.2905703901, -0.08254034380, 0.01055653654, -0.0009064344718,\\
&0.00004896320923, 0.1561591931, -0.04436665650,\\
&0.005668486521, -0.0004886662476, 0.00002846756258,\\
&-0.01109900086, 0.003142952129, -0.0004104191521,\\
&0.00003015241866, -0.000002937610834, -0.001724327839,\\
&0.0004878507042, -0.00006405993087, 0.000004982461017,\\
&-4.244669350\times 10^{-7}, 0.00005940795971, -0.00001721487859,\\
&0.000002697983945, -3.154732603\times 10^{-7}, 0.000001053245678]^T.
\end{aligned}
$$

Therefore, the following solutions will result:

$$
\begin{aligned}
u(x,t) &\approx \left(C_1^T P_t\, P_x^2 + U_2^T\right)\Psi(x,t)\\
&= (0.0004477491972\, t^4 - 0.002945088068\, t^3 + 0.009722265939\, t^2\\
&\quad - 0.01968234776\, t + 0.01973931200)\, x^4 + (-0.004668887903\, t^4\\
&\quad + 0.02818404632\, t^3 - 0.09095738190\, t^2 + 0.1835235781\, t - 0.1837216673)\, x^3\\
&\quad + (0.0001089557110\, t^4 - 0.0008677954496\, t^3 + 0.00303449323\, t^2\\
&\quad - 0.006215940927\, t + 0.006286067622)\, x^2 + (0.02543961817 t^4\\
&\quad - 0.1532928479\, t^3 + 0.4947778201\, t^2 - 0.9984520963\, t\\
&\quad + 0.9990899799)\, x + 0.00008012128465\, t^4 - 0.0001611141175\, t^3\\
&\quad + 0.0001131072956\, t^2 - 0.00005000508800 t + 0.00003037762580,
\end{aligned}
$$

and

$$
\begin{aligned}
v(x,t) &\approx \left(C_2^T P_t\, P_x^2 + V_2^T\right)\Psi(x,t)\\
&= (0.0004465460679\, t^4 - 0.002943337345\, t^3 + 0.009721497859\, t^2\\
&\quad - 0.01968224282\, t + 0.01973930932)\, x^4 + (-0.004668531669\, t^4
\end{aligned}
$$

$$
\begin{aligned}
&+ 0.02818376233\,t^3 - 0.09095738743\,t^2 + 0.1835236021\,t - 0.1837216688)\,x^3 \\
&+ (0.000107920876\,t^4 - 0.0008659152775\,t^3 + 0.003033375132\,t^2 \\
&- 0.006215708200t + 0.006286056882)\,x^2 + (0.025423964414t^4 \\
&- 0.1532928689\,t^3 + 0.4947778196\,t^2 - 0.9984520944\,t + 0.9990899797)\,x \\
&+ 0.00008012032947\,t^4 - 0.0001611133215\,t^3 + 0.0001131072845\,t^2 \\
&- 0.00005000515467t + 0.00003037757586.
\end{aligned}
$$

Figure 3 and Figure 4 show the numerical solution for Equation (30) obtained by two-dimensional Legendre wavelets method, for $x, t \in [0,1]$.

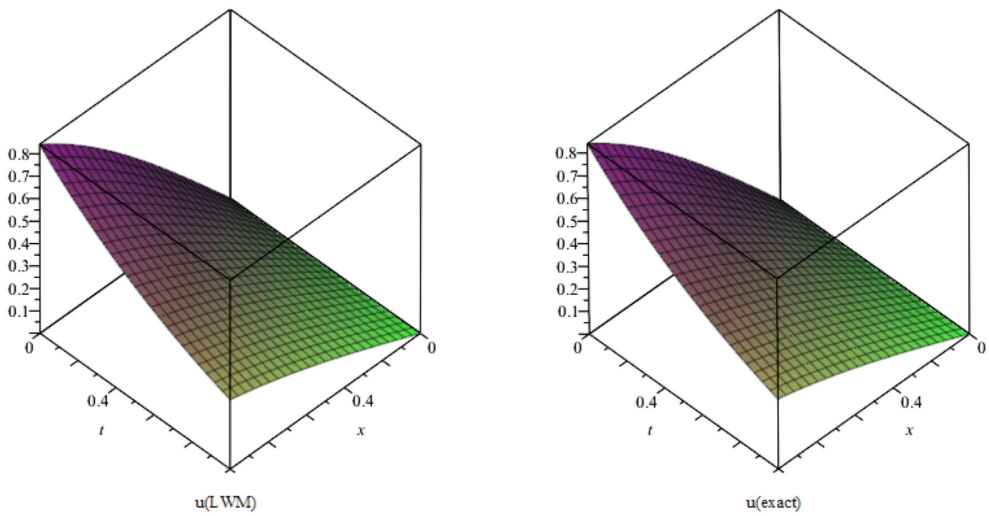

Figure 3. Exact and LWM solution $u(x,t)$ of example 1

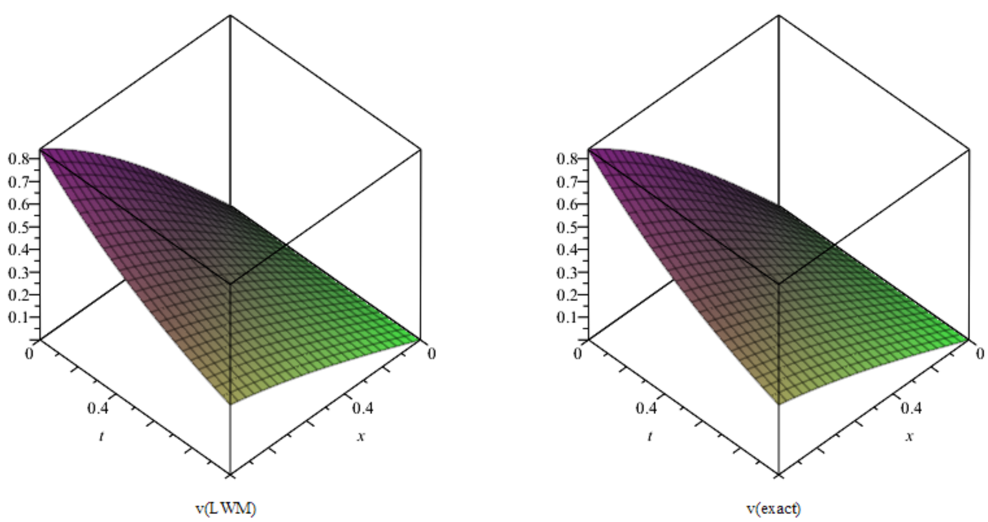

Figure 4. Exact and LWM solution $v(x,t)$ of Example 2

In Tables 2.1-2.5, we show the Comparisons between numerical and analytical solutions of Equation (32) in $t = 0, t = 0.25, t = 0.5, t = 0.75$ and $t = 1$, for various values of x.

Table 2.1. Numerical results of example2 for $t = 0$

x	u_{exact}	u_{LWM}	$\lvert u_{exact} - u_{LWM}\rvert$	v_{exact}	v_{LWM}	$\lvert v_{exact} - v_{LWM}\rvert$
0.0	0.0000000000	0.0000303776	0.0000303776	0.0000000000	0.0000303776	0.0000303776
0.1	0.0998334166	0.0998204886	0.0000129281	0.0998334166	0.0998204884	0.0000129283
0.2	0.1986693308	0.1986616259	0.0000077049	0.1986693308	0.1986616253	0.0000077055
0.3	0.2955202067	0.2955225211	0.0000023144	0.2955202067	0.2955225200	0.0000023133
0.4	0.3894183423	0.3894192801	0.0000009378	0.3894183423	0.3894192781	0.0000009358
0.5	0.4794255386	0.4794153831	0.0000101555	0.4794255386	0.4794153798	0.0000101588
0.6	0.5646424734	0.5646216845	0.0000207889	0.5646424734	0.5646216799	0.0000207935
0.7	0.6442176872	0.6441964136	0.0000212736	0.6442176872	0.6441964136	0.0000212802
0.8	0.7173560909	0.7173451733	0.0000109176	0.7173560909	0.7173451645	0.0000109264
0.9	0.7833269096	0.7833209414	0.0000059682	0.7833269096	0.7833209296	0.0000059800
1.0	0.8414709848	0.8414240698	0.0000469150	0.8414709848	0.8414240547	0.0000469301

Table 2.2. Numerical results of example2 for $t = 0.25$

x	u_{exact}	u_{LWM}	$\lvert u_{exact} - u_{LWM}\rvert$	v_{exact}	v_{LWM}	$\lvert v_{exact} - v_{LWM}\rvert$
0.0	0.0000000000	0.0000227411	0.0000227411	0.0000000000	0.0000227411	0.0000227411
0.1	0.0777503431	0.0777407361	0.0000096070	0.0777503431	0.0777407361	0.0000096070
0.2	0.1547238304	0.1547198174	0.0000040130	0.1547238304	0.1547198174	0.0000040130
0.3	0.2301513684	0.2301567395	0.0000053711	0.2301513684	0.2301567398	0.0000053714
0.4	0.3032793099	0.3032851744	0.0000058645	0.3032793099	0.3032851748	0.0000058649
0.5	0.3733769849	0.3733757107	0.0000012742	0.3733769849	0.3733757113	0.0000012736
0.6	0.4297440005	0.4397358543	0.0000081462	0.4297440005	0.4397358553	0.0000081452
0.7	0.5017172393	0.5017100282	0.0000072111	0.5017172393	0.5017100296	0.0000072097
0.8	0.5586774854	0.5586795722	0.0000020868	0.5586774854	0.5586795740	0.0000020886
0.9	0.6100556106	0.6100627432	0.0000071326	0.6100556106	0.6100627452	0.0000071346
1.0	0.6553382619	0.6553147154	0.0000235465	0.6553382619	0.6553147177	0.0000235442

Table 2.3. Numerical results of example2 for $t = 0.5$

x	u_{exact}	u_{LWM}	$\lvert u_{exact} - u_{LWM}\rvert$	v_{exact}	v_{LWM}	$\lvert v_{exact} - v_{LWM}\rvert$
0.0	0.0000000000	0.0000185202	0.0000185202	0.0000000000	0.0000185202	0.0000185202
0.1	0.0605520281	0.0605452773	0.0000067508	0.0000000000	0.0605452772	0.0000067509
0.2	0.1204990403	0.1204967113	0.0000023290	0.1204990403	0.1204967110	0.0000023293
0.3	0.1792420659	0.1792471729	0.0000051070	0.1792420659	0.1792471725	0.0000051066
0.4	0.2361941641	0.2361997854	0.0000056213	0.2361941641	0.2361997843	0.0000056202
0.5	0.2907862882	0.2907864443	0.0000001561	0.2907862882	0.2907864427	0.0000001545
0.6	0.3424729719	0.3424678177	0.0000051542	0.3424729719	0.3424678156	0.0000051563
0.7	0.3907377788	0.3907333469	0.0000044319	0.3907377788	0.3907333438	0.0000044350
0.8	0.4350984631	0.4351012447	0.0000027816	0.4350984631	0.4351012407	0.0000027776
0.9	0.4751117872	0.4751184970	0.0000067098	0.4751117872	0.4751184917	0.0000067045
1.0	0.5103779515	0.5103608625	0.0000170890	0.5103779515	0.5103608560	0.0000170955

Table 2.4. Numerical results of example2 for $t = 0.75$

x	u_{exact}	u_{LWM}	$\lvert u_{exact} - u_{LWM}\rvert$	v_{exact}	v_{LWM}	$\lvert v_{exact} - v_{LWM}\rvert$
0.0	0.0000000000	0.0000138775	0.0000138775	0.0000000000	0.0000138774	0.0000138775
0.1	0.0471579669	0.0471522029	0.0000057640	0.0471579669	0.0471522028	0.0000057640
0.2	0.0938447469	0.0938426220	0.0000021249	0.0938447469	0.0938426220	0.0000021249
0.3	0.1395938613	0.1395978009	0.0000039396	0.1395938613	0.1395978012	0.0000039399
0.4	0.1839481999	0.1839528350	0.0000046351	0.1839481999	0.1839528355	0.0000046356
0.5	0.2264645889	0.2264652492	0.0000006603	0.2264645889	0.2264652504	0.0000006615
0.6	0.2667182187	0.2667149974	0.0000032213	0.2667182187	0.2667149994	0.0000032193
0.7	0.3043068881	0.3043044631	0.0000024250	0.3043068881	0.3043044660	0.0000024221
0.8	0.3388550237	0.3388584585	0.0000034348	0.3388550237	0.3388584631	0.0000034394
0.9	0.3700174319	0.3700242256	0.0000067937	0.3700174319	0.3700242321	0.0000068002
1.0	0.3974827483	0.3974714360	0.0000113123	0.3974827483	0.3974714453	0.0000113030

Table 2.5. Numerical results of example2 for $t = 1$

x	u_{exact}	u_{LWM}	$\lvert u_{exact} - u_{LWM}\rvert$	v_{exact}	v_{LWM}	$\lvert v_{exact} - v_{LWM}\rvert$
0.0	0.0000000000	0.0000124870	0.0000124870	0.0000000000	0.0000124867	0.0000124870
0.1	0.0367266615	0.0367252801	0.0000013814	0.0367266615	0.0367252800	0.0000013814
0.2	0.0730863624	0.0730893414	0.0000029790	0.0730863624	0.0730893410	0.0000029786
0.3	0.1087158085	0.1087250443	0.0000092358	0.1087158085	0.1087250429	0.0000092344
0.4	0.1432590022	0.1432702379	0.0000112375	0.1432590022	0.1432702344	0.0000112322
0.5	0.1763707992	0.1763802482	0.0000094490	0.1763707992	0.1763802421	0.0000094429
0.6	0.2077203576	0.2077278778	0.0000075202	0.2077203576	0.2077278668	0.0000075092
0.7	0.2369944428	0.2370034060	0.0000089632	0.2369944428	0.2370033872	0.000008944
0.8	0.2639005579	0.2639145881	0.0000140302	0.2639005579	0.2639145577	0.0000139998
0.9	0.2881698658	0.2881866563	0.0000167905	0.2881698658	0.2881866080	0.0000167422
1.0	0.3095598757	0.3095623199	0.0000024442	0.3095598757	0.3095622463	0.0000023706

6. Conclusion

The aim of this paper has been to develop two-dimensional Legendre wavelets for obtaining the solutions of systems of variable-coefficient coupled Burgers' equation. The illustrative examples included demonstrate that we have achieved a method is a very effective and useful technique for finding approximate solutions of these systems. The method is fully described possible error and analyzed. The two-dimensional operational matrices of integration are used to find the solution of the system of variable-coefficient coupled Burgers' equation. In the present method, the problem under study reduces to a system of linear or nonlinear algebraic equations. The two examples presented illustrate the capability and simplicity of the method and the close comparison of the obtained results with those of the exact solutions shows that the proposed method is a highly promising method for various classes of both linear and nonlinear systems of partial differential equations. Here, the computations associated with these examples are performed by the package Maple 13.

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