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## Certain Fractional Integral Operators and the Generalized Incomplete Hypergeometric Functions

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### Abstract

In this paper, we apply a certain general pair of operators of fractional integration involving Appell's function  $F_3$  in their kernel to the generalized incomplete hypergeometric functions  ${}_p\gamma_q[z]$  and  ${}_p\Gamma_q[z]$ , which were introduced and studied systematically by Srivastava *et al.* in the year 2012. Some interesting special cases and consequences of our main results are also considered.

**Keywords:** Gamma function; Incomplete Gamma functions; Decomposition formula; Incomplete Pochhammer symbols; Generalized incomplete hypergeometric functions; Fractional integral operators

**MSC 2010 No.:** Primary 26A33, 33B15, 33B20, 33C05, 33C15, 33C20; Secondary 33B99, 33C99, 60B99

## 1. Introductions and Definitions

Throughout the present investigation, we shall (as usual) denote by  $\mathbb{R}$  and  $\mathbb{C}$  the sets of real and complex numbers, respectively. In terms of the familiar (Euler's) Gamma function  $\Gamma(z)$  which is defined, for  $z \in \mathbb{C} \setminus \mathbb{Z}_0^-$ , by

$$\Gamma(z) = \begin{cases} \int_0^\infty t^{z-1} e^{-t} dt & (\Re(z) > 0) \\ \frac{\Gamma(z+n)}{\prod_{j=0}^{n-1} (z+j)} & (z \in \mathbb{C} \setminus \mathbb{Z}_0^-; n \in \mathbb{N}), \end{cases} \quad (1.1)$$

$$(\mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\}; \mathbb{Z}^- := \{-1, -2, -3, \dots\}; \mathbb{N} := \{1, 2, 3, \dots\}),$$

the Pochhammer symbol  $(\lambda)_\nu$  ( $\lambda, \nu \in \mathbb{C}$ ) is given, in general, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu \in \mathbb{N}; \lambda \in \mathbb{C}), \end{cases} \quad (1.2)$$

it being assumed *conventionally* that  $(0)_0 := 1$  and understood *tacitly* that the  $\Gamma$ -quotient exists (see, for details, (Srivastava and Manocha, 1984, p. 21 *et seq.*)).

The closely-related *incomplete Gamma functions*  $\gamma(z, \kappa)$  and  $\Gamma(z, \kappa)$  defined, respectively, by

$$\gamma(z, \kappa) := \int_0^\kappa t^{z-1} e^{-t} dt \quad (\Re(z) > 0; \kappa \geq 0) \quad (1.3)$$

and

$$\Gamma(z, \kappa) := \int_\kappa^\infty t^{z-1} e^{-t} dt \quad (\kappa \geq 0; \Re(z) > 0 \text{ when } \kappa = 0), \quad (1.4)$$

are known to satisfy the following decomposition formula:

$$\gamma(z, \kappa) + \Gamma(z, \kappa) = \Gamma(z) \quad (\Re(z) > 0). \quad (1.5)$$

The function  $\Gamma(z)$  given by (1.1), and its incomplete versions  $\gamma(z, \kappa)$  and  $\Gamma(z, \kappa)$  given by (1.3) and (1.4), respectively, are known to play important and useful rôles in the study of the analytic solutions of a variety of problems in diverse areas of science and engineering (see, for example, (Abramowitz and (Editors), 1972), (Andrews, 1985), (Chaudhry and Zubair, 2001), (A. Erdélyi and Tricomi, 1953), (N. L. Johnson and Balakrishnan, 1995), (A. A. Kilbas and Trujillo, 2006), (Luke, 1975), (W. Magnus and Soni, 1966), (K. B. Oldham and Spanier, 2009), (F. W. J. Olver and Clark, 2010), (Srivastava and Choi, 2001), (Srivastava and Choi, 2012), (Srivastava and Karlsson, 1985), (Srivastava and Kashyap, 1982), (Temme, 1996), (Watson, 1944) and (Whittaker and Watson, 1973); see also (H. M. Srivastava and Agarwal, 2012) and *especially* the references cited therein).

In view of the great potential for applications in a wide variety of fields, Srivastava *et al.* (H. M. Srivastava and Agarwal, 2012) introduced and studied systematically the following family

of generalized incomplete hypergeometric functions (H. M. Srivastava and Agarwal, 2012, p. 675, Equations (4.1) and (4.2)):

$${}_p\gamma_q \left[ \begin{matrix} (a_1, \kappa), a_2, \dots, a_p; \\ b_1, \dots, b_q; \end{matrix} z \right] := \sum_{n=0}^{\infty} \frac{(a_1; \kappa)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!} \quad (1.6)$$

and

$${}_p\Gamma_q \left[ \begin{matrix} (a_1, \kappa), a_2, \dots, a_p; \\ b_1, \dots, b_q; \end{matrix} z \right] := \sum_{n=0}^{\infty} \frac{[a_1; \kappa]_n (a_2)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}, \quad (1.7)$$

where, in terms of the incomplete Gamma functions  $\gamma(z, \kappa)$  and  $\Gamma(z, \kappa)$  defined by (1.3) and (1.4), respectively, the *incomplete* Pochhammer symbols

$$(\lambda; \kappa)_\nu \quad \text{and} \quad [\lambda; \kappa]_\nu \quad (\lambda, \nu \in \mathbb{C}; x \geq 0)$$

are defined as follows:

$$(\lambda; \kappa)_\nu := \frac{\gamma(\lambda + \nu, \kappa)}{\Gamma(\lambda)} \quad (\lambda, \nu \in \mathbb{C}; \kappa \geq 0) \quad (1.8)$$

and

$$[\lambda; \kappa]_\nu := \frac{\Gamma(\lambda + \nu, \kappa)}{\Gamma(\lambda)} \quad (\lambda, \nu \in \mathbb{C}; \kappa \geq 0). \quad (1.9)$$

so that, obviously, these incomplete Pochhammer symbols  $(\lambda; \kappa)_\nu$  and  $[\lambda; \kappa]_\nu$  satisfy the following decomposition formula:

$$(\lambda; \kappa)_\nu + [\lambda; \kappa]_\nu = (\lambda)_\nu \quad (\lambda, \nu \in \mathbb{C}; \kappa \geq 0), \quad (1.10)$$

where  $(\lambda)_\nu$  is the Pochhammer symbol given by (1.2).

**Remark 1.** As already pointed out by Srivastava *et al.* (H. M. Srivastava and Agarwal, 2012, p. 675, Remark 7), since

$$|(\lambda; \kappa)_n| \leq |(\lambda)_n| \quad \text{and} \quad |[\lambda; \kappa]_n| \leq |(\lambda)_n| \quad (n \in \mathbb{N}_0; \lambda \in \mathbb{C}; \kappa \geq 0), \quad (1.11)$$

the precise (*sufficient*) conditions under which the infinite series in the definitions (1.6) and (1.7) would converge absolutely can be derived from those that are well-documented in the case of the generalized hypergeometric function  ${}_pF_q$  ( $p, q \in \mathbb{N}_0$ ) (see, for details, (Rainville, 1971, pp. 72–73) and (Srivastava and Karlsson, 1985, p. 20); see also (Bailey, 1964), (Carlson, 1977), (Luke, 1975) and (Slater, 1966)). Indeed, in their special case when  $\kappa = 0$ , both  ${}_p\gamma_q$  ( $p, q \in \mathbb{N}_0$ ) and  ${}_p\Gamma_q$  ( $p, q \in \mathbb{N}_0$ ) would reduce immediately to the widely- and extensively-investigated generalized hypergeometric function  ${}_pF_q$  ( $p, q \in \mathbb{N}_0$ ). Furthermore, as an immediate consequence

of the definitions (1.6) and (1.7), we have the following decomposition formula:

$$\begin{aligned} {}_p\gamma_q \left[ \begin{matrix} (a_1, \kappa), a_2, \dots, a_p; \\ b_1, \dots, b_q; \end{matrix} z \right] + {}_p\Gamma_q \left[ \begin{matrix} (a_1, \kappa), a_2, \dots, a_p; \\ b_1, \dots, b_q; \end{matrix} z \right] \\ = {}_pF_q \left[ \begin{matrix} a_1, \dots, a_p; \\ b_1, \dots, b_q; \end{matrix} z \right] \end{aligned} \quad (1.12)$$

in terms of the familiar generalized hypergeometric function  ${}_pF_q$  ( $p, q \in \mathbb{N}_0$ ).

The above-mentioned detailed and systematic investigation by Srivastava *et al.* (H. M. Srivastava and Agarwal, 2012) was indeed motivated largely by the demonstrated potential for applications of the generalized incomplete hypergeometric functions  ${}_p\gamma_q$  and  ${}_p\Gamma_q$  and their special cases in many diverse areas of mathematical, physical, engineering and statistical sciences (see, for details, (H. M. Srivastava and Agarwal, 2012) and the references cited therein). Several *further* properties of each of these generalized incomplete hypergeometric functions and some classes of incomplete hypergeometric polynomials associated with them can be found in the subsequent developments presented in (for example) (Srivastava, 2013b), (Srivastava, 2013a) and (Srivastava and Cho, 2012). Moreover, by using the incomplete Pochhammer symbols given by (1.8) and (1.9), the corresponding incomplete versions of Appell's *two-variable* hypergeometric function  $F_2$  were considered recently in (Çetinkaya, 2013). In the present sequel to these recent works, we propose to derive several image formulas for the generalized incomplete hypergeometric functions  ${}_p\gamma_q$  and  ${}_p\Gamma_q$  by applying a certain general pair of fractional integral operators involving Appell's *two-variable* hypergeometric function  $F_3$ , which we introduce in Section 2 below. We also consider some interesting special cases and consequences of our main results.

## 2. Operators of Fractional Integration and Their Applications

In view of their importance and popularity in recent years, the theory of operators of fractional calculus has been developed widely and extensively (see, for example, each of the research monographs (A. Erdélyi and Tricomi, 1954, Chapter 13), (A. A. Kilbas and Trujillo, 2006), (Kiryakova, 1993), (McBride, 1979), (Miller and Ross, 1993), (Oldham and Spanier, 1974), (Podlubny, 1999) and (S. G. Samko and Marichev, 1993); see also (Srivastava and Saxena, 2001)). Here, in this section, we recall a general pair of fractional integral operators which involve in the kernel Appell's *two-variable* hypergeometric function  $F_3$  defined by (see (Appell and de Fériet, 1926, p. 14))

$$F_3(\alpha, \alpha', \beta, \beta'; \omega; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m (\alpha')_n (\beta)_m (\beta')_n}{(\omega)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \quad (2.1)$$

$$(\max\{|x|, |y|\} < 1).$$

Indeed, for

$$x > 0 \quad \text{and} \quad \alpha, \alpha', \beta, \beta', \omega \in \mathbb{C} \quad (\Re(\omega) > 0),$$

these general operators of fractional integration with the  $F_3$  kernel are defined by

$$\begin{aligned} & \left( I_{0,x}^{\alpha, \alpha', \beta, \beta', \omega} f \right) (x) \\ & := \frac{x^{-\alpha}}{\Gamma(\omega)} \int_0^x (x-t)^{\omega-1} t^{-\alpha'} F_3 \left( \alpha, \alpha', \beta, \beta'; \omega; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) f(t) dt \end{aligned} \quad (2.2)$$

and

$$\begin{aligned} & \left( I_{x,\infty}^{\alpha, \alpha', \beta, \beta', \omega} f \right) (x) \\ & := \frac{x^{-\alpha'}}{\Gamma(\omega)} \int_x^\infty (t-x)^{\omega-1} t^{-\alpha} F_3 \left( \alpha, \alpha', \beta, \beta'; \omega; 1 - \frac{x}{t}, 1 - \frac{t}{x} \right) f(t) dt, \end{aligned} \quad (2.3)$$

where the function  $f(t)$  is so constrained that the defining integrals in (2.2) and (2.3) exist.

The operators or integral transforms in (2.2) and (2.3) were introduced by Marichev (Marichev, 1974) as Mellin type convolution operators with the Appell function  $F_3$  in their kernel. These operators were rediscovered and studied by Saigo (Saigo, 1996) as generalizations of the so-called Saigo fractional integral operators (see also (Kiryakova, 2006) and (Srivastava and Saigo, 1987)). Such further properties as (for example) their relations with the Mellin transform and with the hypergeometric operators (or the Saigo fractional integral operators), together with their decompositional, operational and other properties in the McBride space  $F_{p,\mu}$  (see (McBride, 1979)) were studied by Saigo and Maeda (Saigo and Maeda, 1998) (see also some recent investigations on the subject of fractional calculus in (Agarwal, 2012a; Agarwal, 2012b; Agarwal and Jain, 2011; J. A. T. Machado and Mainardi, 2010; J. A. T. Machado and Mainardi, 2011; S. D. Purohit and Kalla, 2011)).

**Remark 2.** The Appell function  $F_3$  involved in the definitions (2.2) and (2.3) satisfies a system of two linear partial differential equations of the second order and reduces to the Gauss hypergeometric function  ${}_2F_1$  as follows (see (Appell and de Fériet, 1926, p. 25, Eq. (35)) and (Srivastava and Karlsson, 1985, p. 301, Eq. 9.4(87))):

$$F_3(\alpha, \omega - \alpha, \beta, \omega - \beta; \omega; x, y) = {}_2F_1 \left[ \begin{matrix} \alpha, \beta; \\ \omega; \end{matrix} \quad x + y - xy \right]. \quad (2.4)$$

Moreover, it is easily observed that

$$F_3(\alpha, 0, \beta, \beta'; \omega; x, y) = F_3(\alpha, \alpha', \beta, 0; \omega; x, y) = {}_2F_1 \left[ \begin{matrix} \alpha, \beta; \\ \omega; \end{matrix} \quad x \right] \quad (2.5)$$

and

$$F_3(0, \alpha', \beta, \beta'; \omega; x, y) = F_3(\alpha, \alpha', 0, \beta'; \omega; x, y) = {}_2F_1 \left[ \begin{matrix} \alpha', \beta'; \\ \omega; \end{matrix} \middle| y \right]. \quad (2.6)$$

In view of the obvious reduction formula (2.5), the general operators reduce to the aforementioned Saigo operators  $\mathcal{I}_{0,x}^{\alpha,\beta,\omega}$  and  $\mathcal{I}_{x,\infty}^{\alpha,\beta,\omega}$  defined by (see, for details, (Saigo, 1996); see also (Kiryakova, 2006) and (Srivastava and Saigo, 1987) and the references cited therein)

$$\left( \mathcal{I}_{0,x}^{\alpha,\beta,\omega} f \right) (x) := \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} {}_2F_1 \left( \alpha + \beta, -\omega; \alpha; 1 - \frac{t}{x} \right) f(t) dt \quad (2.7)$$

and

$$\left( \mathcal{I}_{x,\infty}^{\alpha,\beta,\omega} f \right) (x) := \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta} {}_2F_1 \left( \alpha + \beta, -\omega; \alpha; 1 - \frac{x}{t} \right) f(t) dt, \quad (2.8)$$

respectively. In fact, we have the following relationships:

$$\left( I_{0,x}^{\alpha,0,\beta,\beta',\omega} f \right) (x) = \left( \mathcal{I}_{0,x}^{\omega,\alpha-\omega,-\beta} f \right) (x) \quad (\omega \in \mathbb{C}) \quad (2.9)$$

and

$$\left( I_{x,\infty}^{\alpha,0,\beta,\beta',\omega} f \right) (x) = \left( \mathcal{I}_{x,\infty}^{\omega,\alpha-\omega,-\beta} f \right) (x) \quad (\omega \in \mathbb{C}). \quad (2.10)$$

In our investigation, we shall make use of each of the following known image formulas which are easy consequences of the definitions in ((Saigo and Maeda, 1998, p. 394)):

$$\begin{aligned} & \left( I_{0,x}^{\alpha,\alpha',\beta,\beta',\omega} t^{\rho-1} \right) (x) \\ &= \frac{\Gamma(\rho)\Gamma(\rho+\omega-\alpha-\alpha'-\beta)\Gamma(\rho-\alpha'+\beta')}{\Gamma(\rho+\beta')\Gamma(\rho+\omega-\alpha-\alpha')\Gamma(\rho+\omega-\alpha'-\beta)} x^{\rho+\omega-\alpha-\alpha'-1} \quad (2.11) \\ & \left( \Re(\omega) > 0; \Re(\rho) > \max \{0, \Re(\alpha+\alpha'+\beta-\omega), \Re(\alpha'-\beta')\} \right) \end{aligned}$$

and

$$\begin{aligned} & \left( I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\omega} t^{\rho-1} \right) (x) \\ &= \frac{\Gamma(1-\rho-\beta)\Gamma(1-\rho-\omega+\alpha+\alpha')\Gamma(1-\rho+\alpha+\beta'-\omega)}{\Gamma(1-\rho)\Gamma(1-\rho+\alpha+\alpha'+\beta'-\omega)\Gamma(1-\rho+\alpha-\beta)} x^{\rho+\omega-\alpha-\alpha'-1} \quad (2.12) \\ & \left( \Re(\omega) > 0; 0 < \Re(\rho) < 1 + \min \{ \Re(-\beta), \Re(\alpha+\alpha'-\omega), \Re(\alpha+\beta'-\omega) \} \right). \end{aligned}$$

We now state and prove our main fractional integral formulas involving each of the generalized incomplete hypergeometric functions  ${}_p\gamma_q$  and  ${}_p\Gamma_q$  defined by (1.6) and (1.7).

**Theorem 1.** Let  $x > 0$  and  $\kappa \geq 0$ . Suppose also that the parameters  $\alpha, \alpha', \beta, \beta', \mu, \omega, \rho \in \mathbb{C}$  are constrained by

$$\Re(\omega) > 0 \quad \text{and} \quad \Re(\rho) > \max \{0, \Re(\alpha + \alpha' + \beta - \omega), \Re(\alpha' - \beta')\}.$$

Then each of the following fractional integral formulas holds true:

$$\left( I_{0,x}^{(\alpha, \alpha', \beta, \beta', \omega)} [t^{\rho-1} {}_p\gamma_q(\mu t)] \right) (x) = x^{\rho+\omega-\alpha-\alpha'-1} \frac{\Gamma(\rho)\Gamma(\rho+\beta'-\alpha')\Gamma(\rho+\omega-\alpha-\beta-\alpha')}{\Gamma(\rho+\beta')\Gamma(\rho+\omega-\alpha-\alpha')\Gamma(\rho+\omega-\beta-\alpha')} \cdot {}_{p+3}\gamma_{q+3} \left[ \begin{matrix} (a_1, \kappa), a_2, \dots, a_p, \rho, \rho+\beta'-\alpha', \rho+\omega-\alpha-\beta-\alpha'; \\ b_1, \dots, b_q, \rho+\beta', \rho+\omega-\alpha-\alpha', \rho+\omega-\beta-\alpha'; \end{matrix} \mu x \right] \quad (2.13)$$

and

$$\left( I_{x,\infty}^{(\alpha, \alpha', \beta, \beta', \omega)} [t^{\rho-1} {}_p\Gamma_q(\mu t)] \right) (x) = x^{\rho+\omega-\alpha-\alpha'-1} \frac{\Gamma(\rho)\Gamma(\rho+\beta'-\alpha')\Gamma(\rho+\omega-\alpha-\beta-\alpha')}{\Gamma(\rho+\beta')\Gamma(\rho+\omega-\alpha-\alpha')\Gamma(\rho+\omega-\beta-\alpha')} \cdot {}_{p+3}\Gamma_{q+3} \left[ \begin{matrix} (a_1, \kappa), a_2, \dots, a_p, \rho, \rho+\beta'-\alpha', \rho+\omega-\alpha-\beta-\alpha'; \\ b_1, \dots, b_q, \rho+\beta', \rho+\omega-\alpha-\alpha', \rho+\omega-\beta-\alpha'; \end{matrix} \mu x \right], \quad (2.14)$$

provided that each member of the assertions (2.13) and (2.14) exists.

*Proof:* For convenience, we denote by  $\Omega(x)$  the left-hand side of the first assertion (2.13) of Theorem 1. Then, by applying the definition (1.6) and changing the order of integration and summation, we get

$$\begin{aligned} \Omega(x) &:= \left( I_{0,x}^{(\alpha, \alpha', \beta, \beta', \omega)} [t^{\rho-1} {}_p\gamma_q(\mu t)] \right) (x) \\ &= \left( I_{0,x}^{(\alpha, \alpha', \beta, \beta', \omega)} \left[ t^{\rho-1} \sum_{n=0}^{\infty} \frac{(a_1; \kappa)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \cdot \frac{(\mu t)^n}{n!} \right] \right) (x) \\ &= \sum_{n=0}^{\infty} \frac{(a_1; \kappa)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{\mu^n}{n!} \cdot \left( I_{0,x}^{(\alpha, \alpha', \beta, \beta', \omega)} [t^{\rho+n-1}] \right) (x), \end{aligned} \quad (2.15)$$

where the inversion of the order of integration and summation can be justified, under the conditions given with Theorem 1, by the absolute convergence of the integral involved and the uniform convergence of the series involved.

Since  $n \in \mathbb{N}_0$ , we can make use of the definition (2.11) with  $\rho$  replaced by  $\rho + n$  ( $n \in \mathbb{N}_0$ ). We thus find from (2.15) that

$$\begin{aligned} \Omega(x) &= x^{\rho+\omega-\alpha-\alpha'-1} \sum_{n=0}^{\infty} \frac{(a_1; \kappa)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \\ &\quad \cdot \frac{\Gamma(\rho+n)\Gamma(\rho+\beta'-\alpha'+n)\Gamma(\rho+\omega-\alpha-\beta-\alpha'+n)}{\Gamma(\rho+\beta'+n)\Gamma(\rho+\omega-\alpha-\alpha'+n)\Gamma(\rho+\omega-\beta-\alpha'+n)} \frac{(\mu x)^n}{n!} \\ &= x^{\rho+\omega-\alpha-\alpha'-1} \frac{\Gamma(\rho)\Gamma(\rho+\beta'-\alpha')\Gamma(\rho+\omega-\alpha-\beta-\alpha')}{\Gamma(\rho+\beta')\Gamma(\rho+\omega-\alpha-\alpha')\Gamma(\rho+\omega-\beta-\alpha')} \sum_{n=0}^{\infty} \frac{(a_1; \kappa)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \\ &\quad \cdot \frac{(\rho)_n (\rho+\beta'-\alpha')_n (\rho+\omega-\alpha-\beta-\alpha')_n}{(\rho+\beta')_n (\rho+\omega-\alpha-\alpha')_n (\rho+\omega-\beta-\alpha')_n} \frac{(\mu x)^n}{n!}. \end{aligned} \quad (2.16)$$



By interpreting the last member of (2.16) by means of the definition (1.6), we obtain the right-hand side of (2.13). Similarly, we can derive the second assertion (2.14) of Theorem 1 by applying the definitions (1.7) and (2.12). This completes the proof of Theorem 1.  $\square$

**Theorem 2.** Let  $x > 0$  and  $\kappa \geq 0$ . Suppose also that the parameters  $\alpha, \alpha', \beta, \beta', \mu, \omega, \rho \in \mathbb{C}$  are constrained by

$$\Re(\omega) > 0 \quad \text{and} \quad 0 < \Re(\rho) < 1 + \min \{ \Re(-\beta), \Re(\alpha + \alpha' - \omega), \Re(\alpha + \beta' - \omega) \}.$$

Then each of the following fractional integral formulas holds true:

$$\begin{aligned} & \left( I_{0,x}^{(\alpha, \alpha', \beta, \beta', \omega)} \left[ t^{\rho-1} {}_p\gamma_q \left( \frac{\mu}{t} \right) \right] \right) (x) = x^{\rho+\omega-\alpha-\alpha'-1} \\ & \cdot \frac{\Gamma(1-\rho-\beta)\Gamma(1-\rho-\omega+\alpha+\alpha')\Gamma(1-\rho-\omega+\alpha+\beta')}{\Gamma(1-\rho)\Gamma(1-\rho-\omega+\alpha+\alpha'+\beta')\Gamma(1-\rho+\alpha-\beta)} \\ & \cdot {}_{p+3}\gamma_{q+3} \left[ \begin{matrix} (a_1, \kappa), a_2, \dots, a_p, 1-\rho-\beta, 1-\rho-\omega+\alpha+\alpha', 1-\rho-\omega+\alpha+\beta'; \\ b_1, \dots, b_q, 1-\rho, 1-\rho-\omega+\alpha+\alpha'+\beta', 1-\rho+\alpha-\beta; \end{matrix} \frac{\mu}{x} \right] \end{aligned} \quad (2.17)$$

and

$$\begin{aligned} & \left( I_{x,\infty}^{(\alpha, \alpha', \beta, \beta', \omega)} \left[ t^{\rho-1} {}_p\Gamma_q \left( \frac{\mu}{t} \right) \right] \right) (x) = x^{\rho+\omega-\alpha-\alpha'-1} \\ & \cdot \frac{\Gamma(1-\rho-\beta)\Gamma(1-\rho-\omega+\alpha+\alpha')\Gamma(1-\rho-\omega+\alpha+\beta')}{\Gamma(1-\rho)\Gamma(1-\rho-\omega+\alpha+\alpha'+\beta')\Gamma(1-\rho+\alpha-\beta)} \\ & \cdot {}_{p+3}\Gamma_{q+3} \left[ \begin{matrix} (a_1, \kappa), a_2, \dots, a_p, 1-\rho-\beta, 1-\rho-\omega+\alpha+\alpha', 1-\rho-\omega+\alpha+\beta'; \\ b_1, \dots, b_q, 1-\rho, 1-\rho-\omega+\alpha+\alpha'+\beta', 1-\rho+\alpha-\beta; \end{matrix} \frac{\mu}{x} \right], \end{aligned} \quad (2.18)$$

provided that each member of the assertions (2.17) and (2.18) exists.

*Proof:* The proofs of the fractional integral formulas (2.17) and (2.18) would run parallel to those of (2.13) and (2.14) asserted by Theorem 1. We, therefore, choose to skip the details involved.  $\square$

### 3. Corollaries and Consequences of Theorems 1 and 2

Upon setting  $\alpha' = 0$  in Theorems 1 and 2, if we use the relationships (2.9) and (2.10), we can deduce the following interesting corollaries involving the generalized incomplete hypergeometric functions  ${}_p\gamma_q$  and  ${}_p\Gamma_q$  defined by (1.6) and (1.7), respectively, and the Saigo fractional integral operators

$$\left( \mathcal{I}_{0,x}^{\alpha, \beta, \omega} f \right) (x) \quad \text{and} \quad \left( \mathcal{I}_{x,\infty}^{\alpha, \beta, \omega} f \right) (x)$$

defined by (2.7) and (2.8), respectively.

**Corollary 1.** Let  $x > 0$  and  $\kappa \geq 0$ . Suppose also that the parameters  $\alpha, \beta, \mu, \omega, \rho \in \mathbb{C}$  are constrained by

$$\Re(\omega) > 0 \quad \text{and} \quad \Re(\rho) > \max \{0, \Re(\omega - \alpha - \beta)\}.$$

Then each of the following fractional integral formulas holds true:

$$\begin{aligned} \left( \mathcal{I}_{0,x}^{(\omega, \alpha-\omega, -\beta)} \left[ t^{\rho-1} {}_p\gamma_q(\mu t) \right] \right) (x) &= x^{\rho+\omega-\alpha-1} \frac{\Gamma(\rho)\Gamma(\rho+\omega-\alpha-\beta)}{\Gamma(\rho+\omega-\alpha)\Gamma(\rho+\omega-\beta)} \\ &\cdot {}_{p+2}\gamma_{q+2} \left[ \begin{matrix} (a_1, \kappa), a_2, \dots, a_p, \rho, \rho+\omega-\alpha-\beta; & \mu x \\ b_1, \dots, b_q, \rho+\omega-\alpha, \rho+\omega-\beta; & \end{matrix} \right] \end{aligned} \quad (3.1)$$

and

$$\begin{aligned} \left( \mathcal{I}_{x,\infty}^{(\omega, \alpha-\omega, -\beta)} \left[ t^{\rho-1} {}_p\Gamma_q(\mu t) \right] \right) (x) &= x^{\rho+\omega-\alpha-1} \frac{\Gamma(\rho)\Gamma(\rho+\omega-\alpha-\beta)}{\Gamma(\rho+\omega-\alpha)\Gamma(\rho+\omega-\beta)} \\ &\cdot {}_{p+2}\Gamma_{q+2} \left[ \begin{matrix} (a_1, \kappa), a_2, \dots, a_p, \rho, \rho+\omega-\alpha-\beta; & \mu x \\ b_1, \dots, b_q, \rho+\omega-\alpha, \rho+\omega-\beta; & \end{matrix} \right], \end{aligned} \quad (3.2)$$

provided that each member of the assertions (3.1) and (3.2) exists.

**Corollary 2.** Let  $x > 0$  and  $\kappa \geq 0$ . Suppose also that the parameters  $\alpha, \beta, \mu, \omega, \rho \in \mathbb{C}$  are constrained by

$$\Re(\omega) > 0 \quad \text{and} \quad 0 < \Re(\rho) < 1 + \min \{ \Re(-\beta), \Re(\alpha - \omega) \}.$$

Then each of the following fractional integral formulas holds true:

$$\begin{aligned} \left( \mathcal{I}_{x,\infty}^{(\omega, \alpha-\omega, -\beta)} \left[ t^{\rho-1} {}_p\gamma_q \left( \frac{\mu}{t} \right) \right] \right) (x) &= x^{\rho+\omega-\alpha-1} \frac{\Gamma(1-\rho-\beta)\Gamma(1-\rho-\omega+\alpha)}{\Gamma(1-\rho)\Gamma(1-\rho+\alpha-\beta)} \\ &\cdot {}_{p+2}\gamma_{q+2} \left[ \begin{matrix} (a_1, \kappa), a_2, \dots, a_p, 1-\rho-\beta, 1-\rho-\omega+\alpha; & \frac{\mu}{x} \\ b_1, \dots, b_q, 1-\rho, 1-\rho+\alpha-\beta; & \end{matrix} \right] \end{aligned} \quad (3.3)$$

and

$$\begin{aligned} \left( \mathcal{I}_{x,\infty}^{(\omega, \alpha-\omega, -\beta)} \left[ t^{\rho-1} {}_p\Gamma_q \left( \frac{\mu}{t} \right) \right] \right) (x) &= x^{\rho+\omega-\alpha-1} \frac{\Gamma(1-\rho-\beta)\Gamma(1-\rho-\omega+\alpha)}{\Gamma(1-\rho)\Gamma(1-\rho+\alpha-\beta)} \\ &\cdot {}_{p+2}\Gamma_{q+2} \left[ \begin{matrix} (a_1, \kappa), a_2, \dots, a_p, 1-\rho-\beta, 1-\rho-\omega+\alpha; & \frac{a}{x} \\ b_1, \dots, b_q, 1-\rho, 1-\rho+\alpha-\beta; & \end{matrix} \right], \end{aligned} \quad (3.4)$$

provided that each member of the assertions (3.3) and (3.4) exists.

**Remark 3.** Several further consequences of Corollaries 1 and 2 of this section can easily be derived by setting (for example)  $\beta = -\alpha$  or  $\beta = 0$ . Such interesting consequences of our results would involve the Erdélyi-Kober fractional integral operators  $\mathcal{E}_{0,x}^{\alpha,\eta}$  and  $\mathcal{K}_{x,\infty}^{\alpha,\eta}$ , the Riemann-Liouville fractional integral operator  $\mathcal{R}_{0,x}^\alpha$  and the Weyl fractional integral operator  $\mathcal{W}_{x,\infty}^\alpha$ . These relatively simpler fractional integral formulas for each of the generalized incomplete

hypergeometric functions  ${}_p\gamma_q$  and  ${}_p\Gamma_q$  defined by (1.6) and (1.7), respectively, can be deduced from Corollaries 1 and 2 by appropriately applying the following relationships (see, for example, (A. M. Mathai and Haubold, 2010)):

$$(\mathcal{R}_{0,x}^\alpha f)(x) := \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt = (\mathcal{I}_{0,x}^{\alpha,-\alpha,\omega} f)(x), \quad (3.5)$$

$$(\mathcal{W}_{x,\infty}^\alpha f)(x) := \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t) dt = (\mathcal{I}_{x,\infty}^{\alpha,-\alpha,\omega} f)(x), \quad (3.6)$$

$$(\mathcal{E}_{0,x}^{\alpha,\omega} f)(x) := \frac{x^{-\alpha-\omega}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^\omega f(t) dt = (\mathcal{I}_{0,x}^{\alpha,0,\omega} f)(x) \quad (3.7)$$

and

$$(\mathcal{K}_{x,\infty}^{\alpha,\omega} f)(x) := \frac{x^\omega}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\omega} f(t) dt = (\mathcal{I}_{x,\infty}^{\alpha,0,\omega} f)(x). \quad (3.8)$$

#### 4. Conclusion

We conclude our present investigation by remarking further that the results obtained here are useful in deriving various fractional integral formulas for each of the families of the generalized incomplete hypergeometric functions  ${}_p\gamma_q$  and  ${}_p\Gamma_q$  defined by (1.6) and (1.7), respectively, involving such relatively more familiar fractional integral operators as (for example) the Riemann-Liouville fractional integral operator  $\mathcal{R}_{0,x}^\alpha$  defined in (3.5), the Weyl fractional integral operator  $\mathcal{W}_{x,\infty}^\alpha$  defined in (3.6), and the Erdélyi-Kober fractional integral operators  $\mathcal{E}_{0,x}^{\alpha,\eta}$  and  $\mathcal{K}_{x,\infty}^{\alpha,\eta}$  defined in (3.7) and (3.8), respectively.

#### REFERENCES

- A. A. Kilbas, H. M. S. and Trujillo, J. J. (2006). *Theory and Applications of Fractional Differential Equations, North-Holland Mathematical Studies, Vol. 204*. Elsevier (North-Holland) Science Publishers, Amsterdam, London and New York.
- A. Erdélyi, W. Magnus, F. O. and Tricomi, F. G. (1954). *Tables of Integral Transforms, Vol. II*. McGraw-Hill Book Company, New York, Toronto and London.
- A. Erdélyi, W. Mangus, F. O. and Tricomi, F. G. (1953). *Higher Transcendental Functions, Vol. I*. McGraw-Hill Book Company, New York, Toronto and London.
- A. M. Mathai, R. K. S. and Haubold, H. J. (2010). *The H-Function: Theory and Applications*. Springer, New York, Dordrecht, Heidelberg and London.
- Abramowitz, M. and (Editors), I. A. S. (1972). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Tenth Printing, National Bureau of Standards*,

- Applied Mathematics Series 55*, National Bureau of Standards, Washington, D.C. Dover Publications, New York.
- Agarwal, P. (2012a). Further results on fractional calculus of saigo operators. *Appl. Appl. Math.*, 7:585–594.
- Agarwal, P. (2012b). Generalized fractional integration of the  $\overline{H}$ -function. *Matematiche (Catania)*, 68:107–118.
- Agarwal, P. and Jain, S. (2011). Further results on fractional calculus of srivastava polynomials. *Bull. Math. Anal. Appl.*, 3:167–174.
- Andrews, L. C. (1985). *Special Functions for Engineers and Applied Mathematicians*. Macmillan Company, New York.
- Appell, P. and de Fériet, J. K. (1926). *Fonctions Hypergéométriques et Hypersphériques; Polynômes d'Hermite*. Gauthier-Villars, Paris.
- Bailey, W. N. (1964). *Generalized Hypergeometric Series, Cambridge Tracts in Mathematics and Mathematical Physics, 32*. Cambridge University Press, Cambridge, London and New York, 1935; Reprinted by Stechert-Hafner Service Agency, New York and London.
- Carlson, B. C. (1977). *Special Functions of Applied Mathematics*. Academic Press, New York, San Francisco and London.
- Çetinkaya, A. (2013). The incomplete second appell hypergeometric functions. *Appl. Math. Comput.*, 219:8332–8337.
- Chaudhry, M. A. and Zubair, S. M. (2001). *On a Class of Incomplete Gamma Functions with Applications*. Chapman and Hall (CRC Press Company), Boca Raton, London, New York and Washington, D.C.
- F. W. J. Olver, D. W. Lozier, R. F. B. and Clark, C. W. (2010). *NIST Handbook of Mathematical Functions, [With 1 CD-ROM (Windows, Macintosh and UNIX)]*, U. S. Department of Commerce, National Institute of Standards and Technology, Washington, D. C. Cambridge University Press, Cambridge, London and New York.
- H. M. Srivastava, M. A. C. and Agarwal, R. P. (2012). The incomplete pochhammer symbols and their applications to hypergeometric and related functions. *Integral Transforms Spec. Funct.*, 23:659–683.
- J. A. T. Machado, V. K. and Mainardi, F. (2010). A poster about the recent history of fractional calculus. *Fract. Calc. Appl. Anal.*, 13:329–334.
- J. A. T. Machado, V. K. and Mainardi, F. (2011). Recent history of fractional calculus. *Commun. Nonlinear Sci. Numer. Simulation*, 16:1140–1153.
- K. B. Oldham, J. M. and Spanier, J. (2009). *An Atlas of Functions with Equator, the Atlas Function Calculator, Second Edition [With 1 CD-ROM (Windows)]*. Springer, Berlin, Heidelberg and New York.
- Kiryakova, V. (1993). *Generalized Fractional Calculus and Applications*. Longman Scientific and Technical, Harlow (Essex).
- Kiryakova, V. (2006). On two saigo's fractional integral operators in the class of univalent functions. *Fract. Calc. Appl. Anal.*, 9:160–176.
- Luke, Y. L. (1975). *Mathematical Functions and Their Approximations*. Academic Press, New York, San Francisco and London.
- Marichev, O. I. (1974). Volterra equation of mellin convolution type with a horn function in the

- kernel (in russian). *Izv. AN BSSR Ser. Fiz.-Mat. Nauk*, 1:128–129.
- McBride, A. C. (1979). *Fractional Calculus and Integral Transforms of Generalized Functions*. Pitman Publishing Limited, London.
- Miller, K. S. and Ross, B. (1993). *An Introduction to the Fractional Calculus and Fractional Differential Equations*. A Wiley-Interscience Publication, John Wiley and Sons, New York, Chichester, Brisbane, Toronto and Singapore.
- N. L. Johnson, S. K. and Balakrishnan, N. (1995). *Continuous Univariate Distributions, Vol. 2*. John Wiley and Sons, New York, Chichester, Brisbane and Toronto.
- Oldham, K. B. and Spanier, J. (1974). *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*. Academic Press, New York and London.
- Podlubny, I. (1999). *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications, Mathematics in Science and Engineering, Vol. 198*. Academic Press, New York, London, Sydney, Tokyo and Toronto.
- Rainville, E. D. (1971). *Special Functions*. Macmillan Company, New York, 1960; Reprinted by Chelsea Publishing Company, Bronx, New York.
- S. D. Purohit, D. L. S. and Kalla, S. (2011). Marichev-saigo-maeda fractional integration operators of the Bessel function. *Matematiche (Catania)* 57, pages 21–32.
- S. G. Samko, A. A. K. and Marichev, O. I. (1993). *Fractional Integrals and Derivatives: Theory and Applications*. Gordon and Breach Science Publishers, Reading, Tokyo, Paris, Berlin and Langhorne (Pennsylvania).
- Saigo, M. (1996). *On generalized fractional calculus operators*. Recent Advances in Applied Mathematics, Proceedings of the International Workshop held at Kuwait University (Kuwait, May 4-7, 1996), Kuwait University, Department of Mathematics and Computer Science, Kuwait.
- Saigo, M. and Maeda, N. (1998). *More generalization of fractional calculus, in: Transform Methods and Special Functions*. Proceedings of the Second International Workshop Dedicated to the 100th Anniversary of the Birth of Nikola Obreschkoff (Varna, August 23-30, 1996), P. Rusev, I. Dimovski and V. Kiryakova, Editors, Bulgarian Academy of Sciences, Institute of Mathematics and Informatics, Sofia.
- Slater, L. J. (1966). *Generalized Hypergeometric Functions*. Cambridge University Press, Cambridge, London and New York.
- Srivastava, H. M. and Choi, J. (2001). *Series Associated with the Zeta and Related Functions*. Kluwer Academic Publishers, Dordrecht, Boston and London.
- Srivastava, H. M. and Choi, J. (2012). *Zeta and q-Zeta Functions and Associated Series and Integrals*. Elsevier Science Publishers, Amsterdam, London and New York.
- Srivastava, H. M. and Karlsson, P. W. (1985). *Multiple Gaussian Hypergeometric Series, Halsted Press (Ellis Horwood Limited, Chichester)*. John Wiley and Sons, New York, Chichester, Brisbane and Toronto.
- Srivastava, H. M. and Kashyap, B. R. K. (1982). *Special Functions in Queuing Theory and Related Stochastic Processes*. Academic Press, New York and London.
- Srivastava, H. M. and Manocha, H. L. (1984). *A Treatise on Generating Functions, Halsted Press (Ellis Horwood Limited, Chichester)*. John Wiley and Sons, New York, Chichester,

Brisbane and Toronto.

- Srivastava, H. M. and Saigo, M. (1987). Multiplication of fractional calculus operators and boundary value problems involving the euler-darboux equation. *J. Math. Anal. Appl.*, 121:325–369.
- Srivastava, H. M. and Saxena, R. K. (2001). Operators of fractional integration and their applications. *Appl. Math. Comput.*, 118:1–52.
- Srivastava, R. (2013a). Some generalizations of pochhammer's symbol and their associated families of hypergeometric functions and hypergeometric polynomials. *Appl. Math. Inform. Sci.*, 7:2195–2206.
- Srivastava, R. (2013b). Some properties of a family of incomplete hypergeometric functions. *Russian J. Math. Phys.*, 20:121–128.
- Srivastava, R. and Cho, N. E. (2012). Generating functions for a certain class of incomplete hypergeometric polynomials. *Appl. Math. Comput.*, 219:3219–3225.
- Temme, N. M. (1996). *Special Functions: An Introduction to Classical Functions of Mathematical Physics*. John Wiley and Sons, New York, Chichester, Brisbane and Toronto.
- W. Magnus, F. O. and Soni, R. P. (1966). *Formulas and Theorems for the Special Functions of Mathematical Physics, Third Enlarged Edition, Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Band 52*. Springer-Verlag, Berlin, Heidelberg and New York.
- Watson, G. N. (1944). *A Treatise on the Theory of Bessel Functions, Second Edition*. Cambridge University Press, Cambridge, London and New York.
- Whittaker, E. T. and Watson, G. N. (1973). *A Course of Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions; with an Account of the Principal Transcendental Functions, Fourth Edition (Reprinted)*. Cambridge University Press, Cambridge, London and New York.