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# Some Geometric Properties of a New Type Metric Space

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## Abstract

In this paper, we define a metric on our new space and then show that this linear metric space is k-nearly uniform convex and has property beta where $p=(p_k)$ is a bounded sequence of positive real numbers. Finally, we give a result about property $(H)$ by using k-nearly uniform convexity.

**Keywords:** Cesàro difference sequence space, Luxemburg norm, extreme point, convex modular, property (*H*)

**AMS-MSC 2010 No:** 40A05, 46A45, 46B20

## 1. Introduction

By a Lacunary sequence $\theta=(k_r)$ where $k_0=0$, we'll mean an increasing sequence of nonnegative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by $\theta$ will be denoted by $I_r = (k_{r-1}, k_r]$. We write $h_r = k_r - k_{r-1}$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by $q_r$. Freedman (1978) defined the space of lacunary strongly convergent sequences as follows:

$$N_\theta = \left\{ x = (x_k) \; : \; \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |x_k - l| = 0, \text{for some } l \right\}.$$

It is well known that there exists a very close connection between the space of lacunary strongly convergent sequences and the space of strongly Cesaro summable sequences. This connection can be found in Das and Patel (1989), Mursaleen and Chishti (1996). Because of this, a lot of geometric properties of Cesaro sequence spaces can be generalized to the lacunary sequence spaces.

Let $\omega$ be the space of all real sequences. Let $p = (p_r)$ be a bounded sequence of positive real numbers. Karakaya (2007) defined the space $\ell(p,\theta)$ as follows:

$$\ell(p,\theta) = \left\{ x = (x_k) \; : \; \sum_{r=1}^{\infty} \left( \frac{1}{h_r} \sum_{k\in I_r} |x_k| \right)^{p_r} < \infty \right\}.$$

The paranorm on $\ell(p,\theta)$ is given by

$$\|x\|_{\ell(p,\theta)} = \left( \sum_{r=1}^{\infty} \left( \frac{1}{h_r} \sum_{k\in I_r} |x_k| \right)^{p_r} \right)^{1/H},$$

where $H = \sup p_r$. If $p_r = p$ for all $r$, we use the notation $\ell_p(\theta)$ in place of $\ell(p,\theta)$. The norm on $\ell_p(\theta)$ is given by

$$\|x\|_{\ell_p(\theta)} = \left( \sum_{r=1}^{\infty} \left( \frac{1}{h_r} \sum_{k\in I_r} |x_k| \right)^{p} \right)^{1/p}.$$

If $\theta = (2^r)$, then $\ell(p,\theta) = ces(p)$. If $\theta = (2^r)$ and $p_r = p$ for all $r$, then $\ell(p,\theta) = ces_p$.

For $x \in \ell(p,\theta)$, let

$$\sigma(x) = \sum_{r=1}^{\infty} \left( \frac{1}{h_r} \sum_{k\in I_r} |x_k| \right)^{p_r}$$

and define the Luxemburg norm on $\ell(p,\theta)$ by $\|x\|_L = \inf\left\{\rho > 0 \ : \ \sigma\left(\frac{x}{\rho}\right) \le 1\right\}$. The Luxemburg norm on $\ell_p(\theta)$ can be reduced to the usual norm on $\ell_p(\theta)$, that is, $\|x\|_L = \|x\|_{\ell_p(\theta)}$.

Ahuja et al. (1977) introduced the notions of strict convexity and U.C.I (uniform convexity) in linear metric spaces which are generalizations of the corresponding concepts in linear normed spaces. Later, Sastry and Naidu (1979) introduced the notions of U.C.II and U.C.III in linear metric spaces and showed that these three forms are not always equivalent. Further, Junde and Chen (1994), Junde et al. (1995) showed that if a linear metric space is complete and U.C.I , then it is reflexive.

Let $X$ be a vector space over the scalar field of real numbers and $d$ be an invariant metric on $X$. We denote $B_d(X)$ and $S_d(X)$ as follows:

$$B_d(X) = \{x \in X \ : \ d(x,\mathbf{0}) \le r\} \text{ and } S_d(X) = \{x \in X \ : \ d(x,\mathbf{0}) = r\}.$$

Let $(X,d)$ be a linear metric space and $B_d(X)$( resp., $S_d(X)$) be the closed unit ball (resp., the unit sphere) of $X$. A linear metric space $(X,d)$ has property $(\beta)$ if and only if for each $r > 0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for each element $x \in B_d(0,r)$ and each sequence $(x_n)$ in $B_d(0,r)$ with $sep(x_n) \ge \varepsilon$, there is an index $k$ for which

$$d\left(\frac{x + x_k}{2}, \mathbf{0}\right) \le 1 - \delta ,$$

where $sep(x_n) = \inf\{d(x_n, x_m) \ : \ n \ne m\} > \varepsilon$, Sanhan and Mongkolkeha (2011). If for each $x \in S_d(0,r)$ and $(x_n) \subset S_d(0,r)$, $x_n \xrightarrow{w} x$ implies $x_n \to x$, a linear metric space $(X,d)$ is said to have property $(H)$.

Let $k \ge 2$ be an integer. A linear metric space $(X,d)$ is said to be k-nearly uniform convex (k-NUC) if for every $\varepsilon > 0$ and $r > 0$, there exists $\delta > 0$ such that for any sequence $(x_n) \subset B_d(0,r)$ with $sep(x_n) \ge \varepsilon$, there are $s_1, s_2, ..., s_k$ such that

$$d\left(\frac{x_{s_1} + x_{s_2} + ... + x_{s_k}}{k}, \mathbf{0}\right) \le r - \delta ,$$

Junde and Narang (2000).

## 2. Main Results

In this section, our goal is to define a metric on $\ell(p,\theta)$ and show that $\ell(p,\theta)$ has property $(\beta)$ and $k-NUC$ under the metric. Let $\omega$ be the space of all real sequences and $p=(p_r)$ be a bounded sequence of real numbers with $p_r > 1$ for all $r \in \mathrm{N}$. The mapping

$$d(x,y)=\left(\sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}|x(k)-y(k)|\right)^{p_r}\right)^{1/H}$$

is a metric on the space $\ell(p,\theta)$, where $H=\sup p_r$, since the function $|t|^p$ is convex for $p>1$. First, we will show that the space $\ell(p,\theta)$ has property $(\beta)$ under the above metric. To do this, we need the following two lemmas.

**Lemma 2.1.**

Let $y,z\in(\ell(p,\theta),d)$. If $\beta\in(0,1)$, then

$$(d(y+z,\mathbf{0}))^M \le (d(y,\mathbf{0}))^M + 2^M\beta(d(y,\mathbf{0}))^M + \frac{2^M}{\beta^{M-1}}(d(z,\mathbf{0}))^M .$$

***Proof:***

Let $y,z\in(\ell(p,\theta),d)$ and $0<\beta<1$. Then,

$$\begin{aligned}(d(y+z,\mathbf{0}))^M &= \sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}|y(k)+z(k)|\right)^{p_r}\\
&\le \sum_{r=1}^{\infty}\left((1-\beta)\frac{1}{h_r}\sum_{k\in I_r}|y(k)|+\beta\frac{1}{h_r}\sum_{k\in I_r}\left|y(k)+\frac{z(k)}{\beta}\right|\right)^{p_r}\\
&\le (1-\beta)\sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}|y(k)|\right)^{p_r}+\beta\sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}\left|y(k)+\frac{z(k)}{\beta}\right|\right)^{p_r}\\
&\le \sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}|y(k)|\right)^{p_r}+2^M\beta\sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}|y(k)|\right)^{p_r}\\
&\quad +2^M\sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}\left|\frac{z(k)}{\beta}\right|\right)^{p_r}\end{aligned}$$

$$\leq \sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}|y(k)|\right)^{p_r} + 2^M\beta\sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}|y(k)|\right)^{p_r}$$

$$+\frac{2^M}{\beta^{M-1}}\sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}|z(k)|\right)^{p_r}$$

$$=(d(y,\mathbf{0}))^M + 2^M\beta(d(y,\mathbf{0}))^M + \frac{2^M}{\beta^{M-1}}(d(z,\mathbf{0}))^M.$$

**Lemma 2.2.**

Let $y,z\in(\ell(p,\theta),d)$. Then, for any $\varepsilon>0$ and $L>0$ there exists $\delta>0$ such that

$$\left|(d(y+z,\mathbf{0}))^M-(d(y,\mathbf{0}))^M\right|<\varepsilon,$$

where

$$(d(y,\mathbf{0}))^M\leq L \quad\text{and}\quad (d(z,\mathbf{0}))^M\leq\delta.$$

***Proof:***

Let $\varepsilon>0$ and $L>0$. For $\beta=\dfrac{\varepsilon}{2^{M+1}(L+\varepsilon)}$, we take $\delta=\dfrac{\varepsilon\beta^{M-1}}{2^{M+1}}$. By using previous Lemma 2.1, we have

$$\begin{aligned}(d(y+z,\mathbf{0}))^M &\leq (d(y,\mathbf{0}))^M + 2^M\beta(d(y,\mathbf{0}))^M+\frac{2^M}{\beta^{M-1}}(d(z,\mathbf{0}))^M\\ &\leq (d(y,\mathbf{0}))^M + 2^M\beta L+\frac{2^M}{\beta^{M-1}}\delta\\ &\leq (d(y,\mathbf{0}))^M + 2^M\frac{\varepsilon}{2^{M+1}}\frac{L}{L+\varepsilon}+\frac{2^M}{\beta^{M-1}}\frac{\varepsilon\beta^{M-1}}{2^{M+1}} \qquad (2.1)\\ &\leq (d(y,\mathbf{0}))^M+\frac{\varepsilon}{2}+\frac{\varepsilon}{2}\\ &\leq (d(y,\mathbf{0}))^M+\varepsilon\end{aligned}$$

and

$$
\begin{aligned}
(d(y,\mathbf{0}))^M &\le (d(y+z,\mathbf{0}))^M + 2^M \beta (d(y+z,\mathbf{0}))^M + \frac{2^M}{\beta^{M-1}}(d(-z,\mathbf{0}))^M \\
&\le (d(y+z,\mathbf{0}))^M + 2^M \beta\left((d(y,\mathbf{0}))^M + \varepsilon\right) + \frac{2^M}{\beta^{M-1}}\delta \\
&\le (d(y+z,\mathbf{0}))^M + 2^M \beta(L+\varepsilon) + \frac{2^M}{\beta^{M-1}}\frac{\varepsilon\beta^{M-1}}{2^{M+1}} \\
&= (d(y+z,\mathbf{0}))^M + 2^M \frac{\varepsilon}{2^{M+1}(L+\varepsilon)}(L+\varepsilon) + \frac{\varepsilon}{2} \\
&= (d(y+z,\mathbf{0}))^M + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= (d(y+z,\mathbf{0}))^M + \varepsilon
\end{aligned}
\tag{2.2}
$$

From $(2.1)$ and $(2.2)$, we obtain that $\left|(d(y+z,\mathbf{0}))^M - (d(y,\mathbf{0}))^M\right| < \varepsilon.$

**Theorem 2.3.**

The space $(\ell(p,\theta),d)$ has property $(\beta)$.

***Proof:***

Let $\varepsilon > 0$ and $(x_n) \subset B(\ell(p,\theta),d)$ such that $sep(x_n) \ge \varepsilon$ and $x \in B(\ell(p,\theta),d)$. We take

$$
y^N = \left(0,0,\dots,0,\sum_{k=1}^{N} y(k), y(N+1), y(N+2),\dots\right).
$$

By using diagonal method, we can find a subsequence $(x_{n_j})$ of $(x_n)$ for each $N \in \mathbb{N}$ such that $(x_{n_j}(k))$ converges for each $k \in \mathbb{N}$ with $1 \le k \le N$, since $(x_n(k))_{k=1}^{\infty}$ is bounded for each $k \in \mathbb{N}$. Therefore, there is $r_N \in \mathbb{N}$ for each $N \in \mathbb{N}$ such that $sep\left((x_n^N)_{j>r_N}\right) \ge \varepsilon$. So, there is a sequence of positive integers $(r_N)_{n=1}^{\infty}$ with $r_1 < r_2 < r_3 \dots$ such that $d\left(x_{r_N}^N, \mathbf{0}\right) \ge \frac{\varepsilon}{2}$ for all $N \in \mathbb{N}$. Then, there exists $\kappa > 0$ such that for all $N \in \mathbb{N}$,

$$
\sum_{r=N}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}\left|x_{r_N}\right|\right)^{p_r} \ge \kappa . \tag{2.3}
$$

By Lemma 2.2, there exists $\delta_0$ such that

$$\left|(d(y+z,\mathbf{0}))^M - (d(y,\mathbf{0}))^M\right| < \frac{\kappa}{2^m}, \tag{2.4}$$

where

$$(d(y,\mathbf{0}))^M < r^M \text{ and } (d(z,\mathbf{0}))^M \leq \delta_0.$$

There exists $N_1 \in \mathrm{N}$ such that $\left(d\left(x^{N_1},\mathbf{0}\right)\right)^M \leq \delta_0$ if $x \in B(\ell(p,\theta))$ and $(d(x,\mathbf{0}))^M \leq \delta_0$. Let us take $y = x_{r_{N_1}}^{N_1}$ and $z = x^{N_1}$. Hence, we have

$$\sum_{r=N_1}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}\left|\frac{x(k)+x_{r_{N_1}}(k)}{2}\right|\right)^{p_r} \leq \sum_{r=N_1}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}\left|\frac{x_{r_{N_1}}(k)}{2}\right|\right)^{p_r} + \frac{\kappa}{2^M}. \tag{2.5}$$

From $(2.3),(2.4),(2.5)$ and by using convexity of the function $f(t) = |t|^{p_r}$, for all $r \in \mathrm{N}$, we obtain that

$$\begin{aligned}
\left(d\left(\frac{y+z}{2},\mathbf{0}\right)\right)^M &= \sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}\left|\frac{x(k)+x_{r_{N_1}}(k)}{2}\right|\right)^{p_r} \\
&= \sum_{r=1}^{N_1-1}\left(\frac{1}{h_r}\sum_{k\in I_r}\left|\frac{x(k)+x_{r_{N_1}}(k)}{2}\right|\right)^{p_r} + \sum_{r=N_1}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}\left|\frac{x(k)+x_{r_{N_1}}(k)}{2}\right|\right) \\
&\leq \sum_{r=1}^{N_1-1}\left(\frac{1}{h_r}\sum_{k\in I_r}\left|\frac{x(k)+x_{r_{N_1}}(k)}{2}\right|\right)^{p_r} + \sum_{r=N_1}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}\left|\frac{x_{r_{N_1}}(k)}{2}\right|\right)^{p_r} + \frac{\kappa}{2^M} \\
&\leq \frac{1}{2}\sum_{r=1}^{N_1-1}\left(\frac{1}{h_r}\sum_{k\in I_r}|x(k)|\right)^{p_r} + \frac{1}{2}\sum_{r=1}^{N_1-1}\left(\frac{1}{h_r}\sum_{k\in I_r}\left|x_{r_{N_1}}(k)\right|\right)^{p_r} \\
&\quad + \frac{1}{2^M}\sum_{r=N_1}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}\left|x_{r_{N_1}}(k)\right|\right)^{p_r} + \frac{\kappa}{2^M}
\end{aligned}$$

$$\leq \frac{1}{2}\sum_{r=1}^{N_1-1}\left(\frac{1}{h_r}\sum_{k\in I_r}|x(k)|\right)^{p_r} + \frac{1}{2}\sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}\left|x_{r_{N_1}}(k)\right|\right)^{p_r}$$

$$-\frac{2^M-2}{2^{M+1}}\sum_{r=N_1}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}\left|x_{r_{N_1}}(k)\right|\right)^{p_r} + \frac{\kappa}{2^M}$$

$$< \frac{r^M}{2} + \frac{r^M}{2} - \frac{2^M-2}{2^{M+1}}\kappa + \frac{\kappa}{2^M}$$

$$= r^M - \frac{\kappa}{2}.$$

Therefore, we have $d\left(\frac{y+z}{2},\mathbf{0}\right) < \left(r^M - \frac{\kappa}{2}\right)^{1/M}$ whenever $\delta \in \left(0, r - \left(r^M - \frac{\kappa}{2}\right)^{1/M}\right)$. Consequently, the space $(\ell(p,\theta), d)$ posses property $(\beta)$.

Now, we'll examine $k-NUC$ property of the space $(\ell(p,\theta), d)$.

**Theorem 2.4.**

The space $\ell(p,\theta)$ is $k-NUC$ for any integer $k \geq 2$.

***Proof:***

Let $\varepsilon > 0$ and $(x_n) \subset B_d(\ell(p,\theta))$ with $sep(x_n) \geq \varepsilon$. For each $m \in \mathrm{N}$, let

$$x_n^m = (0, 0, ..., x_n(m), x_n(m+1), ...). \tag{2.6}$$

Since the sequence $(x_n(i))_{i=1}^{\infty}$ is bounded for each $i \in \mathrm{N}$, by using diagonal method, we can find a subsequence $(x_{n_j})$ of $(x_n)$ such that $(x_{n_j}(k))$ converges for each $k \in \mathrm{N}$. Therefore, there is an increasing sequence $t_m$ with $sep\left(\left(x_{n_j}^m\right)_{j>t_m}\right) \geq \varepsilon$. Hence, there exists a sequence of positive integers $(r_m)_{m=1}^{\infty}$ with $r_1 < r_2 < r_3 < ...$ such that $d\left(x_{r_m}^m, \mathbf{0}\right) \geq \frac{\varepsilon}{2}$, for all $m \in \mathrm{N}$. Then, there is $\zeta > 0$ such that

$$\sum_{r=m}^{\infty}\left(\frac{1}{h_r}\sum_{i\in I_r}\left|x_{r_m}\right|\right)^{p_r} \geq \zeta. \tag{2.7}$$

Let $\alpha > 0$ such that $1 < \alpha < \liminf_{r\to\infty} p_r$ . Let $\varepsilon_1 = \frac{k^{\alpha-1}-1}{(k-1)k^{\alpha}}\frac{\zeta}{2}$ for $k \geq 2$. From Lemma 2.2, there is a $\delta > 0$ such that

$$\left|(d(y+z,\mathbf{0}))^M - (d(y,\mathbf{0}))^M\right| < \varepsilon_1 \ , \tag{2.8}$$

where $(d(y,\mathbf{0}))^M < r^M$ and $(d(z,\mathbf{0}))^M \leq \delta$. Then, there exist positive integers $m_i\,(i = 1,2,..,k-1)$ with $m_1 < m_2 < \ldots < m_{k-1}$ such that $d\left(x_i^{m_i},\mathbf{0}\right) \leq \delta$. Now, define $m_k = m_{k-1}+1$. Then, we have $d\left(x_{r_{m_k}}^{m_k},\mathbf{0}\right) \geq \zeta$ for all $m \in \mathrm{N}$. For $1 \leq i \leq k-1$, let $t_i = i$ and $t_k = r_{m_k}$. By using $(2.6),(2.7),(2.8)$ and the convexity of function $f_i(u) = |u|^{p_i}\,(i \in \mathrm{N})$, we obtain

$$\begin{aligned}
\left(d\left(\frac{x_{s_1}+x_{s_2}+\ldots+x_{s_k}}{k},\mathbf{0}\right)\right)^M &= \sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{i\in I_r}\left|\frac{x_{s_1}(i)+\ldots+x_{s_k}(i)}{k}\right|\right)^{p_r} \\
&= \sum_{r=1}^{m_1}\left(\frac{1}{h_r}\sum_{i\in I_r}\left|\frac{x_{s_1}(i)+\ldots+x_{s_k}(i)}{k}\right|\right)^{p_r} + \sum_{r=m_1+1}^{\infty}\left(\frac{1}{h_r}\sum_{i\in I_r}\left|\frac{x_{s_1}(i)+\ldots+x_{s_k}(i)}{k}\right|\right)^{p_r} \\
&\leq \sum_{r=1}^{m_1}\left(\frac{1}{h_r}\sum_{i\in I_r}\left|\frac{x_{s_1}(i)+\ldots+x_{s_k}(i)}{k}\right|\right)^{p_r} \\
&\quad + \sum_{r=m_1+1}^{\infty}\left(\frac{1}{h_r}\sum_{i\in I_r}\left|\frac{x_{s_2}(i)+\ldots+x_{s_k}(i)}{k}\right|\right)^{p_r} + \varepsilon_1 \\
&\leq \sum_{r=1}^{m_1}\frac{1}{k}\sum_{j=1}^{k}\left(\frac{1}{h_r}\sum_{i\in I_r}\left|x_{s_j}(i)\right|\right)^{p_r} + \sum_{r=m_1+1}^{m_2}\left(\frac{1}{h_r}\sum_{i\in I_r}\left|\frac{x_{s_2}(i)+x_{s_3}(i)+\ldots+x_{s_k}(i)}{k}\right|\right)^{p_r} \\
&\quad + \sum_{r=m_2+1}^{\infty}\left(\frac{1}{h_r}\sum_{i\in I_r}\left|\frac{x_{s_3}(i)+x_{s_4}(i)+\ldots+x_{s_k}(i)}{k}\right|\right)^{p_r} + 2\varepsilon_1 \\
&\leq \sum_{r=1}^{m_1}\frac{1}{k}\sum_{j=1}^{k}\left(\frac{1}{h_r}\sum_{i\in I_r}\left|x_{s_j}(i)\right|\right)^{p_r} + \sum_{r=m_1+1}^{m_2}\frac{1}{k}\sum_{j=2}^{k}\left(\frac{1}{h_r}\sum_{i\in I_r}\left|x_{s_j}(i)\right|\right)^{p_r} \\
&\quad + \sum_{r=m_2+1}^{m_3}\frac{1}{k}\sum_{j=3}^{k}\left(\frac{1}{h_r}\sum_{i\in I_r}\left|x_{s_j}(i)\right|\right)^{p_r} \\
&\quad + \ldots + \sum_{r=m_{k-1}+1}^{m_k}\frac{1}{k}\sum_{j=k-1}^{k}\left(\frac{1}{h_r}\sum_{i\in I_r}\left|x_{s_j}(i)\right|\right)^{p_r} + \sum_{r=m_k+1}^{\infty}\left(\frac{1}{h_r}\sum_{i\in I_r}\left|\frac{x_{s_k}(i)}{k}\right|\right)^{p_r} + (k-1)\varepsilon_1
\end{aligned}$$

$$\leq \left( \frac{\left(d\left(x_{s_1},\theta\right)^M + d\left(x_{s_2},\theta\right)^M + \ldots + d\left(x_{s_k},\theta\right)^M\right)}{k} \right.$$

$$\left. + \frac{1}{k}\sum_{r=1}^{m_k}\left(\frac{1}{h_r}\sum_{i\in I_r}\left|x_{s_k}(i)\right|\right)^{p_r} + \sum_{r=m_k+1}^{\infty}\left(\frac{1}{h_r}\sum_{i\in I_r}\left|\frac{x_{s_k}(i)}{k}\right|\right)^{p_r} \right) + (k-1)\varepsilon_1$$

$$\leq \frac{k-1}{k}r^M + \frac{1}{k}\sum_{r=1}^{m_k}\left(\frac{1}{h_r}\sum_{i\in I_r}\left|x_{s_k}(i)\right|\right)^{p_r} + \frac{1}{k^\alpha}\sum_{r=m_k+1}^{\infty}\left(\frac{1}{h_r}\sum_{i\in I_r}\left|\frac{x_{s_k}(i)}{k}\right|\right)^{p_r} + (k-1)\varepsilon_1$$

$$\leq r^M - \frac{r^M}{k} + \frac{1}{k}\left(r^M - \sum_{r=m_k+1}^{\infty}\left(\frac{1}{h_r}\sum_{i\in I_r}\left|x_{s_k}(i)\right|\right)^{p_r}\right)$$

$$+ \frac{1}{k^\alpha}\sum_{r=m_k+1}^{\infty}\left(\frac{1}{h_r}\sum_{i\in I_r}\left|\frac{x_{s_k}(i)}{k}\right|\right)^{p_r} + (k-1)\varepsilon_1$$

$$\leq r^M + (k-1)\varepsilon_1 - \left(\frac{k^{\alpha-1}-1}{k^\alpha}\right)\zeta$$

$$\leq r^M + (k-1)\frac{k^{\alpha-1}-1}{k^\alpha(k-1)}\left(\frac{\zeta}{2}\right) - \left(\frac{k^{\alpha-1}-1}{k^\alpha}\right)\zeta = r^M - \left(\frac{k^{\alpha-1}-1}{k^\alpha}\right)\left(\frac{\zeta}{2}\right).$$

Thus, we have

$$d\left(\frac{x_{s_1}(i) + x_{s_2}(i) + \ldots + x_{s_k}(i)}{k}, \mathbf{0}\right) < \left(r^M - \left(\tfrac{k^{\alpha-1}-1}{k^\alpha}\right)\tfrac{\zeta}{2}\right)^{1/M} < r - \delta$$

$$\text{for } \delta \in \left(0, r - \left(r^M - \left(\tfrac{k^{\alpha-1}-1}{k^\alpha}\right)\tfrac{\zeta}{2}\right)^{1/M}\right).$$

Hence, $(\ell(p,\theta), d)$ is $k-NUC$.

Since $k-NUC$ implies $NUC$ and $NUC$ implies property $(H)$, by using previous theorem, we give the following result:

**Corollary 2.5.**

The space $(\ell(p,\theta), d)$ has property $(H)$.

## 3. Conclusion

Many mathematicians are interested in $\ell(p)$-type sequence spaces. Then, some geometric properties on these spaces were considered equipped with the Luxemburg norm. In linear metric spaces, the notion of uniform convexity,strict convexity or rotundity was introduced in 1977. Later, the relation between these properties and property (H) was investigated. From this point of view, we defined a metric on the space $\ell(p,\theta)$ as an $\ell(p)$-type sequence space. Then, we studied the geometric structure of this space and showed that the linear metric space $(\ell(p,\theta),d)$ is k-NUC, has property (H) and possesses property $(\beta)$.

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