Applications and Applied Mathematics: An International Journal (AAM)

Volume 8 | Issue 1 Article 5

6-2013

Stability of Multiwavelet Frames with Different Matrix Dilations and Matrix Translations

F. A. Shah
University of Kashmir

Sunita Goyal
JVMGRR College

Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 8, Issue 1 (June 2013), pp. 151 - 160

Stability of Multiwavelet Frames with Different Matrix Dilations and Matrix Translations

F. A. Shah
Department of Mathematics
University of Kashmir
South Campus, Anantnag-192101
Jammu and Kashmir, India
fashah79@gmail.com

Sunita Goyal
Department of Mathematics
JVMGRR College, Dadri -127306
Haryana, India
sunitagoel2011@gmail.com

Received: December 03, 2012; Accepted: May 11, 2013

Abstract

In this paper, we study the stability of multiwavelet frames with different matrix dilations and matrix translations by means of operator theory and show that these frames remain stable over some kinds of perturbations of the basic generators.

Keywords: Wavelet; multiwavelet frame; linear operator; matrix dilation; matrix translation

MSC 2010 No.: 42C40; 42C15; 41R58; 65T60

1. Introduction

Frames were first introduced by Duffin and Schaeffer (1952) in the context of non-harmonic Fourier series. Outside of signal processing, frames did not seem to generate much interest until the seminal work of Daubechies, Grossmann, and Meyer (1986). They combined the theory of continuous wavelet transforms with the theory of frames to introduce wavelet (affine) frames for $L^2(\mathbb{R})$. Since then the theory of frames began to be more widely investigated, and now it is found to be useful in signal processing, image processing, harmonic analysis, sampling theory, data transmission with erasures, quantum computing and medicine. Recently, more applications of the theory of frames are found in diverse areas including optics, filter banks, signal detection and in the study of Bosev spaces and Banach spaces. We refer [Christensen (2003), Kovacevic and Chebira (2007a, b)] for an introduction to frame theory and its applications.

The wavelet frames and orthonormal wavelets, which are developed in parallel, are the two main theories in wavelet analysis. Wavelet frames are different from the orthonormal wavelets because of redundancy. By sacrificing orthonormality and allowing redundancy, the wavelet frames become much easier to construct than the orthonormal wavelets. An important problem in practice is therefore to determine conditions for the wavelet system

$$\{ \psi_{j,k} =: a^{j/2}\, \psi(a^j x - bk):\ a > 1,\ b > 0,\ j,\ k \in \mathbb{Z} \} \text{ to be a frame for } L^2(\mathbb{R}).$$

Therefore, it is important to study the stability of wavelet frames and a wavelet frame $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is called a *stable wavelet frame*, if there is any small perturbation on either a, b, j, k *or* ψ, the system $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is still wavelet frame for $L^2(\mathbb{R})$. Many results in this direction have been established during last two decades, for example, Favier and Zalik (1995) studied under what conditions the wavelet frames are stable if ψ is replaced by another function ϕ or the k are replaced by approximations $\{\lambda_{j,k}\}$, Balan (1997) studied the perturbation of translation parameter b which was first considered by Daubechies (1992) for the Meyer wavelet basis. More result can be found in [Jing (1999), Zhu (2001), Wang and Cheng (2006)] and the references therein.

In recent years, considerable attention has been given to multiwavelet frames as an important generalization of wavelet frames. Multiwavelet frames have more desired properties than wavelet frames and have found their applications in signal processing, image compression, computer graphics, tomography and so on. Chui and Shi (2000) studied multiwavelet frames and establishes a complete characterization to multiwavelet frames for an arbitrary real dilation factor $a > 1$ and real translation factor $b > 0$. On the other hand, Bownik (2001) presented a new approach to characterize multiwavelet frames by means of basic equations in the Fourier domain whereas Chui and his colleagues (2002) continued this work, and gave a complete characterization formula for multiwavelet frames with arbitrary matrix dilations A.

Recently, Li *et al*. (2009) extended the results of Bownik (2001), Chui *et al.* (2002) and Chui and Shi (2000) to the case when all the dilation factors a_ℓ are different and all the translation factors b_ℓ are not same. Subsequently, Li and Shi (2010) studied these result for different matrix dilations A_ℓ and matrix translations B_ℓ.

Motivated and inspired by the one-dimensional fundamental works in [Balan (1997), Chui and Shi (2000), Favier and Zalik (1995)] and the higher dimension works in [Chui *et al.* (2002), Li *et al.* (2009), Li and Shi (2010)], in this article, we shall investigate the stability of multiwavelet frames with different matrix dilations and matrix translations and some meaningful results are obtained by means of operator theory.

2. Main Results

We begin this section with the definition of multiwavelet frame and in the rest of this paper, we use $\mathbb{Z}$ and $\mathbb{R}^d$ to denote the sets of all integers and d-tuples of real numbers, respectively. Let $\boldsymbol{A} = (A_1, A_2, A_3, \dots, A_\ell)$ and $\boldsymbol{B} = (B_1, B_2, B_3, \dots, B_\ell)$ be two given matrix vectors, where A_ℓ, $\ell = 1, 2, \dots, L$ are $d \times d$ dilation matrices i.e., all the eigenvalues λ of A_ℓ satisfy $|\lambda| > 1$ and B_ℓ, $\ell = 1, 2, \dots, L$ are $d \times d$ non-singular matrices. For given $\boldsymbol{\Psi} := \{\psi^1, \psi^2, \dots, \psi^\ell\} \subset L^2(\mathbb{R}^d)$, define the multiwavelet system

$$\boldsymbol{\Psi}(\boldsymbol{A}, \boldsymbol{B}) := \{\psi^\ell_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^d, \ell = 1, 2, \dots, L\}, \qquad (2.1)$$

where $\psi^\ell_{j,k} = |A_\ell|^{j/2} \psi^\ell(A_\ell{}^j . - B_\ell k)$. The multiwavelet system $\boldsymbol{\Psi}(\boldsymbol{A}, \boldsymbol{B})$ is called a *multiwavelet frame* with respect to dilation $\boldsymbol{A}$ and translation $\boldsymbol{B}$, if there exist positive numbers $0 < C \le D < \infty$ such that for all $f \in L^2(\mathbb{R}^d)$

$$C \, \| f \|^2 \le \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \left| \langle f, \psi^\ell_{j,k} \rangle \right|^2 \le D \, \| f \|^2 . \qquad (2.2)$$

The largest constant C and the smallest constant D for which (2.2) holds are called *multiwavelet frame bounds*. A multiwavelet frame is a *tight multiwavelet frame* if C and D are chosen so that $C = D$ and if only the upper bound holds in the above inequality, then $\boldsymbol{\Psi}(\boldsymbol{A}, \boldsymbol{B})$ is said to be a *Bessel sequence* with Bessel constant D.

Let $S : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$ be the surjective operator such that

$$S\psi^\ell(f) = \tilde{\psi}^\ell(f), \quad \ell = 1, 2, \dots, L .$$

Then, similar to $\boldsymbol{\Psi}$ and $\psi^\ell_{j,k}$, we have $\widetilde{\boldsymbol{\Psi}} := \{\tilde{\psi}^1, \tilde{\psi}^2, \dots, \tilde{\psi}^\ell\} \subset L^2(\mathbb{R}^d)$ and

$$\widetilde{\boldsymbol{\Psi}}(\boldsymbol{A}, \boldsymbol{B}) := \{\tilde{\psi}^\ell_{j,k} : j \in \mathbb{Z}, \ k \in \mathbb{Z}^d, \ell = 1, 2, \dots, L\}, \qquad (2.3)$$

where

$$\tilde{\psi}^\ell_{j,k} = |A_\ell|^{j/2} \tilde{\psi}^\ell(A_\ell{}^j . - B_\ell k).$$

We need the following lemma to prove our results (Christensen (2003)).

Lemma 2.1.

Let H_1 and H_2 be Hilbert spaces, $\lambda_1, \lambda_2 \in [0, 1]$ and $P, Q : H_1 \longrightarrow H_2$ are linear operators such that

$$\|P(x) - Q(x)\| \le \lambda_1 \|P(x)\| + \lambda_2 \|Q(x)\|, \quad \text{for all } x \in H_1 .$$

Then,

$$codim\ H_2\ \overline{P(x)} = codim\ H_2\ \overline{Q(x)}\ .$$

Theorem 2.2.

Let $\boldsymbol{\Psi}\ (\boldsymbol{A}, \boldsymbol{B})$ defined by (2.1) is a multiwavelet frame for $L^2(\mathbb{R}^d)$. Then, the system $\widetilde{\boldsymbol{\Psi}}\ (\boldsymbol{A}, \boldsymbol{B})$ defined by (2.3) is also a multiwavelet frame for $L^2(\mathbb{R}^d)$.

Proof:

Since $\boldsymbol{\Psi}\ (\boldsymbol{A}, \boldsymbol{B})$ is a multiwavelet frame for $L^2(\mathbb{R}^d)$ and by virtue of the operator S, $S\psi^{\ell}_{j,k}(f) = \widetilde{\psi}^{\ \ell}_{j,k}(f)$, $j \in \mathbb{Z}, k \in \mathbb{Z}^d, \ell = 1, 2, \dots, L,$ we have

$$\sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \left| \langle g,\ \widetilde{\psi}^{\ \ell}_{j,k}(f) \rangle \right|^2 = \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \left| \langle g,\ S\psi^{\ell}_{j,k}(f) \rangle \right|^2$$

$$= \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \left| \langle S^* g,\ \psi^{\ell}_{j,k}(f) \rangle \right|^2$$

$$\leq\ D\ \|S^* g\|_2^2$$

$$\leq\ D\ \|S^*\|_2^2\ \ \|g\|_2^2. \qquad (2.4)$$

Similarly,

$$\sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \left| \langle g,\ \widetilde{\psi}^{\ \ell}_{j,k}(f) \rangle \right|^2 = \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \left| \langle g,\ S\psi^{\ell}_{j,k}(f) \rangle \right|^2$$

$$= \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \left| \langle S^* g,\ \psi^{\ell}_{j,k}(f) \rangle \right|^2$$

$$\geq\ C\ \|S^* g\|_2^2$$

$$\geq\ C\ \|(S^*)^{-1}\|_2^2\ \ \|g\|_2^2. \qquad (2.5)$$

Combining (2. 4) and (2. 5), we get the desired result.

Theorem 2.3.

Let $\boldsymbol{\Psi}\,(\boldsymbol{A}\,,\boldsymbol{B})$ as in (2.1) be a multiwavelet frame for $L^2(\mathbb{R}^d)$ with frame bounds C and D. Suppose that $M > 0, \alpha \geq 0,\ 0 \leq \beta \leq 1$, and $(1-\alpha)\sqrt{C} > \sqrt{M}$, the collection $\widetilde{\boldsymbol{\Psi}}\,(\boldsymbol{A}\,,\boldsymbol{B})$ defined by (2.3) satisfies

$$\left\| \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} G_{j,k}^{\ell} \left[\psi_{j,k}^{\ell}(f) - \widetilde{\psi}_{j,k}^{\ell}(f)\right] \right\|_2 \leq \alpha \left\| \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} G_{j,k}^{\ell}\, \psi_{j,k}^{\ell}(f) \right\|_2 + \beta \left\| \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} G_{j,k}^{\ell}\, \widetilde{\psi}_{j,k}^{\ell}(f) \right\|_2 + \sqrt{M} \left[\sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \left| G_{j,k}^{\ell} \right|^2 \right]^{1/2}. \quad (2.6)$$

Then, $\widetilde{\boldsymbol{\Psi}}\,(\boldsymbol{A}\,,\boldsymbol{B})$ forms a multiwavelet frame for $L^2(\mathbb{R}^d)$ with lower and upper frame bounds

$$\frac{\left[(1-\alpha)\sqrt{C} - \sqrt{M}\right]^2}{(1+\beta)^2} \quad and \quad \frac{\left[(1+\alpha)\sqrt{D} + \sqrt{M}\right]^2}{(1-\beta)^2},$$

respectively.

Proof:

By equation (2. 6), we have

$$\left\| \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} G_{j,k}^{\ell}\, \widetilde{\psi}_{j,k}^{\ell}(f) \right\|_2 \leq \frac{1+\alpha}{1-\beta} \left\| \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} G_{j,k}^{\ell}\, \psi_{j,k}^{\ell}(f) \right\|_2 + \frac{\sqrt{M}}{1-\beta} \left[\sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \left| G_{j,k}^{\ell} \right|^2 \right]^{1/2}.$$

Also, by virtue of *Hahn-Banach theorem*, there exist $f^* \in (L^2(\mathbb{R}^d))^*$ with $\|f^*\|_2 = 1$ such that

$$\left\| \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} G_{j,k}^{\ell}\, \psi_{j,k}^{\ell}(f) \right\|_2 = f^* \left[\sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} G_{j,k}^{\ell}\, \psi_{j,k}^{\ell}(f) \right]$$

$$\leq \left[\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\left|G_{j,k}^{\ell}\right|^2\right]^{1/2}\left[\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\left|f^*\left(\psi_{j,k}^{\ell}(f)\right)\right|^2\right]^{1/2}$$

$$\leq \sqrt{D}\left[\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\left|G_{j,k}^{\ell}\right|^2\right]^{1/2}.$$

Hence,

$$\left\|\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}G_{j,k}^{\ell}\,\tilde{\psi}_{j,k}^{\ell}(f)\right\|_2 \leq \frac{\left[(1+\alpha)\sqrt{D}+\sqrt{M}\right]}{(1-\beta)}\left[\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\left|G_{j,k}^{\ell}\right|^2\right]^{\frac{1}{2}}.$$

Moreover from the above formula, for every $f^*\in(L^2(\mathbb{R}^d))^* = L^2(\mathbb{R}^d)$ one may get

$$\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\left|g^*\left(\tilde{\psi}_{j,k}^{\ell}(f)\right)\right|^2 = g^*\left[\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\frac{\left|g^*\left(\tilde{\psi}_{j,k}^{\ell}(f)\right)\right|^2}{g^*\left(\tilde{\psi}_{j,k}^{\ell}(f)\right)}\,\tilde{\psi}_{j,k}^{\ell}(f)\right]$$

$$\leq \|g^*\|_2\,\frac{\left[(1+\alpha)\sqrt{D}+\sqrt{M}\right]}{(1-\beta)}\left[\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\left|g^*\left(\tilde{\psi}_{j,k}^{\ell}(f)\right)\right|^2\right]^{1/2}$$

$$\leq \|g\|_2^2\,\frac{\left[(1+\alpha)\sqrt{D}+\sqrt{M}\right]^2}{(1-\beta)^2}.$$

Therefore, we have

$$\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\left|\langle f,\ \tilde{\psi}_{j,k}^{\ell}\rangle\right|^2 \leq \|f\|_2^2\,\frac{\left[(1+\alpha)\sqrt{D}+\sqrt{M}\right]^2}{(1-\beta)^2}, \tag{2.7}$$

for all $f\in L^2(\mathbb{R}^d)$.

Now, in order to find the lower bound for the system $\tilde{\boldsymbol{\Psi}}(\boldsymbol{A},\boldsymbol{B})$ to be frame for $L^2(\mathbb{R}^d)$, we define an operator $P: L^2(\mathbb{R}^d)\longrightarrow l^2(\mathbb{Z}^d)$ by

$$P(f) = \{ \langle f, \ \psi^{\ell}_{j,k}(f) \rangle \ : \ j \in \mathbb{Z}, k \in \mathbb{Z}^d, \ell = 1,2,\dots,L \}$$

such that

$$\sqrt{C} \, \| f \|_2^2 \leq \| P(f) \|_2^2 \leq \sqrt{D} \, \| f \|_2^2 , \quad \text{for all } f \in L^2(\mathbb{R}^d).$$

Suppose $\Omega = P(L^2(\mathbb{R}^d))$, then it is easy to verify that P^{-1} is a linear homeomorphism with $\| P^{-1} \| \leq \sqrt{C^{-1}}$ and Ω is closed subspace of $l^2(\mathbb{Z}^d)$, so we let $Q : l^2(\mathbb{Z}^d) \longrightarrow \Omega$ to be an orthogonal projection. Then, we have

$$\| T \| \leq \| P^{-1} \| \, \| Q \| \leq \sqrt{C^{-1}},$$

where $T = P^{-1} Q : l^2(\mathbb{Z}^d) \longrightarrow L^2(\mathbb{R}^d)$. Suppose that $T^* : L^2(\mathbb{R}^d) \longrightarrow l^2(\mathbb{Z}^d)$ be the conjugate of T such that

$$T^*(f) = \{ G^{\ell}_{j,k}(f) \ : \ j \in \mathbb{Z}, k \in \mathbb{Z}^d, \ell = 1,2,\dots,L \}.$$

Then, we have

$$\left[\sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \left| G^{\ell}_{j,k} \right|^2 \right]^{1/2} \leq \| T \| \, \| f \|_2 \leq \sqrt{C^{-1}} \, \| f \|_2 . \tag{2.8}$$

Also for every $g \in L^2(\mathbb{R}^d)$, we obtain

$$\begin{aligned} \left\langle \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} G^{\ell}_{j,k}(f) \, \psi^{\ell}_{j,k}(f) \, , \ g \right\rangle &= \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} G^{\ell}_{j,k}(f) \langle \psi^{\ell}_{j,k}(f), \ g \rangle \\ &= \langle T^*(f), \ P(g) \rangle \\ &= \langle f, \ TP(g) \rangle \\ &= \langle f, \ P^{-1} QP(g) \rangle \\ &= \langle f, \ g \rangle . \end{aligned}$$

or equivalently

$$f = \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} G^{\ell}_{j,k}(f) \, \psi^{\ell}_{j,k}(f) .$$

Define a linear operator $U : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$ by

$$U(f) = \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} G_{j,k}^{\ell}(f)\, \tilde{\psi}_{j,k}^{\ell}(f)\,.$$

Then, by virtue of (2. 6) and (2. 8), we get

$$\| f - U(f) \|_2 \le \left[\alpha + \frac{\sqrt{M}}{\sqrt{C}} \right] \| f \|_2 + \beta \, \| U(f) \|_2$$

and

$$\frac{(1-\alpha)\sqrt{C} - \sqrt{M}}{(1+\beta)\sqrt{C}} \; \| f \|_2 \le \| U(f) \|_2 \le \frac{(1+\alpha)\sqrt{C} + \sqrt{M}}{(1-\beta)\sqrt{C}} \; \| f \|_2\,.$$

Moreover, it is clear from above that U is one-one bounded linear operator and $U(L^2(\mathbb{R}^d))$ is a closed subspace of $L^2(\mathbb{R}^d)$. Therefore, by Lemma 2.1, it follows that U is an invertible bounded linear operator with

$$\| U^{-1} \|_2 \le \frac{(1+\beta)\sqrt{C}}{\left[(1-\alpha)\sqrt{C} - \sqrt{M}\,\right]}\,.$$

Thus, for every $f, g \in L^2(\mathbb{R}^d)$, we have

$$\begin{aligned}
|\langle f, U^*(g) \rangle| &= |\langle U(f), g \rangle| \\
&= \left| \textstyle\sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} G_{j,k}^{\ell}(f) \langle \tilde{\psi}_{j,k}^{\ell}(f), g \rangle \right| \\
&\le \left[\textstyle\sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \left| G_{j,k}^{\ell} \right|^2 \right]^{1/2} \left[\textstyle\sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \left| \langle g, \tilde{\psi}_{j,k}^{\ell}(f) \rangle \right|^2 \right]^{1/2} \\
&\le \sqrt{C^{-1}} \| f \|_2 \left[\textstyle\sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \left| \langle g, \tilde{\psi}_{j,k}^{\ell}(f) \rangle \right|^2 \right]^{1/2}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\| U^*(g) \|_2 &= sup_{\|f\| \le 1} \; |\langle f, U^*(g) \rangle| \\
&\le \sqrt{C^{-1}} \left[\textstyle\sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \left| \langle g, \tilde{\psi}_{j,k}^{\ell}(f) \rangle \right|^2 \right]^{1/2}.
\end{aligned}$$

Further,

$$\begin{aligned}
\| g \|_2 &= \| (U^*)^{-1} U^*(g) \|_2 \\
&\le \| U^{-1} \|_2 \, \| U^*(g) \|_2
\end{aligned}$$

$$\leq \frac{(1+\beta)\sqrt{C}}{\left[(1-\alpha)\sqrt{C}-\sqrt{M}\,\right]}\sqrt{C^{-1}}\left[\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\left|\langle g,\ \tilde{\psi}_{j,k}^{\ell}(f)\rangle\right|^2\right]^{1/2}.$$

Therefore, for every $f \in L^2(\mathbb{R}^d)$, we have

$$\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\left|\langle f,\ \tilde{\psi}_{j,k}^{\ell}\rangle\right|^2 \geq \|f\|_2^2\ \frac{\left[(1-\alpha)\sqrt{C}-\sqrt{M}\,\right]^2}{(1+\beta)^2}. \tag{2.9}$$

The proof of theorem is followed by (2. 7) and (2. 9).

3. Conclusion

The notion of multiwavelet frame associated with different matrix dilations and matrix translations is introduced. Their stability properties are investigated by means operator theory method. We give a proof that these multiwavelet frames remain stable over some kinds of perturbations of the basic generators.

REFERENCES

Balan, R. (1997). Stability theorems for Fourier frames and wavelet Riesz basis, *J. Fourier Anal. Appl.* Vol. 3, pp. 499-504.

Bownik, M. (2001). On the characterization of multiwavelets in $L^2(\mathbb{R}^n)$, *Proc. Amer. Math. Soc.* Vol. 129(11), pp. 3265-3274.

Christensen, O. (2003). *An Introduction to Frames and Riesz Bases,* Birkhauser, Boston.

Chui, C. K., Czaja, W., Maggioni, M. and G. Weiss. (2002). Characterization of general tight wavelet frames with matrix dilations and tightness preserving oversampling, *J. Fourier Anal. Appl.* Vol. 8(2), pp. 174-200.

Chui, C. K. and Shi, X. L. (2000). Orthonormal wavelets and tight frames with arbitrary real dilation, *Appl. Comput. Harmon. Anal.* Vol. 9(2), pp. 243-264.

Daubechies, I. (1992). *Ten Lectures on Wavelets*, CBMS-NSF Regional Conferences in Applied Mathematics, SIAM, Philadelphia.

Daubechies, I., Grossmann, A. and Meyer, Y. (1986). Painless non-orthogonal expansions, *J. Math. Phys.* Vol. 27(5), pp. 1271-1283.

Duffin, R. J. and Shaeffer, A. C. (1952). A class of non-harmonic Fourier series, *Trans. Amer. Math. Soc.* Vol. 72, pp. 341-366.

Favier, S. J. and Zalik, R. A. (1995). On the stability of frames and Riesz basis, *Appl. Comput. Harmon. Anal.* Vol. 2, pp. 160-173.

Jing, Z. (1999). On the stability of wavelet and Gabor frames (Riesz bases), *J. Fourier Anal. Appl.* Vol. 1, pp. 105-125.

Kovacevic, J. and Chebira, A. (2007a). Life beyond bases: the advent of frames-I, *IEEE Sig. Process. Magz.* Vol. 24(4), pp. 86-104.

Kovacevic, J. and Chebira, A. (2007b). Life beyond bases: the advent of frames-II, *IEEE Sig. Process. Magz.* Vol. 24(5) (2007), pp. 115-125.

Li, B. B., Li, D. F., and Shi, X. L. (2009). Characterization of multiwavelets with different real dilations and translations, *Acta Math. Hungar.* Vol. 124, pp. 41-57.

Li, D. F. and Shi, X. L. (2010). Tight multiwavelet frames with different matrix dilations and matrix translations, *Numer. Funct. Anal. Optimiz.* Vol. 31(7), pp. 798-813.

Wang, G. and Cheng, Z. (2006). On the stability of multiwavelet frames, in *Wavelet Analysis and Applications,* T. Qian, M. Vai and X. Yuesheng eds. Birkhauser, Bessel, pp. 83-90.

Zhu, Y. C. (2001). On the stability of q-frames and *p*-Riesz basis in Banach spaces, *Chinese Anal. Math.* Vol. 22 A (3), pp. 359-364.