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New Existence Results to Solution of Fractional Boundary Value Problems

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Abstract

In this article, we verify the existence of solution to boundary value problem of nonlinear fractional differential equation involving Caputo fractional derivatives. We obtain new existence results based on nonlinear alternative of Leray-Schauder type and Krasnoselskiis fixed point theorem. At the end, two illustrative examples have been presented.

Keywords: Fractional boundary value problem, Leray-Schauder fixed point theorem, Krasnoselskiis fixed point theorem, Existence results

AMS-MSC 2010 No.: 26A33, 34A12, 34A40.

1. Introduction

In the recent years, fractional calculus is one of the interest issues that attract many scientists, specially mathematicians and engineer scientists. Many natural phenomena can be present by boundary value problems of fractional differential equations. Many authors in different fields such as chemical physics, fluid flows, electrical networks, viscoelasticity, try to modeling of these phenomena by boundary value problems of fractional differential equations [Oldham (1974), Ross (1975), Tatom (1995), Nonnenmacher and Metzler (1995)]. For achieve extra information in fractional calculus, specially boundary value problems, reader can refer to more valuable papers or books that are written by authors such as Samko et al. (1993), Kilbas et al. (2006), Miller (1993), Podlubny (1999), Lakshmikantham et al. (2009), Agarwal et al. (2009), Ahmad (2010), Bai (2010), Benchohara et al. (2009), Nieto (2010), Zhang (2006), Baleanu (2012) and Trujillo (2008).

Bai and Qui (2009) considered the existence of a positive solution to a singular BVPs nonlinear FDE

$$(D_{0^+}^{\alpha}u)(t) = f\big(t, u(t)\big), \quad 2 < \alpha \leq 3$$

$$u(0) = u'(1) = u''(0) = 0,$$

where $D_{0^+}^{\alpha}$ denotes Caputo derivative of order $\alpha, f : (0,1] \times [0,+\infty) \to [0,+\infty)$, $lim_{t\to 0^+} f(t,.) = +\infty$. Their analysis was based on Leray-Schauder type and Krasnoselskiis fixed point theorem. In (2011), Khan et al., established the existence and uniqueness of solution for the nonlinear FDE

$$(D_{0^+}^{q}u)(t) = f(t, u(t), (D_{0^+}^{\sigma}u)(t)), \quad 1 < q \leq 2$$

subject to integral boundary conditions

$$\alpha u(0) - \beta u'(0) = \int_0^T g\big(s, u(s)\big)ds, \quad \gamma u(1) - \delta u'(1) = \int_0^T h\big(s, u(s)\big)ds,$$

where $0 < \sigma \leq 1, \alpha, \delta > 0, \ \beta, \gamma \geq 0$ and $D_{0^+}^{\alpha}$ denotes Caputo derivative, by using nonlinear alternative of Leray-Schauder type and Banach fixed point theorem.

In the present work, we prove the existence of solution to a BVP of nonlinear FDE

$$\begin{cases} (D_{0^+}^{\alpha}x)(t) = f(t, x(t), x'(t), (D_{0^+}^{q}x)(t)), \\ \qquad x(0) = -x(1), \ x'(0) = 0, \\ \qquad (D_{0^+}^{p}x)(t) = -(D_{0^+}^{p}x)(1), \end{cases} \tag{1.1}$$

where $2 < \alpha \leq 3, \ 1 < p < 2, \ \alpha - p - 1 > 0, \ 1 < q < 2$ and $D_{0^+}^{\alpha}, D_{0^+}^{p}, D_{0^+}^{q}$ are Caputo fractional derivatives and $f: [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous function. The main tools for achieving this goal are nonlinear alternative of Leray-Schauder type and Krasnoselskiis fixed point theorem in a cone.

Throughout this paper, we assume that the following conditions hold:

(H1) f is continuous.

(H2) There exists a positive constant r such that

$$|f(t,u_2,v_2,w_2)-f(t,u_1,v_1,w_1)|\le r(|u_2-u_1|+|v_2-v_1|+|w_2-w_1|),$$

for any $t\in[0,1]$ and $u_1,v_1,w_1,u_2,v_2,w_2\in\mathbb{R}$.

(H3) $f(t,x,y,z)\le\varrho\lambda_1$ for $(t,x,y,z)\in[0,1]\times[-\varrho,\varrho]^3$.

(H4) $f(t,x,y,z)\ge\rho\lambda_2$ for $(t,x,y,z)\in[0,1]\times[-\rho,\rho]^3$,

where λ_1,λ_2 are determined later.

(H5) There exists a continuous, nondecreasing function $\varphi:\mathbb{R}\to\mathbb{R}_0^+$ with

$$|f(t,x,y,z)|\le\varphi(|x|+|y|+|z|)$$

for

$$(t,x,y,z)\in[0,1]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}.$$

(H6) There exists $R>0$ with $\frac{R}{\varphi(R)}>\Delta$, where Δ is determined later.

2. Main Results

We now give definitions, lemmas and theorems that will be used in the remainder of this paper.

Throughout the paper, we denote $\mathbb{R}^3=\mathbb{R}\times\mathbb{R}\times\mathbb{R}$, $\mathbb{R}^+=[0,+\infty]$, $\mathbb{R}_0^+=(0,+\infty)$ and

$$f_.=f(\cdot,x(\cdot),x'(\cdot),(D_{0^+}^q x)(\cdot)).$$

Definition 2.1.

The Riemann-Liouville fractional integral of order $\alpha>0$, of a function $g:\mathbb{R}_0^+\to\mathbb{R}$ is defined by

$$I_{0^+}^\alpha g(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}g(s)ds\,,\quad n-1<\alpha\le n, \tag{2.1}$$

provided that right-hand side is point wise defined on $\mathbb{R}_0^+$.

Definition 2.2.

The Caputo fractional derivative of order $\alpha>0$, of a function $g:\mathbb{R}_0^+\to\mathbb{R}$ is defined by

$$D_{0^+}^{\alpha}g(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1} g^{(n)}(s)ds\,, \quad n-1<\alpha\leq n. \tag{2.2}$$

Lemma 2.3.

Let $n-1<\alpha\leq n$, $(n\in\mathbb{N})$. Then fractional differential equation

$$D_0^+ x(t) = 0$$

has solution

$$x(t) = c_0 + c_1 t + \cdots + c_{n-1}t^{n-1},\ \ c_i\in\mathbb{R},\ \ i=0,1,\cdots n-1,\ \ n=[\alpha]+1.$$

Lemma 2.4.

Let $n-1<\alpha\leq n$, $(n\in\mathbb{N})$. Then

$$I_0^+ D_0^+ x(t) = c_0 + c_1 t + \cdots + c_{n-1}t^{n-1},$$

for some $c_i\in\mathbb{R},\ i=0,1,\cdots n-1,\ n=[\alpha]+1$.

Lemma 2.5.

For any $h\in C[0,1]$ and $2<\alpha\leq 3$, a unique solution of the problem

$$x(t) - h(t) = 0, \tag{2.3}$$

$$\begin{cases} x(0) = -x(1),\ \ x'(0) = 0, \\ (D_{0^+}^p x)(0) = -(D_{0^+}^p x)(1), \end{cases} \tag{2.4}$$

is

$$x(t) = \int_0^1 G(t,s)h(s)ds, \tag{2.5}$$

where $G(t,s)$is the Green function given by

$$G(t,s) = \begin{cases} \frac{(t-s)^{\alpha-1}-\frac{1}{2}(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(2t^2-1)}{4}\frac{\Gamma(3-p)}{\Gamma(\alpha-p)}(1-s)^{\alpha-p-1}, & 0\leq s\leq t\leq 1 \\ -\frac{1}{2\Gamma(\alpha)}(1-s)^{\alpha-1} + \frac{(2t^2-1)}{4}\frac{\Gamma(3-p)}{\Gamma(\alpha-p)}(1-s)^{\alpha-p-1}. & 0\leq s\leq t\leq 1. \end{cases} \tag{2.6}$$

Proof:

According to lemma 2.4, Equation (1.1) is equivalent to the following integral equation:

$$x(t) = I_{0^+}^{\alpha} h(t) + c_0 + c_1 t + c_2 t^2, \tag{2.7}$$

for some $c_0, c_1, c_2 \in \mathbb{R}$. So, we get

$$x'(t) = DI_{0^+}^{\alpha}h(t) + c_1 + 2c_2t = I_{0^+}^{\alpha-1}h(t) + c_1 + 2c_2t, \quad (2.8)$$

and

$$\begin{aligned}(D_{0^+}^{p}x)(t) &= D_{0^+}^{p}I_{0^+}^{\alpha}h(t) + c_1\frac{t^{1-p}}{\Gamma(2-p)} + c_2\frac{t^{2-p}}{\Gamma(3-p)}, \\ &= I_{0^+}^{\alpha-p}h(t) + c_1\frac{t^{1-p}}{\Gamma(2-p)} + c_2\frac{t^{2-p}}{\Gamma(3-p)}.\end{aligned} \quad (2.9)$$

From (2.4), we obtain

$$c_0 = \frac{-1}{2\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}h(s)ds - \frac{\Gamma(3-p)}{4\Gamma(\alpha-p)}\int_0^1 (1-s)^{\alpha-p-1}h(s)ds,$$

$$c_1 = 0, \;\; c_2 = \frac{\Gamma(3-p)}{4\Gamma(\alpha-p)}\int_0^1 (1-s)^{\alpha-p-1}h(s)ds.$$

Substituting c_0, c_1 and c_2 into (2.7), we obtain (2.5) where $G(t,s)$ is (2.6). The proof is complete.

Lemma 2.6. [Bai and Qiu (2009)]

Let E be a Banach space, $P \subseteq E$ a cone and Ω_1, Ω_2 are two bounded open balls of Ecentered at the origin with $\overline{\Omega}_1 \subseteq \Omega_2$.Let $A: P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ is a completely continuous operator such that either

(i) $\|Ax\| \leq \|x\|, x \in P \cap \partial\Omega_1$ and $\|Ax\| \geq \|x\|, x \in P \cap \partial\Omega_2$,

(ii) $\|Ax\| \geq \|x\|, x \in P \cap \partial\Omega_1$ and $\|Ax\| \leq \|x\|, x \in P \cap \partial\Omega_2$,

holds. Then, A has a fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Lemma 2.7. [Khan, et al. (2011)]

Let E be a Banach space, $M \subseteq E$, closed and convex. Assume that U is a relatively open subset of M with $0 \in U$ and $A: \overline{U} \to M$ is a continuous compact map. Then either

(i') A has a fixed point in $\overline{U}$, or

(ii') There exists $u \in \partial U$, and $\lambda \in (0,1)$ with $u = \lambda Au$.

Now, let $X = \{x(t); x(t) \in C^1[0,1], D_{0^+}^{q}x(t) \in C^1[0,1]\}$ equipped with the norm

$$\|x\|_* = max_{t\in[0,1]}|x(t)| + max_{t\in[0,1]}|x'(t)| + max_{t\in[0,1]}\left|D_{0^+}^{q}x(t)\right|.$$

It is easy to see that X is a Banach space. Define operator $A : X \to X$, as

$$Ax(t) = \int_0^1 G(t,s) f\left(s, x(s), x'(s), D_{0^+}^q x(s)\right) ds.$$

Lemma 3.1.

Assume that the hypotheses **(H1)-(H2)** are satisfied. Then, $A: X \to X$ is completely continuous.

Proof:

Let (x_m) be a sequence such that $x_m \to x$ in $C^1[0;\ 1]$, i.e. $\|x_m - x\|_* \to 0$. From **(H2)** we have

$$|Ax_m(t) - Ax(t)|$$

$$= \left|\int_0^1 G(t,s)\left\{f\left(s, x_m(s), x_m{}'(s), D_{0^+}^q x_m(s)\right) - f\left(s, x(s), x'(s), D_{0^+}^q x(s)\right)\right\}\right| ds$$

$$\le r\int_0^1 |G(t,s)|\left\{|x_m(s) - x(s)| + \left|x_m^{'(s)} - x'(s)\right| + \left|D_{0^+}^q x_m(s) D_{0^+}^q x(s)\right|\right\} ds$$

$$\le r\|x_m - x\|_* \int_0^1 |G(t,s)| ds$$

$$\le r\|x_m - x\|_* \left\{\int_0^{\mathrm{t}} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \mathrm{ds} + \int_0^1 \left|\frac{(1-s)^{\alpha-1}}{2\Gamma(\alpha)} + \frac{(2t^2-1)\Gamma(3-p)}{4\Gamma(\alpha-p)}(1-s)^{\alpha-p-1}\right| \mathrm{ds}\right\}$$

$$\le r\|x_m - x\|_* \left\{\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{1}{2\Gamma(\alpha+1)} + \frac{|2t^2-1|\Gamma(3-p)}{4\Gamma(\alpha-p+1)}\right\},$$

$$\left|(Ax_m)'(t) - (Ax)'(t)\right|$$

$$= \left|\int_0^1 \frac{\partial G(t,s)}{\partial t}\left\{f\left(s, x_m(s), x_m{}'(s), D_{0^+}^q x_m(s)\right) - f\left(s, x(s), x'(s), D_{0^+}^q x(s)\right)\right\}\right| ds$$

$$\le r\int_0^1 \left|\frac{\partial G(t,s)}{\partial t}\right|\left\{|x_m(s) - x(s)| + |x_m{}'(s) - x'(s)| + \left|D_{0^+}^q x_m(s) - D_{0^+}^q x(s)\right|\right\} ds$$

$$\le r\|x_m - x\|_* \int_0^1 \left|\frac{\partial G(t,s)}{\partial t}\right| ds$$

$$\le r\|x_m - x\|_* \left\{\int_0^{\mathrm{t}} \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} \mathrm{ds} + \int_0^1 \frac{t\Gamma(3-p)}{\Gamma(\alpha-p)}(1-s)^{\alpha-p-1} \mathrm{ds}\right\}$$

$$\le r\|x_m - x\|_* \left\{\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t\Gamma(3-p)}{\Gamma(\alpha-p)}\right\},$$

and

$$\left|(D_{0^+}^q Ax_m)(t) - ((D_{0^+}^q Ax)(t)\right|$$

$$= \left|\int_0^1 \frac{(t-s)^{q-1}}{\Gamma(2-q)}\{(Ax_m)''(s) - (Ax)''(s)\}\right| ds$$

$$= \int_0^1 \frac{(t-s)^{q-1}}{\Gamma(2-q)} \ \left|(Ax_m)''(s) - (Ax)''(s)\right| ds$$

$$\leq r\|x_m - x\|_* \left\{\int_0^t \frac{(t-s)^{1-q}}{\Gamma(2-q)}\{\int_0^s \frac{(s-\tau)^{\alpha-3}}{\Gamma(\alpha-2)} d\tau + \frac{\Gamma(3-p)}{\Gamma(\alpha-p)}\int_0^1 (1-\tau)^{\alpha-p-1} d\tau\}\right\} ds$$

$$\leq r\|x_m - x\|_* \int_0^1 \frac{(1-s)^{1-q}s^{\alpha-2}}{\Gamma(2-q)\Gamma(\alpha-1)} ds + r\|x_m - x\|_* \frac{\Gamma(3-p)}{\Gamma(\alpha-p+1)}\int_0^1 \frac{(1-s)^{1-q}}{\Gamma(2-q)} ds,$$

$$\leq r\|x_m - x\|_* \left\{\frac{1}{\Gamma(\alpha-q+1)} + \frac{\Gamma(3-p)}{\Gamma(3-q)\Gamma(\alpha-p+1)}\right\}.$$

Hence, we get

$$\|Ax_m - Ax\|_* = \max_{t\in[0,1]}|Ax_m(t) - Ax(t)| + \max_{t\in[0,1]}|(Ax_m)'(t) - (Ax)'(t)|$$

$$+ \max_{t\in[0,1]}\left|(D_{0^+}^q Ax_m)(t) - (D_{0^+}^q Ax)(t)\right|.$$

$$\leq r\|x_m - x\|_* \left\{\frac{3}{2\Gamma(\alpha+1)} + \frac{\Gamma(3-p)}{4\Gamma(\alpha-p+1)}\right\} + r\|x_m - x\|_* \left\{\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(3-p)}{\Gamma(\alpha-p)}\right\}$$

$$+r\|x_m - x\|_* \left\{\frac{1}{\Gamma(\alpha-q+1)} + \frac{\Gamma(3-p)}{\Gamma(3-q)\Gamma(\alpha-p+1)}\right\}$$

$$\leq r\|x_m - x\|_* \Delta,$$

where

$$\Delta = \frac{3}{2\Gamma(\alpha+1)} + \frac{\Gamma(3-p)}{4\Gamma(\alpha-p+1)} + \frac{1}{\Gamma(\alpha)} + \frac{\Gamma(3-p)}{\Gamma(\alpha-p+1)} + \frac{1}{\Gamma(\alpha-q+1)} + \frac{\Gamma(3-p)}{\Gamma(3-q)\Gamma(\alpha-p+1)}.$$

Therefore, A is continuous.

Let $M \subseteq X$ be bounded, i.e. there exists a positive constant b such that $\|x\| \leq b;\ \forall x \in M$: From (H1) f is continuous in $[0;\ 1] \times \mathbb{R}^3$. We assume that

$$L = \max_{t\in[0,1]}\left|f\left(t, x_m(t), x_m{}'(t), D_{0^+}^q x_m(t)\right)\right| + 1.$$

Thus,

$$|(Ax)(t)| = \left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_s ds\right|$$

$$+\int_0^1 \left(\frac{-(1-s)^{\alpha-1}}{2\Gamma(\alpha)}+\frac{(2t^2-1)\Gamma(3-p)}{4\Gamma(\alpha-p)}(1-s)^{\alpha-p-1}\right)f_s ds|$$

$$\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f_s|ds+\int_0^1 \left(\frac{(1-s)^{\alpha-1}}{2\Gamma(\alpha)}|f_s|ds\right.$$

$$+\frac{|2t^2-1|\Gamma(3-p)}{4\Gamma(\alpha-p)}\int_0^1 (1-s)^{\alpha-p-1}|f_s|ds$$

$$\leq L\left(\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{1}{2\Gamma(\alpha+1)}+\frac{|2t^2-1|\Gamma(3-p)}{4\Gamma(\alpha-p+1)}\right),$$

$$\begin{aligned}|(Ax)'(t)| &= \left|\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}f_s ds+\int_0^1 \frac{t\Gamma(3-p)}{\Gamma(\alpha-p)}(1-s)^{\alpha-p-1}f_s ds\right| \\ &\leq \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f_s|ds+\frac{t\Gamma(3-p)}{\Gamma(\alpha-p)}\int_0^1 (1-s)^{\alpha-p-1}|f_s|\,ds \\ &\leq L\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}+\frac{t\Gamma(3-p)}{4\Gamma(\alpha-p+1)}\right),\end{aligned}$$

and

$$\begin{aligned}\left|(D_{0^+}^q Ax(t)\right| &= \left|\int_0^t \frac{(t-s)^{1-q}}{\Gamma(2-q)}\left\{\int_0^s \frac{(s-\tau)^{\alpha-3}}{\Gamma(\alpha-2)}f_\tau d\tau+\frac{\Gamma(3-p)}{\Gamma(\alpha-p)}\int_0^1 (1-\tau)^{\alpha-p-1}f_\tau d\tau\right\}ds\right| \\ &= \int_0^t \frac{(t-s)^{1-q}}{\Gamma(2-q)}\left\{\int_0^s \frac{(s-\tau)^{\alpha-3}}{\Gamma(\alpha-2)}|f_\tau|d\tau+\frac{\Gamma(3-p)}{\Gamma(\alpha-p)}\int_0^1 (1-\tau)^{\alpha-p-1}|f_\tau|d\tau\right\}ds \\ &\leq L\left(\int_0^t \frac{(t-s)^{1-q}s^{\alpha-2}}{\Gamma(2-q)\Gamma(\alpha-1)}ds+\frac{\Gamma(3-p)}{\Gamma(\alpha-p+1)}\int_0^t \frac{(t-s)^{1-q}}{\Gamma(2-q)}ds\right. \\ &\leq L\left(\int_0^1 \frac{(1-s)^{1-q}s^{\alpha-2}}{\Gamma(2-q)\Gamma(\alpha-1)}ds+\frac{\Gamma(3-p)}{\Gamma(\alpha-p+1)}\int_0^1 \frac{(1-s)^{1-q}}{\Gamma(2-q)}ds\right) \\ &= L\left\{\frac{1}{\Gamma(\alpha-q+1)}+\frac{\Gamma(3-p)}{\Gamma(3-q)\Gamma(\alpha-p+1)}\right\}.\end{aligned}$$

Therefore

$$\|Ax\|_* = max_{t\in[0,1]}|Ax(t)|+max_{t\in[0,1]}|(Ax)'(t)|+max_{t\in[0,1]}\left|(D_{0^+}^q Ax)(t)\right| \leq L\Delta.$$

This follows that $A(M)$ is bounded. $\forall\epsilon > 0$, there exists

$$\delta = \min\left\{\frac{\frac{\epsilon}{3}}{\frac{L(\alpha+2)}{\Gamma(\alpha+1)}+\frac{L\Gamma(3-p)}{\Gamma(\alpha-p+1)}},\frac{\frac{\epsilon}{3}}{\frac{6L}{\Gamma(\alpha+1)}+\frac{L\Gamma(3-p)}{\Gamma(\alpha-p+1)}},\frac{\frac{\epsilon}{3}}{\frac{L(\alpha+1)}{\Gamma(\alpha)}+\frac{L\Gamma(3-p)}{\Gamma(\alpha-p+1)}},\right.$$

$$\left.\frac{\frac{\epsilon}{3}}{\frac{6L}{\Gamma(\alpha)}+\frac{L\Gamma(3-p)}{\Gamma(\alpha-p+1)}},\left(\frac{\frac{\epsilon}{3}}{\frac{L(4-q)}{\Gamma(\alpha-1)}+\frac{L(4-q)\Gamma(3-p)}{\Gamma(\alpha-p+1)}}\right)^{\frac{1}{2-q}},\left(\frac{2^{q-2}\frac{\epsilon}{3}}{\frac{3L}{\Gamma(\alpha-1)\Gamma(3-q)}+\frac{3L\Gamma(3-p)\Gamma(3-q)}{\Gamma(\alpha-p+1)}}\right)^{\frac{1}{2-q}}\right\},$$

for all $x \in M$ and t_1, t_2 with $t_1 < t_2$ and $0 < t_2 - t_1 < \delta$, we have

$$|Ax(t_2)-Ax(t_1)|\,ds = \int_0^{t_1} \frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}|f_s|ds+\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}|f_s|ds$$

$$+\frac{[(t_2-t_1)(t_2+t_1)]\Gamma(3-p)}{2\Gamma(\alpha-p+1)}\int_0^1(1-s)^{\alpha-p-1}|f_s|ds$$

$$\leq L\left(\frac{2(t_2-t_1)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(t_2{}^{\alpha}-t_1{}^{\alpha})}{\Gamma(\alpha+1)}+\frac{(t_2-t_1)\Gamma(3-p)}{4\Gamma(\alpha-p+1)}\right).$$

Case 1. For $\delta \leq t_1 < t_2 < 1$, by means value theorem, $\exists c_1 \in (t_1,t_2)$ such that

$$|Ax(t_2)-Ax(t_1)| \leq L(\frac{2\delta^{\alpha}}{\Gamma(\alpha+1)}+\frac{\alpha c_1{}^{\alpha-1}(t_2-t_1)}{\Gamma(\alpha+1)}+\frac{\delta\Gamma(3-p)}{4\Gamma(\alpha-p+1)})$$

$$\leq L(\frac{(\alpha+2)}{\Gamma(\alpha+1)}+\frac{\Gamma(3-p)}{\Gamma(\alpha-p+1)})\ \delta.$$

Case 2. For $0 \leq t_1 \leq \delta,\ t_2 < 2\delta$, by means value theorem, $\exists c_1 \in (t_1,t_2)$ such that

$$|Ax(t_2)-Ax(t_1)| \leq L(\frac{3t_2{}^{\alpha}}{\Gamma(\alpha+1)}+\frac{(t_2-t_1)\Gamma(3-p)}{4\Gamma(\alpha-p+1)})$$

$$\leq L\left(\frac{3(2\delta)^{\alpha}}{\Gamma(\alpha+1)}+\frac{\delta\Gamma(3-p)}{\Gamma(\alpha-p+1)}\right),$$

$$\leq L\left(\frac{6}{\Gamma(\alpha+1)}+\frac{3\Gamma(3-p)}{\Gamma(\alpha-p+1)}\right)\delta.$$

Similarly, we have

$$\left|(Ax)'(t_2)-(Ax)'(t_1)\right| = \left|\int_0^{t_2}\frac{(t_2-s)^{\alpha-2}}{\Gamma(\alpha-1)}f_s ds+\int_0^{t_1}\frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}f_s ds\right|$$

$$+\frac{(t_2-t_1)\Gamma(3-p)}{\Gamma(\alpha-p)\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-p-1}f_s ds|$$

$$=\int_0^{t_1}\frac{(t_2-s)^{\alpha-2}-(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f_s|ds+\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f_s|ds$$

$$+\frac{(t_2-t_1)\Gamma(3-p)}{\Gamma(\alpha-p+1)\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-p-1}|f_s|ds$$

$$\leq L(\frac{2(t_2-t_1)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(t_2{}^{\alpha-1}-t_1{}^{\alpha-1})}{\Gamma(\alpha)}+\frac{(t_2-t_1)\Gamma(3-p)}{4\Gamma(\alpha-p+1)}).$$

Case 1. For $\delta \leq t_1 < t_2 < 1$, by means value theorem, $\exists c_1 \in (t_1,t_2)$ such that

$$\left|(Ax)'(t_2)-(Ax)'(t_1)\right| \leq L(\frac{2\delta^{\alpha-1}}{\Gamma(\alpha)}+\frac{(\alpha-1)c_1{}^{\alpha-2}(t_2-t_1)}{\Gamma(\alpha)}+\frac{\delta\Gamma(3-p)}{4\Gamma(\alpha-p+1)})$$

$$\leq L(\frac{(\alpha+1)}{\Gamma(\alpha)}+\frac{\Gamma(3-p)}{\Gamma(\alpha-p+1)})\ \delta.$$

Case 2. For $0 \leq t_1 \leq \delta,\ t_2 < 2\delta$,

$$|(Ax)'(t_2) - (Ax)'(t_1)| \leq L(\frac{3{t_2}^{\alpha-1}}{\Gamma(\alpha)} + \frac{(t_2-t_1)\Gamma(3-p)}{4\Gamma(\alpha-p+1)})$$

$$\leq L(\frac{3(2\delta)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\delta\Gamma(3-p)}{\Gamma(\alpha-p+1)})$$

$$\leq L\left(\frac{6}{\Gamma(\alpha)} + \frac{3\Gamma(3-p)}{\Gamma(\alpha-p+1)}\right)\delta,$$

and

$$\left|(D_{0^+}^q Ax)(t_2) - ((D_{0^+}^q Ax)(t_1)\right|$$

$$= \left|\int_0^{t_2} \frac{(t_2-s)^{1-q}}{\Gamma(2-q)}(Ax)''(s)ds - \int_0^{t_1} \frac{(t_1-s)^{1-q}}{\Gamma(2-q)}(Ax)''(s)ds\right|$$

$$= |\int_0^{t_1} \frac{(t_2-s)^{1-q}}{\Gamma(2-q)}\{\int_0^s \frac{(s-\tau)^{\alpha-3}}{\Gamma(\alpha-2)} f_\tau d\tau + \frac{\Gamma(3-p)}{\Gamma(\alpha-p)}\int_0^1 (1-\tau)^{\alpha-p-1} f_\tau d\tau\}$$

$$+ \int_{t_1}^{t_2} \frac{(t_2-s)^{1-q}}{\Gamma(2-q)}\{\int_0^s \frac{(s-\tau)^{\alpha-3}}{\Gamma(\alpha-2)} f_\tau d\tau + \frac{\Gamma(3-p)}{\Gamma(\alpha-p)}\int_0^1 (1-\tau)^{\alpha-p-1} f_\tau d\tau\}$$

$$- \int_0^{t_1} \frac{(t_1-s)^{1-q}}{\Gamma(2-q)}\{\int_0^s \frac{(s-\tau)^{\alpha-3}}{\Gamma(\alpha-2)} f_\tau d\tau + \frac{\Gamma(3-p)}{\Gamma(\alpha-p)}\int_0^1 (1-\tau)^{\alpha-p-1} f_\tau d\tau\}|$$

$$\leq \int_0^{t_1}(\frac{(t_2-s)^{1-q}}{\Gamma(2-q)} - \frac{(t_1-s)^{1-q}}{\Gamma(2-q)})$$

$$\times \left(\int_0^s \frac{(s-\tau)^{\alpha-3}}{\Gamma(\alpha-2)}|f_\tau| d\tau + \frac{\Gamma(3-p)}{\Gamma(\alpha-p)}\int_0^1 (1-\tau)^{\alpha-p-1}|f_\tau| d\tau\right) ds$$

$$+ \int_{t_1}^{t_2} \frac{(t_2-s)^{1-q}}{\Gamma(2-q)}\{\int_0^s \frac{(s-\tau)^{\alpha-3}}{\Gamma(\alpha-2)}|f_\tau| d\tau + \frac{\Gamma(3-p)}{\Gamma(\alpha-p)}\int_0^1 (1-\tau)^{\alpha-p-1}|f_\tau| d\tau$$

$$\leq L\int_0^{t_1}(\frac{(t_2-s)^{1-q}}{\Gamma(2-q)} - \frac{(t_1-s)^{1-q}}{\Gamma(2-q)})(\frac{s^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{\Gamma(3-p)}{\Gamma(\alpha-p+1)})ds$$

$$+ L\int_{t_1}^{t_2} \frac{(t_2-s)^{1-q}}{\Gamma(2-q)}(\frac{s^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{\Gamma(3-p)}{\Gamma(\alpha-p+1)})ds$$

$$\leq L\left(\frac{s^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{\Gamma(3-p)}{\Gamma(\alpha-p+1)}\right)\left(\frac{2(t_2-t_1)^{2-q} + {t_2}^{2-q} - {t_1}^{2-q}}{\Gamma(3-q)}\right).$$

Case 1. For $\delta \leq t_1 < t_2 < 1$ by means value theorem, since $1-q<0$ we have

$$\left|(D_{0^+}^q Ax)(t_2) - ((D_{0^+}^q Ax)(t_1)\right| \leq L\left(\frac{s^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{\Gamma(3-p)}{\Gamma(\alpha-p+1)}\right)\left(\frac{2(t_2-t_1)^{2-q}+t_2{}^{2-q}-t_1{}^{2-q}}{\Gamma(3-q)}\right)$$

$$\leq L\left(\frac{1}{\Gamma(\alpha-1)} + \frac{\Gamma(3-p)}{\Gamma(\alpha-p+1)}\right)(2\delta^{2-q} + (2-q)\delta^{1-q}(t_2 - t_1))$$

$$\leq L\left(\frac{1}{\Gamma(\alpha-1)} + \frac{\Gamma(3-p)}{\Gamma(\alpha-p+1)}\right)(4-q)\delta^{2-q}.$$

Case 2. For $0 \leq t_1 \leq \delta,\ t_2 < 2\delta$,

$$\left|(D_{0^+}^q Ax)(t_2) - ((D_{0^+}^q Ax)(t_1)\right| \leq L\left(\frac{s^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{\Gamma(3-p)}{\Gamma(\alpha-p+1)}\right)\left(\frac{3t_2{}^{2-q}}{\Gamma(3-q)}\right)$$

$$\leq L\left(\frac{1}{\Gamma(\alpha-1)} + \frac{\Gamma(3-p)}{\Gamma(\alpha-p+1)}\right)\left(\frac{3(2\delta)^{2-q}}{\Gamma(3-q)}\right).$$

Therefore, $A(M)$ is equicontinuous. By the Arzela-Ascoli theorem $A(M)$ is compact. Consequently, $A : X \to X$ is completely continuous.

Theorem 3.2.

Let $2 < \alpha \leq 3$, $1 < q \leq 2$, and $f : [0;\ 1] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous. Suppose that there exist two distinct positive constants $\varrho, \rho, (\varrho > \rho)$ such that:

(H3) $f(t,x,y,z) \leq \varrho\lambda_1$ for $(t,x,y,z) \in [0,1] \times [-\varrho,\varrho]^3$,

(H4) $f(t,x,y,z) \geq \rho\lambda_2$ for $(t,x,y,z) \in [0,1] \times [-\rho,\rho]^3$,

where

$$\lambda_1 = \left(\frac{3}{2\Gamma(\alpha+1)} + \frac{\Gamma(3-p)}{4\Gamma(\alpha-p+1)} + \frac{1}{\Gamma(\alpha)} + \frac{\Gamma(3-p)}{\Gamma(\alpha-p+1)} + \frac{1}{\Gamma(\alpha-q+1)} + \frac{\Gamma(3-p)}{\Gamma(3-q)\Gamma(\alpha-p+1)}\right)^{-1},$$

and

$$\lambda_2 = \left(\frac{1}{2\Gamma(\alpha+1)} + \frac{\Gamma(3-p)}{4\Gamma(\alpha-p+1)} + \frac{1}{\Gamma(\alpha)} + \frac{\Gamma(3-p)}{\Gamma(\alpha-p+1)} + \frac{1}{\Gamma(\alpha-q+1)} + \frac{\Gamma(3-p)}{\Gamma(3-q)\Gamma(\alpha-p+1)}\right)^{-1}.$$

Then, boundary value problem (1.1) has at least one solution.

Proof:

From Lemma 3.1 we know operator $A: X \to X$ is completely continuous. We present the proof into two steps.

Step 1. We suppose that $\Omega_1 = \{x \in X; \|x\|_* < \rho\}$. For $\in\cap\ \partial\Omega_1$, $\forall t \in [0,1]$, from **(H4)**, we get

$$Ax(1) = \int_0^1 G(1,s) f_s ds$$

$$= \int_0^1 (\frac{(1-s)^{\alpha-1}}{2\Gamma(\alpha)} + \frac{\Gamma(3-p)}{4\Gamma(\alpha-p)}(1-\tau)^{\alpha-p-1}) f_s ds$$

$$\geq \rho\lambda_2 \int_0^1 (\frac{(1-s)^{\alpha-1}}{2\Gamma(\alpha)} + \frac{\Gamma(3-p)}{4\Gamma(\alpha-p)}(1-\tau)^{\alpha-p-1}) ds$$

$$= \rho\lambda_2 \left(\frac{1}{2\Gamma(\alpha+1)} + \frac{\Gamma(3-p)}{4\Gamma(\alpha-p+1)}\right),$$

$$(Ax)'(1) = \int_0^1 \frac{\partial G(1,s)}{\partial t} f_s ds$$

$$= \int_0^1 (\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{\Gamma(3-p)}{\Gamma(\alpha-p)}(1-\tau)^{\alpha-p-1}) f_s ds$$

$$\geq \rho\lambda_2 \int_0^1 (\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{\Gamma(3-p)}{\Gamma(\alpha-p)}(1-\tau)^{\alpha-p-1}) ds$$

$$= \rho\lambda_2 \left(\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(3-p)}{\Gamma(\alpha-p+1)}\right),$$

and

$$(D_{0^+}^q Ax)(1) = \int_0^1 \frac{(1-s)^{1-q}}{\Gamma(2-q)} \{\int_0^s \frac{(s-\tau)^{\alpha-3}}{\Gamma(\alpha-2)} f_\tau d\tau + \frac{\Gamma(3-p)}{\Gamma(\alpha-p)} \int_0^1 (1-\tau)^{\alpha-p-1} f_\tau d\tau\} ds$$

$$\geq \rho\lambda_2 (\int_0^1 \frac{(1-s)^{1-q} s^{\alpha-2}}{\Gamma(2-q)\Gamma(\alpha-1)} ds + \frac{\Gamma(3-p)}{\Gamma(\alpha-p+1)} \int_0^1 \frac{(1-s)^{1-q}}{\Gamma(2-q)} ds)$$

$$= \rho\lambda_2 \left(\frac{1}{\Gamma(\alpha-q+1)} + \frac{\Gamma(3-p)}{\Gamma(3-q)\Gamma(\alpha-p+1)}\right).$$

Hence,

$$\|Ax\|_* = max_{t\in[0,1]}|Ax(t)| + max_{t\in[0,1]}|(Ax)'(t)| + max_{t\in[0,1]}\left|(D_{0^+}^q Ax)(t)\right|$$

$$= \rho\lambda_2 \left(\frac{1}{2\Gamma(\alpha+1)} + \frac{\Gamma(3-p)}{4\Gamma(\alpha-p+1)}\right) + \rho\lambda_2 \left(\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(3-p)}{\Gamma(\alpha-p+1)}\right)$$

$$+\rho\lambda_2 \left(\frac{1}{\Gamma(\alpha-q+1)} + \frac{\Gamma(3-p)}{\Gamma(3-q)\Gamma(\alpha-p+1)}\right)$$

$$= \rho.$$

Consequently, we derive $\|Ax\|_* \geq \|x\|_*$ for all $x \in \partial\Omega_1$.

Step 2. We suppose that $\Omega_2 = \{x \in X; \|x\|_* < \varrho\}$**.** For $\in \cap\ \partial\Omega_2$, $\forall t \in [0,1]$, from **(H3)**, we get

$$|(Ax)(t)| = \left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_s ds + \int_0^1 \left(\frac{-(1-s)^{\alpha-1}}{2\Gamma(\alpha)} + \frac{(2t^2-1)\Gamma(3-p)}{4\Gamma(\alpha-p)}(1-s)^{\alpha-p-1}\right) f_s ds\right|$$

$$\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f_s| ds + \int_0^1 \left(\frac{(1-s)^{\alpha-1}}{2\Gamma(\alpha)} |f_s| ds + \frac{|2t^2-1|\Gamma(3-p)}{4\Gamma(\alpha-p)} \int_0^1 (1-s)^{\alpha-p-1} |f_s|\, ds\right.$$

$$\leq \varrho\lambda_1 \left(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{1}{2\Gamma(\alpha+1)} + \frac{|2t^2-1|\Gamma(3-p)}{4\Gamma(\alpha-p+1)}\right),$$

$$|(Ax)'(t)| = \left|\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} f_s ds + \int_0^1 \frac{t\Gamma(3-p)}{\Gamma(\alpha-p)} (1-s)^{\alpha-p-1} f_s ds\right|$$

$$\leq \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f_s| ds + \frac{t\Gamma(3-p)}{\Gamma(\alpha-p)} \int_0^1 (1-s)^{\alpha-p-1} |f_s|\, ds$$

$$\leq \varrho\lambda_1 \left(\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t\Gamma(3-p)}{4\Gamma(\alpha-p+1)}\right),$$

and

$$\begin{aligned}
\left|(D_{0^+}^q Ax(t)\right| &= \left|\int_0^t \frac{(t-s)^{1-q}}{\Gamma(2-q)} \left\{\int_0^s \frac{(s-\tau)^{\alpha-3}}{\Gamma(\alpha-2)} f_\tau d\tau + \frac{\Gamma(3-p)}{\Gamma(\alpha-p)} \int_0^1 (1-\tau)^{\alpha-p-1} f_\tau d\tau\right\} ds\right| \\
&= \int_0^t \frac{(t-s)^{1-q}}{\Gamma(2-q)} \left\{\int_0^s \frac{(s-\tau)^{\alpha-3}}{\Gamma(\alpha-2)} |f_\tau| d\tau + \frac{\Gamma(3-p)}{\Gamma(\alpha-p)} \int_0^1 (1-\tau)^{\alpha-p-1} |f_\tau| d\tau\right\} ds \\
&\leq \varrho\lambda_1 \left(\int_0^t \frac{(t-s)^{1-q} s^{\alpha-2}}{\Gamma(2-q)\Gamma(\alpha-1)} ds + \frac{\Gamma(3-p)}{\Gamma(\alpha-p+1)} \int_0^t \frac{(t-s)^{1-q}}{\Gamma(2-q)} ds\right. \\
&\leq \varrho\lambda_1 \left(\int_0^1 \frac{(1-s)^{1-q} s^{\alpha-2}}{\Gamma(2-q)\Gamma(\alpha-1)} ds + \frac{\Gamma(3-p)}{\Gamma(\alpha-p+1)} \int_0^1 \frac{(1-s)^{1-q}}{\Gamma(2-q)} ds\right) \\
&= \varrho\lambda_1 \left\{\frac{1}{\Gamma(\alpha-q+1)} + \frac{\Gamma(3-p)}{\Gamma(3-q)\Gamma(\alpha-p+1)}\right\}.
\end{aligned}$$

Therefore,

$$\|Ax\|_* = max_{t\in[0,1]} |Ax(t)| + max_{t\in[0,1]} |(Ax)'(t)| + max_{t\in[0,1]} \left|(D_{0^+}^q Ax)(t)\right|$$

$$\leq \varrho\lambda_1 \left(\frac{3}{2\Gamma(\alpha+1)} + \frac{\Gamma(3-p)}{4\Gamma(\alpha-p+1)}\right) + \rho\lambda_2 \left(\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(3-p)}{\Gamma(\alpha-p+1)}\right)$$

$$+\varrho\lambda_1 \left(\frac{1}{\Gamma(\alpha-q+1)} + \frac{\Gamma(3-p)}{\Gamma(3-q)\Gamma(\alpha-p+1)}\right)$$

$$\leq \varrho.$$

Consequently, we derive $\|Ax\|_* \leq \|x\|_*$ for all $x \in \partial\Omega_2$.

Now, by (ii) from Lemma 2.6 the proof is complete.

Theorem 3.3.

Let $f\,:\,[0;\,1]\times\mathbb{R}^3\to\mathbb{R}$ is continuous. Suppose that the following hypotheses are satisfied:

(H5) There exists a continuous, nondecreasing function $\varphi:\mathbb{R}\to\mathbb{R}_0^+$ with

$$|f(t,x,y,z)|\le\varphi(|x|+|y|+|z|),$$

for

$$(t,x,y,z)\in[0,1]\times\mathbb{R}^3.$$

(H6) There exists $R>0$ with $\frac{R}{\varphi(R)}>\Delta$, where Δ is determined later.

Then, boundary value problem (1.1) has one solution.

Proof:

We consider $U=\{x\in X;\|x\|_*<R\}$. By Lemma 3.1, we know $A:\overline{U}\to X$ is completely continuous. If $\exists x\in\partial U$, and $\lambda\in(0,1)$ such that

$$x\ =\lambda Ax, \tag{3.1}$$

then

$$x'=\lambda(Ax)', \tag{3.2}$$

and

$$D_{0^+}^q x\ =\lambda(D_{0^+}^q(Ax). \tag{3.3}$$

From **(H5)** and (3.1)-(3.3), for $t\in[0,1]$ we have

$$|x(t)|=|\lambda Ax(t)|\le\left|\int_0^1 G(t,s)f(s,x(s),x'(s),D_{0^+}^q x(s))ds\right|$$

$$\le\int_0^1 G(t,s)\varphi(|x(s)|+|x'(s)|+|D_{0^+}^q x(s)|)ds$$

$$\le\varphi(\|x\|_*)\ \int_0^1 G(t,s)ds$$

$$\le\varphi(\|x\|_*)(\frac{3}{2\Gamma(\alpha+1)}+\frac{\Gamma(3-p)}{4\Gamma(\alpha-p+1)}),$$

$$|x'(t)|=|\lambda(Ax)'(t)|\le\left|\int_0^1\frac{\partial G(t,s)}{\partial t}f(s,x(s),x'(s),D_{0^+}^q x(s))ds\right|$$

$$\leq \int_0^1 \frac{\partial G(t,s)}{\partial t}\varphi(|x(s)| + |x'(s)| + |D_{0^+}^q x(s)|)ds$$

$$\leq \varphi(\|x\|_*) \int_0^1 \frac{\partial G(t,s)}{\partial t} ds$$

$$\leq \varphi(\|x\|_*)(\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(3-p)}{\Gamma(\alpha-p+1)}),$$

and

$$\left|(D_{0^+}^q x(t)\right| = \left|(\lambda D_{0^+}^q Ax(t)\right|$$

$$= |\int_0^t \frac{(t-s)^{1-q}}{\Gamma(2-q)}\{\int_0^s \frac{(s-\tau)^{\alpha-3}}{\Gamma(\alpha-2)} f\left(\tau, x(\tau), x'(\tau), D_{0^+}^q x(\tau)\right) d\tau$$

$$+\frac{\Gamma(3-p)}{\Gamma(\alpha-p)}\int_0^1 (1-\tau)^{\alpha-p-1} f(\tau, x(\tau), x'(\tau), D_{0^+}^q x(\tau))d\tau\}ds\,|$$

$$\leq \int_0^t \frac{(t-s)^{1-q}}{\Gamma(2-q)}\{\int_0^s \frac{(s-\tau)^{\alpha-3}}{\Gamma(\alpha-2)}\varphi(|x(\tau)| + |x'(\tau)| + |D_{0^+}^q x(\tau)|)d\tau\}\mathrm{ds}$$

$$+\frac{\Gamma(3-p)}{\Gamma(\alpha-p)}\int_0^1 (1-\tau)^{\alpha-p-1}\varphi(|x(\tau)| + |x'(\tau)| + |D_{0^+}^q x(\tau)|)d\tau\}ds$$

$$\leq \varphi(\|x\|_*)\left(\frac{1}{\Gamma(\alpha-q+1)} + \frac{\Gamma(3-p)}{\Gamma(3-q)\Gamma(\alpha-p+1)}\right).$$

So,

$$\|Ax\|_* = max_{t\in[0,1]}|x(t)| + max_{t\in[0,1]}|x'(t)| + max_{t\in[0,1]}\left|(D_{0^+}^q x)(t)\right|$$

$$\leq \varphi(\|x\|_*)\left(\frac{3}{2\Gamma(\alpha+1)} + \frac{\Gamma(3-p)}{4\Gamma(\alpha-p+1)}\right) \quad + \varphi(\|x\|_*)\left(\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(3-p)}{\Gamma(\alpha-p+1)}\right)$$

$$+\varphi(\|x\|_*)\left(\frac{1}{\Gamma(\alpha-q+1)} + \frac{\Gamma(3-p)}{\Gamma(3-q)\Gamma(\alpha-p+1)}\right)$$

$$\leq \varphi(\|x\|_*)\Delta.$$

Therefore,

$$\frac{\|x\|_*}{\varphi(\|x\|_*)} \leq \Delta. \tag{3.4}$$

Combining **(H6)** and (3.4) we imply that $\|x\|_* \neq R$; a contradiction with $x \in \partial U$. By Lemma 2.7, A has a fixed point $x \in \overline{U}$. Therefore, the BVP (1.1) has one solution. The proof is complete.

Example 3.4. Consider the BVP

$$\begin{cases} (D_{0^+}^{\frac{5}{2}}x)(t) = \frac{1}{t+3}\{\frac{|x(t)+x'(t)+(D_{0^+}^{\frac{3}{2}}x)(t)|}{1+|x(t)+x'(t)+(D_{0^+}^{\frac{3}{2}}x)(t)|}\}, \\ x(0) = -x(1), \;\; x'(0) = 0, \\ (D_{0^+}^{\frac{3}{2}}x)(t) = -(D_{0^+}^{\frac{3}{2}}x)(1). \end{cases} \tag{3.5}$$

To show that boundary value problem (3.5) has at least one solution, we apply theorem (3.2) with

$$\alpha = \frac{5}{2},\; p,q = \frac{3}{2}, f\left(t, x(t), x'(t), D_{0^+}^{q}x(t)\right) = \frac{1}{t+3}\left\{\frac{|x(t)+x'(t)+(D_{0^+}^{\frac{3}{2}}x)(t)|}{1+|x(t)+x'(t)+(D_{0^+}^{\frac{3}{2}}x)(t)|}\right\},$$

$$\lambda_1 = 0.23,\; \lambda_2 = 0.23,\; \varrho = 1.43,\; \rho = 1.$$

It is easy to show that the conditions of theorem (3.2) are satisfied. In conclusion, problem (3.5) has at least one solution on [0,1].

Example 3.5.

Consider the BVP

$$\begin{cases} (D_{0^+}^{\frac{5}{2}}x)(t) = \frac{t}{6}|x(t) + x'(t) + (D_{0^+}^{\frac{3}{2}}x)(t)|, \\ x(0) = -x(1), \;\; x'(0) = 0, \\ (D_{0^+}^{\frac{3}{2}}x)(t) = -(D_{0^+}^{\frac{3}{2}}x)(1). \end{cases} \tag{3.6}$$

To show that boundary value problem (3.6) has at least one solution, we apply Theorem (3.3) with

$$\alpha = \frac{5}{2},\; p,q = \frac{3}{2}, f\left(t, x(t), x'(t), D_{0^+}^{q}x(t)\right) = \frac{t}{6}|x(t) + x'(t) + (D_{0^+}^{\frac{3}{2}}x)(t)|,$$

$$\varphi(u) = \frac{|u|}{5}\;, \Delta = 4.275.$$

It is easy to show that the conditions of theorem (3.3) are satisfied. In conclusion, problem (3.5) has at least one solution on [0,1].

4. Conclusion

In this paper, by using nonlinear alternative of Leray-Schauder type and Krasnoselskiis fixed point theorem, we obtain conditions to prove existence and uniqueness of solutions of (1.1) (see

Theorems 3.2 and 3.3). From examples 3.4 and 3.5, it is easy to see that our results are new and interesting.

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REFERENCES

Agarwal, R.P., Benchohara, M. and Hamani, S. (2009). Boundary value problems for fractional differential equations, J. Georgian, Mathd, Vol. 16, pp. 401-411.

Agarwal, R.P., Benchohara, M. and Slimani, B.A. (2008). Existence results for differential equations with fractional order and impulses, Mem. Diff. Equ. Math. Phys, Vol. 44, pp. 1-21.

Ahmad, B. (2010). Existence of solutions for irregular boundary value problems of nonlinear fractional differential equations, Appl. Math. Lett, Vol. 23, pp. 390-394.

Ahmad, B. and Nieto, J.J. (2009). Existence of solutions for nonlocal boundary value problems of higher order nonlinear fractional di_erential equations, Abstr. Appl. Anal, Art ID 494720.

Ahmad, B. and Nieto, J.J. (2009). Existence results for a coupled system of nonlinear fractional differential equations with three point boundary conditions, Comput. Math. Appl, Vol. 58, pp. 1838-1843.

Ahmad, B. and Sivasundaram, S. (2010). On four-point nonlocal boundary value problems of nonlinear integro-differential equations of fractional order, Appl. Math. Comput, Vol. 217, pp. 480-487.

Bai, Z. (2010). On positive solutions of a nonlocal fractional boundary value problems, Non-linear. Anal, Vol. 72, pp. 916-924.

Bai, Z. and Qiu, T. (2009). Existence of positive solution for singular fractional differential equations, Appl. Math. Comput, Vol. 215, pp. 2761-2767.

Baleanu, D., Diethelm, K., Scalas, E. and Trujillo, J.J. (2012). Fractional calculus models and numerical methods (series on complexity, nonlinearity and chaos), World Scientific.

Baleanu, D. and Trujillo, J.J. (2008). On exact solutions of a class of fractional Euler-Lagrange equations, Nonlinear Dynamics, Vol. 52, No. 4, pp. 331-335.

Benchohara, M., Hamani, S. and Ntouyas, S.K. (2009). Boundary value problems for differential equations with fractional order and nonlocal conditions, Nonlinear, Anal, Vol. 71, pp. 2391-2396.

Granas, A. and Dugundji, J. (2003). Fixed point theory, Springer-Verlag New York.

Khan, R.A., Rehman, M.U. and Henderson, J. (2011). Existence and uniqueness of solutions for nonlinear fractional differential equations with integral boundary conditions, Fractional Differential Calculus, Vol. 1, No 1, pp. 29-43.

Kilbas, A.A., Srivastava, H.M. and Trujillo, J.J. (2006). Theory and application of fractional diffferential equations, Elsevier B.V, Netherlands.

Krasnoselskii, M.A. (1964). Positive solution of operator equation, Noordhood, Groningen.

Lakshmikantham, V., Leela, S. and Vasundhara, J. (2009). Theory of fractional dynamic systems, Cambridge Academic Publishers, Cambridge.

Miller, K.S., B. and Ross, B. (1993). An introduction to the fractional calculus and fractional differential equation, John Wiley and Sons, New York.

Nonnenmacher, T.F. and Metzler, R. (1995). On the Riemann-Liouville fractional calculus and some recent applications, Fractals, Vol. 3, pp. 557-566.

Nieto, J.J. (2010). Maximum principles for fractional di_erentia equations derived from Mittag-Leffer functions, Appl. Math. Lett, Vol. 23, pp. 1248-1251.

Oldham, K.B. and Spanier, J. (1974). The fractional calculus, Academic press, New York and London.

Podlubny, I. (1999). Fractional differential equations, Academic Press, San Diego, CA.

Richard, M., Xu, F. and Baleanu, D. (2009). Solving the fractional order Bloch equation, Concept in Magnetic Resonance, Vol. 34, No. 1, 16-23.

Ross(Ed.), B. (1974). The fractional calculus and its application, in: Lecture notes in mathematics, vol.475, Springer-Verlag, Berlin.

Samko, S.G., Kilbas, A.A. and Marichev, O.I. (1993). Fractional integral and derivatives (theory and application), Gordon and Breach, Switzerland.

Tatom, F.B. (1995). The relationship between fractional calculus and fractals, Fractals, Vol. 3, pp. 217-229.

Zhang, S. (2006). Existence of solutions for a boundary value problems of fractional order, Acta. Math. Sci, Vol. 26B, pp. 220-228.