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Further Results on Fractional Calculus of Saigo Operators

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Abstract

A significantly large number of earlier works on the subject of fractional calculus give interesting account of the theory and applications of fractional calculus operators in many different areas of mathematical analysis (such as ordinary and partial differential equations, integral equations, special functions, summation of series, et cetera). The main object of the present paper is to study and develop the Saigo operators. First, we establish two results that give the image of the product of multivariable H-function and a general class of polynomials in Saigo operators. On account of the general nature of the Saigo operators, multivariable H-function and a general class of polynomials a large number of new and Known Images involving Riemann-Liouville and Erde'lyi-Kober fractional integral operators and several special functions notably generalized Wright hypergeometric function, Mittag-Leffler function, Whittaker function follow as special cases of our main findings. Results given by Kilbas, Kilbas and Sebastian, Saxena et al. and Gupta et al., follow as special cases of our findings.

Keywords: Fractional integral operators by Saigo; Riemann-Liouville and Erde'lyi-Kober; multivariable H-function; general class of polynomials; Mittag-Leffler functions

MSC 2010 No.: 26A33, 33C45, 33C60, 33C70

1. Introduction

The fractional integral operator involving various special functions, have found significant importance and applications in various sub-field of applicable mathematical analysis. Since last four decades, a number of workers like Love(1967), McBride (1982), Kalla (1969, 1969), Kalla and Saxena (1969, 1974), Saxena et al. (2009), Saigo (1978, 1979, 1980), Kilbas (2005), Kilbas and Sebastian (2008) and Kiryakova (1994, 2008), etc. have studied in depth, the properties, applications and different extensions of various hypergeometric operators of fractional integration. A detailed account of such operators along with their properties and applications can be found in the research monographs by Smako, Kilbas and Marichev (1993), Miller and Ross (1993), Kiryakova (1994, 2008), Kilbas, Srivastava and Trujillo (2006) and Debnath and Bhatta (2006).

A useful generalization of the hypergeometric fractional integrals, including the Saigo operators (1978, 1979, 1980), has been introduced by Marichev (1974) [see details in Samko et al. (1993) and also see Kilbas and Saigo (2004, p.258)] as follows:

Let α,β,η be complex numbers and $x > 0$, than the generalized fractional integral operators [the Saigo operators (1978)] involving Gaussian hypergeometric function are defined by the following equations:

$$\left(I_{0^+}^{\alpha,\beta,\eta} f\right)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x}\right) f(t)\,dt, \qquad (\Re(\alpha)>0), \tag{1}$$

and

$$\left(I_{-}^{\alpha,\beta,\eta} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta} {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{x}{t}\right) f(t)\,dt, \qquad (\Re(\alpha)>0). \tag{2}$$

where ${}_2F_1(.)$ is the Gaussian hypergeometric function defined by:

$${}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}. \tag{3}$$

When $\beta = -\alpha$, the above equations (1) and (2) reduce to the following classical Riemann-Liouville fractional integral operator [see Samko et al. (1993, p. 94), equations (5.1), (5.3)]:

$$\left(I_{0^+}^{\alpha,-\alpha,\eta} f\right)(x) = \left(I_{0^+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\,dt, \qquad (x>0) \tag{4}$$

and

$$\left(I_{-}^{\alpha,-\alpha,\eta} f\right)(x) = \left(I_{-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{\infty}(t-x)^{\alpha-1} f(t)\,dt, \qquad (x>0). \tag{5}$$

Again, if $\beta = 0$, the equations (1) and (2) reduce to the following Erde'lyi-Kober fractional integral operator [see Samko et al. (1993, p. 322), equations (18.5), (18.6)]:

$$\left(I_{0^+}^{\alpha,0,\eta} f\right)(x) = \left(I_{\eta,\alpha}^{+} f\right)(x) = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)}\int_{0}^{x}(x-t)^{\alpha-1} t^{\eta} f(t)\,dt, (x>0) \tag{6}$$

and

$$\left(I_{-}^{\alpha,0,\eta} f\right)(x) = \left(K_{\eta,\alpha}^{-} f\right)(x) = \frac{x^{\eta}}{\Gamma(\alpha)}\int_{x}^{\infty}(t-x)^{\alpha-1} t^{-\alpha-\eta} f(t)\,dt, (x>0) \tag{7}$$

Recently, Gupta et al. (2010) have obtained the images of the product of two H-functions in the Saigo operator given by (1) and (2) thereby generalizing several important results obtained earlier by Kilbas, Kilbas and Sebastian, Saxena et al., as mentioned in the papers cited above. It has recently become a subject of interest for many researchers in the field of fractional calculus and its applications. Motivated by these avenues of applications, a number of workers have made use of the fractional calculus operators to obtain the image formulas. The aim of the present paper is to obtain two results that give the images of the product of multivariable H-function and a general class of polynomials in Saigo operators.

The H-function of several variables is defined and represented as follows; Srivastava et al. [1982, pp. 251-252, equations (C.1)-(C.3)]:

$$H[z_1,...,z_r] \equiv H_{P,Q:P_1,Q_1;...;P_r,Q_r}^{0,N:M_1,N_1;...;M_r,N_r}\left[\begin{array}{c|l} z_1 & (a_j;\alpha_j',...,\alpha_j^{(r)})_{1,p}:(c_j',\gamma_j')_{1,p_1};...;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r} \\ \vdots & \\ z_r & (b_j;\beta_j',...,\beta_j^{(r)})_{1,q}:(d_j',\delta_j')_{1,q_1};...;(f_j^{(r)},F_j^{(r)})_{1,q_r} \end{array}\right]$$

$$= \left(\frac{1}{2\pi i}\right)^r \int_{L_1}\cdots\int_{L_r} \phi_1(\xi_1)...\phi_r(\xi_r)\psi(\xi_1,...\xi_r) z_1^{\xi_1}...z_r^{\xi_r} d\xi_1...d\xi_r, \tag{8}$$

where

$$\phi_i(\xi_i) = \frac{\prod_{j=1}^{M_i}\Gamma(d_j^{(i)}-\delta_j^{(i)}\xi_i)\prod_{j=1}^{N_i}\Gamma(1-c_j^{(i)}+\gamma_j^{(i)}\xi_i)}{\prod_{j=M_i+1}^{Q_i}\Gamma(1-d_j^{(i)}+\delta_j^{(i)}\xi_i)\prod_{j=N_i+1}^{P_i}\Gamma(c_j^{(i)}-\gamma_j^{(i)}\xi_i)}, \tag{9}$$

and

$$\psi(\xi_1,...,\xi_r) = \frac{\prod_{j=1}^{N}\Gamma(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}\xi_i)}{\prod_{j=1}^{Q}\Gamma(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}\xi_i)\prod_{j=N+1}^{P}\Gamma(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}\xi_i)}, \forall i \in \{1,...,r\}. \tag{10}$$

It is assumed that the various H-functions of several variables occurring in the paper always satisfy the appropriate existence and convergence conditions corresponding appropriately to those recorded in the book by Srivastava et al. [(1982), pp. 251-253, equations (C.4)-(C.6)]. In case $r = 2$, equation (8) reduces to the H-function of two variables; Srivastava et al. [(1982), p. 82, equation (6.1.1)].

Also, $S_n^m[x]$ occurring in the sequel denotes the general class of polynomials introduced by Srivastava [1972, p. 1, equation (1)]:

$$S_n^m[x] = \sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}x^k, n = 0,1,2,\cdots, \tag{11}$$

where m is an arbitrary positive integer and the coefficients $A_{n,k}\,(n,k \geq 0)$ are arbitrary constants, real or complex. On suitably specializing the coefficients $A_{n,k}, S_n^m[x]$ yields a number of known polynomials as its special cases. These include, among others, the Hermite polynomials, the Jacobi polynomials, the Laguerre polynomials, the Bessel polynomials, the Gould–Hopper polynomials, the Brafman polynomials and several others [see Srivastava and Singh (1983), pp. 158–161].

2. Preliminary Lemmas

The following lemmas will be required to establish our main results.

Lemma 1. Kilbas and Sebastain [(2008), p. 871, equations (15) to (18)].

Let $\alpha,\beta,\eta\in\mathbb{C}$ be such that $\left[\mathrm{Re}(\alpha)>0\right]$ and $\left[\mathrm{Re}(\mu)>\max\{0,\mathrm{Re}(\beta-\eta)\}\right]$, then there holds the following relation:

$$\left(I_{0^+}^{\alpha,\beta,\eta}t^{\mu-1}\right)(x)=\frac{\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu+\alpha+\eta)\Gamma(\mu-\beta)}x^{\mu-\beta-1}. \tag{12}$$

In particular, if $\beta=-\alpha$ and $\beta=0$ in (11), we have:

$$\left(I_{0^+}^{\alpha}t^{\mu-1}\right)(x)=\frac{\Gamma(\mu)}{\Gamma(\mu+\alpha)}x^{\mu+\alpha-1},\qquad \mathrm{Re}(\alpha)>0,\mathrm{Re}(\mu)>0, \tag{13}$$

$$\left(I_{\eta,\alpha}^{+}t^{\mu-1}\right)(x)=\frac{\Gamma(\mu+\eta)}{\Gamma(\mu+\alpha+\eta)}x^{\mu-1},\qquad \mathrm{Re}(\alpha)>0,\mathrm{Re}(\mu)>-\mathrm{Re}(\eta). \tag{14}$$

Lemma 2. Kilbas and Sebastain [(2008), p. 872, equations (21) to (24)].

Let $\alpha,\beta,\eta\in\mathbb{C}$ be such that $\mathrm{Re}(\alpha)>0$ and $\left[\mathrm{Re}(\mu)<1+\min\{\mathrm{Re}(\beta),\mathrm{Re}(\eta)\}\right]$, then there holds the following relation:

$$\left(I_{-}^{\alpha,\beta,\eta}t^{\mu-1}\right)(x)=\frac{\Gamma(\beta-\mu+1)\Gamma(\eta-\mu+1)}{\Gamma(1-\mu)\Gamma(\alpha+\beta+\eta-\mu+1)}x^{\mu-\beta-1}. \tag{15}$$

In particular, if $\beta=-\alpha$ and $\beta=0$ in (14), author has

$$\left(I_{-}^{\alpha}t^{\mu-1}\right)(x)=\frac{\Gamma(1-\alpha-\mu)}{\Gamma(1-\mu)}x^{\mu+\alpha-1},\qquad 1-\mathrm{Re}(\mu)>\mathrm{Re}(\alpha)>0. \tag{16}$$

$$\left(K_{\eta,\alpha}^{-}t^{\mu-1}\right)(x)=\frac{\Gamma(\eta-\mu+1)}{\Gamma(1-\mu+\alpha+\eta)}x^{\mu-1},\qquad \mathrm{Re}(\mu)<1+\mathrm{Re}(\eta). \tag{17}$$

3. Main Results

Image (1):

$$\left\{I_{0+}^{\alpha,\beta,\eta}\left(t^{\mu-1}(b-at)^{-\nu}\prod_{j=1}^{s}S_{n_j}^{m_j}\left[c_j t^{\lambda_j}(b-at)^{-\delta_j}\right]H\left[z_1 t^{\sigma_1}(b-at)^{-\omega_1}\ldots z_r t^{\sigma_r}(b-at)^{-\omega_r}\right]\right)\right\}(x)$$
$$=b^{-\nu}x^{\mu-\beta-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1k_1}\ldots(-n_s)_{m_sk_s}}{k_1!\ldots k_s!}A'_{n_1,m_1}\ldots A^{(s)}_{n_s,m_s}c_1^{k_1}\ldots c_s^{k_s}(b)^{-\sum_{j=1}^{s}\delta_jk_j}(x)^{\sum_{j=1}^{s}\lambda_jk_j}$$
$$H^{0,N+3:\,M_1,N_1;\ldots;M_r,N_r;1,0}_{P+3,Q+3:P_1,Q_1;\ldots;P_r,Q_r;0,1}\left[\begin{matrix}z_1\dfrac{x^{\sigma_1}}{b^{\omega_1}}\\ \vdots\\ z_r\dfrac{x^{\sigma_r}}{b^{\omega_r}}\\ -\dfrac{a}{b}x\end{matrix}\,\middle|\,\begin{matrix}A:C\\ B:D\end{matrix}\right], \tag{18}$$

where

$$A=\left(1-\nu-\sum_{j=1}^{s}\delta_jk_j;\omega_1,\ldots,\omega_r,1\right),\left(1-\mu-\sum_{j=1}^{s}\lambda_jk_j;\sigma_1,\ldots,\sigma_r,1\right),$$
$$\left(1-\mu-\eta+\beta-\sum_{j=1}^{s}\lambda_jk_j;\sigma_1,\ldots,\sigma_r,1\right),(\mathrm{a}_j;\alpha'_j,\ldots,\alpha_j^{(r)})_{1,P}$$
$$B=\left(1-\nu-\sum_{j=1}^{s}\delta_jk_j;\omega_1,\ldots,\omega_r,0\right),\left(1-\mu+\beta-\sum_{j=1}^{s}\lambda_jk_j;\sigma_1,\ldots,\sigma_r,1\right),$$
$$\left(1-\mu-\alpha-\eta-\sum_{j=1}^{s}\lambda_jk_j;\sigma_1,\ldots,\sigma_r,1\right),,(\mathrm{b}_j;\beta'_j,\ldots,\beta_j^{(r)})_{1,Q} \tag{19}$$
$$C=(\mathrm{c}'_j,\gamma'_j)_{1,P_1};\ldots;(c_j^{(r)},\gamma_j^{(r)})_{1,P_r};-$$
$$D=(d'_j,\delta'_j)_{1,Q_1};\ldots;(f_j^{(r)},F_j^{(r)})_{1,Q_r};(0,1)$$

The sufficient conditions of validity of (18) are

(i) $\alpha,\beta,\eta,\mu,\nu,\delta_j,\omega_i,a,b,c,z_i \in C$ and $\lambda_j,\sigma_i>0\,\forall i\in\{1,...,r\}$ & $j\in\{1,...,s\}$

(ii) $|\arg z_i|<\frac{1}{2}\Omega_i\pi$ and $\Omega_i>0$,

where $\Omega_i=-\sum_{j=N+1}^{P}\alpha_j^{(i)}-\sum_{j=1}^{Q}\beta_j^{(i)}+\sum_{j=1}^{N_i}\gamma_j^{(i)}-\sum_{j=N_i+1}^{P_i}\gamma_j^{(i)}+\sum_{j=1}^{M_i}\delta_j^{(i)}-\sum_{j=M_i+1}^{Q_i}\delta_j^{(i)};\forall i\in\{1,...,r\}$.

(iii) $\mathrm{Re}(\alpha)>0$ and

$$\mathrm{Re}(\mu)+\sum_{i=1}^{r}\sigma_i \min_{1\le j\le M_i}\mathrm{Re}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>\max\{0,\mathrm{Re}(\beta-\eta)\},$$

$$\mathrm{Re}(\nu)+\sum_{i=1}^{r}\omega_i \min_{1\le j\le M_i}\mathrm{Re}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>\max\{0,\mathrm{Re}(\beta-\eta)\}.$$

(iv) $\left|\frac{a}{b}x\right|<1$.

***Proof*:**

In order to prove (18), we first express the product of a general class of polynomials occurring on its left-hand side in the series form given by (11), replace the multivariable H-functions occurring therein by its well-known Mellin–Barnes contour integral given by (8), interchange the order of summations, $(\xi_1,...,\xi_r)$-integrals and taking the fractional integral operator inside (which is permissible under the conditions stated) and make a little simplification. Next, we express the terms $(b-ax)^{-(\nu+\sum_{j=1}^{s}\delta_j k_j+\sum_{i=1}^{r}\omega_i\xi_i)}$ so obtained in terms of Mellin-Barnes contour integral Srivastava et al. [(1982), p. 18, equation (2.6.3); p. 10, equation (2.1.1)] it takes the following form (Say I) after a little simplification:

$$I=(b)^{-\nu}\sum_{k_1=0}^{[n_1/m_1]}...\sum_{k_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1k_1}...(-n_s)_{m_sk_s}}{k_1!...k_s!}A'_{n_1,m_1}...A^{(s)}_{n_s,m_s}c_1^{k_1}...c_s^{k_s}(b)^{-\sum_{j=1}^{s}\delta_jk_j}$$
$$\frac{1}{(2\pi i)^{r+1}}\int_{L_1}...\int_{L_r}\psi(\xi_1,...,\xi_r)\prod_{i=1}^{r}\phi_i(\xi_i)z_i^{\xi_i}(b)^{-\sum_{i=1}^{r}\omega_i\xi_i}\int_L\frac{\Gamma\left(\nu+\sum_{j=1}^{s}\delta_jk_j+\sum_{i=1}^{r}\omega_i\xi_i+\xi\right)}{\Gamma\left(\nu+\sum_{j=1}^{s}\delta_jk_j+\sum_{i=1}^{r}\omega_i\xi_i\right)\Gamma(1+\xi)}$$
$$\left(-\frac{a}{b}\right)^{\xi}d\xi\left(I_{0^+}^{\alpha,\beta,\eta}t^{\mu+\sum_{j=1}^{s}\lambda_jk_j+\sum_{i=1}^{r}\sigma_i\xi_i+\xi-1}\right)(x). \tag{20}$$

Finally, applying the lemma 1 and re-interpreting the Mellin-Barnes contour integral thus obtain in terms of the multivariable H-function defined by (8), we arrive at the right hand side of (18) after a little simplifications.

If we put $\beta=-\alpha$ in Image 1, we arrive at the following new and interesting corollary concerning Riemann-Liouville fractional integral operator defined by (4) and using (13).

Corollary 1.1:

$$\left\{I_{0+}^{\alpha}\left(t^{\mu-1}(b-at)^{-\nu}\prod_{j=1}^{s}S_{n_j}^{m_j}\left[c_jt^{\lambda_j}(b-at)^{-\delta_j}\right]H\left[z_1t^{\sigma_1}(b-at)^{-\omega_1}...z_rt^{\sigma_r}(b-at)^{-\omega_r}\right]\right\}(x)$$
$$=b^{-\nu}x^{\mu+\alpha-1}\sum_{k_1=0}^{[n_1/m_1]}...\sum_{k_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1k_1}...(-n_s)_{m_sk_s}}{k_1!...k_s!}A'_{n_1,m_1}...A^{(s)}_{n_s,m_s}c_1^{k_1}...c_s^{k_s}(b)^{-\sum_{j=1}^{s}\delta_jk_j}(x)^{\sum_{j=1}^{s}\lambda_jk_j}$$
$$H^{0,N+2:\,M_1,N_1;...;M_r,N_r;1,0}_{P+2,Q+2:P_1,Q_1;...;P_r,Q_r;0,1}\left[\begin{array}{c|c} z_1\frac{x^{\sigma_1}}{b^{\omega_1}} & \\ \vdots & A':C \\ z_r\frac{x^{\sigma_r}}{b^{\omega_r}} & B':D \\ -\frac{a}{b}x & \end{array}\right], \tag{21}$$

where

$$A' = \left(1-\nu-\sum_{j=1}^{s}\delta_j k_j;\omega_1,...,\omega_r,1\right),\left(1-\mu-\sum_{j=1}^{s}\lambda_j k_j;\sigma_1,...,\sigma_r,1\right),(\mathrm{a}_j;\alpha_j',...,\alpha_j^{(r)})_{1,P}$$

$$B' = \left(1-\nu-\sum_{j=1}^{s}\delta_j k_j;\omega_1,...,\omega_r,0\right),\left(1-\mu-\alpha-\sum_{j=1}^{s}\lambda_j k_j;\sigma_1,...,\sigma_r,1\right),,(\mathrm{b}_j;\beta_j',...,\beta_j^{(r)})_{1,Q} \tag{22}$$

and where C and D are same as given in (18) and the conditions of existence of the above corollary follow easily with the help of Image 1.

Again, if we put $\beta = 0$ in Image 1, we get the following result which is also believed to be new and pertains to Erde'lyi-Kober fractional integral operators defined by (6) and using (14).

Corollary 1.2.

$$\left\{I_{\eta,\alpha}^{+}\left(t^{\mu-1}(b-at)^{-\nu}\prod_{j=1}^{s}S_{n_j}^{m_j}\left[c_j t^{\lambda_j}(b-at)^{-\delta_j}\right]H\left[z_1 t^{\sigma_1}(b-at)^{-\omega_1}...z_r t^{\sigma_r}(b-at)^{-\omega_r}\right]\right\}(x)$$

$$= b^{-\nu}x^{\mu-1}\sum_{k_1=0}^{[n_1/m_1]}...\sum_{k_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1k_1}...(-n_s)_{m_sk_s}}{k_1!...k_s!}A_{n_1,m_1}...A_{n_s,m_s}^{(s)}c_1^{k_1}...c_s^{k_s}(b)^{-\sum_{j=1}^{s}\delta_j k_j}(x)^{\sum_{j=1}^{s}\lambda_j k_j}$$

$$H_{P+2,Q+2:P_1,Q_1;...;P_r,Q_r;0,1}^{0,N+2:\;M_1,N_1;...;M_r,N_r;1,0}\left[\begin{array}{c|c} z_1\frac{x^{\sigma_1}}{b^{\omega_1}} & \\ \vdots & A'':C \\ z_r\frac{x^{\sigma_r}}{b^{\omega_r}} & B'':D \\ -\frac{a}{b}x & \end{array}\right], \tag{23}$$

where C and D are same as given in (18) and

$$A'' = \left(1-\nu-\sum_{j=1}^{s}\delta_j k_j;\omega_1,...,\omega_r,1\right),\left(1-\mu-\eta-\sum_{j=1}^{s}\lambda_j k_j;\sigma_1,...,\sigma_r,1\right),(\mathrm{a}_j;\alpha_j',...,\alpha_j^{(r)})_{1,P}$$

$$B'' = \left(1-\nu-\sum_{j=1}^{s}\delta_j k_j;\omega_1,...,\omega_r,0\right),\left(1-\mu-\alpha-\eta-\sum_{j=1}^{s}\lambda_j k_j;\sigma_1,...,\sigma_r,1\right),(\mathrm{b}_j;\beta_j',...,\beta_j^{(r)})_{1,Q}. \tag{24}$$

The sufficient conditions of validity of (23) are:

(i) $\mathrm{Re}(\alpha) > 0$ *and*

$$\mathrm{Re}(\mu)+\sum_{i=1}^{r}\sigma_i \min_{1\le j\le M_i}\mathrm{Re}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -\mathrm{Re}(\eta)$$

$$\mathrm{Re}(\nu)+\sum_{i=1}^{r}\omega_i \min_{1\le j\le M_i}\mathrm{Re}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -\mathrm{Re}(\eta)$$

and the conditions (i), (ii) and (iv) in Image 1are also satisfied.

Image (2):

$$\left\{I_{-}^{\alpha,\beta,\eta}\left(t^{\mu-1}(b-at)^{-\nu}\prod_{j=1}^{s}S_{n_j}^{m_j}\left[c_j t^{\lambda_j}(b-at)^{-\delta_j}\right]H\left[z_1 t^{\sigma_1}(b-at)^{-\omega_1}...z_r t^{\sigma_r}(b-at)^{-\omega_r}\right]\right\}(x)$$

$$= b^{-\nu} x^{\mu-\beta-1} \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_s=0}^{[n_s/m_s]} \frac{(-n_1)_{m_1k_1} \ldots (-n_s)_{m_sk_s}}{k_1! \ldots k_s!} A'_{n_1,m_1} \ldots A^{(s)}_{n_s,m_s} c_1^{k_1} \ldots c_s^{k_s} (b)^{-\sum_{j=1}^{s}\delta_j k_j} (x)^{\sum_{j=1}^{s}\lambda_j k_j}$$

$$H^{0,N+3:\ M_1,N_1;\ldots;M_r,N_r;1,0}_{P+3,Q+3:P_1,Q_1;\ldots;P_r,Q_r;0,1} \left[\begin{array}{c|c} z_1 \frac{x^{\sigma_1}}{b^{\omega_1}} & \\ \vdots & A^*:C \\ z_r \frac{x^{\sigma_r}}{b^{\omega_r}} & B^*:D \\ -\frac{a}{b}x & \end{array} \right], \tag{25}$$

where C and D are given by (18) and

$$\begin{aligned} A^* &= \left(1-\nu-\sum_{j=1}^{s}\delta_j k_j;\omega_1,\ldots,\omega_r,1\right), \left(\mu-\beta+\sum_{j=1}^{s}\lambda_j k_j;\sigma_1,\ldots,\sigma_r,1\right), \\ &\left(\mu-\eta+\sum_{j=1}^{s}\lambda_j k_j;\sigma_1,\ldots,\sigma_r,1\right), (\mathrm{a_j};\alpha'_j,\ldots,\alpha_j^{(r)})_{1,P} \\ B^* &= \left(1-\nu-\sum_{j=1}^{s}\delta_j k_j;\omega_1,\ldots,\omega_r,0\right), \left(\mu+\sum_{j=1}^{s}\lambda_j k_j;\sigma_1,\ldots,\sigma_r,1\right), \\ &\left(\mu-\alpha-\beta-\eta+\sum_{j=1}^{s}\lambda_j k_j;\sigma_1,\ldots,\sigma_r,1\right),, (\mathrm{b_j};\beta'_j,\ldots,\beta_j^{(r)})_{1,Q}. \end{aligned} \tag{26}$$

The sufficient conditions of validity of (25) are

(i) $\mathrm{Re}(\alpha) > 0$ *and*

$$\mathrm{Re}(\mu) - \sum_{i=1}^{r}\sigma_i \min_{1\le j\le M_i} \mathrm{Re}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) < 1 + \min\{\mathrm{Re}(\beta), \mathrm{Re}(\eta)\}$$

$$\mathrm{Re}(\nu) + \sum_{i=1}^{r}\omega_i \min_{1\le j\le M_i} \mathrm{Re}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) < 1 + \min\{\mathrm{Re}(\beta), \mathrm{Re}(\eta)\}$$

and the conditions (i), (ii) and (iv) in Image 1are also satisfied.

***Proof*:**

We easily obtain the Image 2 after a little simplification on making use of similar lines as adopted in Image 1 and using Lemma 2.

If we put $\beta = -\alpha$ and $\beta = 0$ in Image 2 and using (16) and (17), in succession we shall easily arrive at the corresponding corollaries concerning Riemann-Liouville and Erde'llyi-Kober fractional integral operators respectively.

Corollary 1.3.

$$\left\{ I_-^{\alpha} \left(t^{\mu-1}(b-at)^{-\nu} \prod_{j=1}^{s} S_{n_j}^{m_j}\left[c_j t^{\lambda_j}(b-at)^{-\delta_j}\right] H\left[z_1 t^{\sigma_1}(b-at)^{-\omega_1} \ldots z_r t^{\sigma_r}(b-at)^{-\omega_r}\right] \right) \right\}(x)$$

$$= b^{-\nu} x^{\mu+\alpha-1} \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_s=0}^{[n_s/m_s]} \frac{(-n_1)_{m_1k_1} \ldots (-n_s)_{m_sk_s}}{k_1! \ldots k_s!} A'_{n_1,m_1} \ldots A^{(s)}_{n_s,m_s} c_1^{k_1} \ldots c_s^{k_s} (b)^{-\sum_{j=1}^{s}\delta_j k_j} (x)^{\sum_{j=1}^{s}\lambda_j k_j}$$

$$H^{0,N+2:\ M_1,N_1;\ldots;M_r,N_r;1,0}_{P+2,Q+2:P_1,Q_1;\ldots;P_r,Q_r;0,1} \left[\begin{array}{c|c} z_1 \frac{x^{\sigma_1}}{b^{\omega_1}} & \\ \vdots & A^{**}:C \\ z_r \frac{x^{\sigma_r}}{b^{\omega_r}} & B^{**}:D \\ -\frac{a}{b}x & \end{array} \right], \tag{27}$$

where C and D are given by (18) and conditions of validity are same as (25) and

$$A^{**}=\left(1-\nu-\sum_{j=1}^{s}\delta_j k_j;\omega_1,...,\omega_r,1\right),\left(\alpha+\mu+\sum_{j=1}^{s}\lambda_j k_j;\sigma_1,...,\sigma_r,1\right),(\mathrm{a}_j;\alpha'_j,...,\alpha_j^{(r)})_{1,P}$$

$$B^{**}=\left(1-\nu-\sum_{j=1}^{s}\delta_j k_j;\omega_1,...,\omega_r,0\right),\left(\mu+\sum_{j=1}^{s}\lambda_j k_j;\sigma_1,...,\sigma_r,1\right),(\mathrm{b}_j;\beta'_j,...,\beta_j^{(r)})_{1,Q}. \tag{28}$$

Corollary 1.4.

$$\left\{K_{\eta,\alpha}^{-}\left(t^{\mu-1}(b-at)^{-\nu}\prod_{j=1}^{s}S_{n_j}^{m_j}\left[c_j t^{\lambda_j}(b-at)^{-\delta_j}\right]H\left[\mathrm{z}_1 t^{\sigma_1}(b-at)^{-\omega_1}...z_r t^{\sigma_r}(b-at)^{-\omega_r}\right]\right\}(x)$$

$$=b^{-\nu}x^{\mu-1}\sum_{k_1=0}^{[n_1/m_1]}...\sum_{k_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1k_1}...(-n_s)_{m_sk_s}}{k_1!...k_s!}A'_{n_1,m_1}...A^{(s)}_{n_s,m_s}c_1^{k_1}...c_s^{k_s}(b)^{-\sum_{j=1}^{s}\delta_j k_j}(x)^{\sum_{j=1}^{s}\lambda_j k_j}$$

$$H^{0,N+2:\ M_1,N_1;...;M_r,N_r;1,0}_{P+2,Q+2:P_1,Q_1;...;P_r,Q_r;0,1}\left[\begin{array}{c|c} z_1\frac{x^{\sigma_1}}{b^{\omega_1}} & \\ \vdots & A^{***}:C \\ z_r\frac{x^{\sigma_r}}{b^{\omega_r}} & B^{***}:D \\ -\frac{a}{b}x & \end{array}\right], \tag{29}$$

where C and D are given by (18) and

$$A^{***}=\left(1-\nu-\sum_{j=1}^{s}\delta_j k_j;\omega_1,...,\omega_r,1\right),\left(\mu-\eta+\sum_{j=1}^{s}\lambda_j k_j;\sigma_1,...,\sigma_r,1\right),(\mathrm{a}_j;\alpha'_j,...,\alpha_j^{(r)})_{1,P}$$

$$B^{***}=\left(1-\nu-\sum_{j=1}^{s}\delta_j k_j;\omega_1,...,\omega_r,0\right),\left(\mu-\alpha-\eta+\sum_{j=1}^{s}\lambda_j k_j;\sigma_1,...,\sigma_r,1\right),(\mathrm{b}_j;\beta'_j,...,\beta_j^{(r)})_{1,Q}. \tag{30}$$

The conditions of validity of the above results follow easily from the conditions given with Image 2.

4. Special Cases and Applications

The generalized fractional integral operator Images 1 and 2 established here are unified in nature and act as key formulae. Thus the product of the general class of polynomials involved in Image1 and 2 reduce to a large spectrum of polynomials listed by Srivastava and Singh [(1983), pp. 158– 161], and so from Image 1and 2 we can further obtain various fractional integral results involving a number of simpler polynomials. Again, the multivariable H-function occurring in these Images can be suitably specialized to a remarkably wide variety of useful functions (or product of several such functions) which are expressible in terms of the generalized Wright hypergeometric function, the generalized Mittag-Leffler function and Bessel functions of one variable.

Example 1.

If we reduce the multivariable H-function in to the product of two Fox H-functions in Image 1 and then reduce one H-function to the exponential function by taking $\sigma_1=1,\quad \omega_1\to 0$,we get the following result after a little simplification which is believed to be new:

$$\left\{I_{0^+}^{\alpha,\beta,\eta}\left(t^{\mu-1}(b-at)^{-\nu}\prod_{j=1}^{s}S_{n_j}^{m_j}\left[c_j t^{\lambda_j}(b-at)^{-\delta_j}\right]e^{-z_1t}H^{M_2,N_2}_{P_2,Q_2}\left[\mathrm{z}_2 t^{\sigma_2}(b-at)^{-\omega_2}\left|\begin{array}{l}(c_j,\gamma_j)_{1,P_2}\\(d_j,\delta_j)_{1,Q_2}\end{array}\right.\right]\right\}(x)$$

$$=b^{-\nu}x^{\mu-\beta-1}\sum_{k_1=0}^{[n_1/m_1]}...\sum_{k_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1k_1}...(-n_s)_{m_sk_s}}{k_1!...k_s!}A'_{n_1,m_1}...A^{(s)}_{n_s,m_s}c_1^{k_1}...c_s^{k_s}(b)^{-\sum_{j=1}^{s}\delta_j k_j}(x)^{\sum_{j=1}^{s}\lambda_j k_j}$$

$$H^{0,3:1,0;M_2,N_2;1,0}_{3,3:0,1;P_2,Q_2;0,1}\left[\begin{array}{c|l} z_1x & \left(1-\nu-\sum_{j=1}^{s}\delta_jk_j;1,\omega_2,1\right),\left(1-\mu-\sum_{j=1}^{s}\lambda_jk_j;1,\sigma_2,1\right),\left(1-\mu-\eta+\beta-\sum_{j=1}^{s}\lambda_jk_j;1,\sigma_2,1\right): \\ z_2\frac{x^{\sigma_2}}{b^{\omega_2}} & \\ -\frac{a}{b}x & \left(1-\nu-\sum_{j=1}^{s}\delta_jk_j;1,\omega_2,0\right),\left(1-\mu+\beta-\sum_{j=1}^{s}\lambda_jk_j;1,\sigma_2,1\right),\left(1-\mu-\alpha-\eta-\sum_{j=1}^{s}\lambda_jk_j;1,\sigma_2,1\right): \end{array}\right.$$

$$\left.\begin{array}{l} -;(c_j,\gamma_j)_{1,P_2};- \\ (0,1);(d_j,\delta_j)_{1,Q_2};(0,1)\end{array}\right]. \tag{31}$$

The conditions of validity of the above result easily follow from (18).

If we put $\beta=-\alpha$ *and* $\nu,\omega_2=0$ *and* $S_{n_j}^{m_j}=1$ and make suitable adjustment in the parameters in the equation (31), we arrive at the known result [see Kilbas and Saigo (2004), p. 52, equation (2.7.9)].

If we put $\nu, \omega_2, z_1 = 0$ *and* $S_{n_j}^{m_j} = 1$ and make suitable adjustment in the parameters in the equation (31), we arrive at the known result [see Gupta et al. (2010), p. 209, equation (25)].

Example 2.

If we reduce the multivariable H-function involved in (18) to the product of r different Whittaker functions Srivastava et al. [(1982), p.18, equation (2.6.7)] and taking $\sigma_i = 1, \quad \omega_i \to 0$, we arrive at the following new and interesting result:

$$\left\{ I_{0^+}^{\alpha,\beta,\eta} \left(t^{\mu-1} (b-at)^{-\nu} \prod_{j=1}^{s} S_{n_j}^{m_j} \left[c_j t^{\lambda_j} (b-at)^{-\delta_j} \right] \prod_{i=1}^{r} e^{-\left(\frac{z_i t}{2}\right)} W_{\lambda_i,\mu_i}(z_i t) \right) \right\}(x)$$

$$= b^{-\nu} x^{\mu-\beta-1} \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_s=0}^{[n_s/m_s]} \frac{(-n_1)_{m_1 k_1} \cdots (-n_s)_{m_s k_s}}{k_1! \ldots k_s!} A'_{n_1,m_1} \ldots A^{(s)}_{n_s,m_s} c_1^{k_1} \ldots c_s^{k_s} (b)^{-\sum_{j=1}^{s} \delta_j k_j} (x)^{\sum_{j=1}^{s} \lambda_j k_j}$$

$$H_{3,3:1,2;\ldots;1,2;0,1}^{0,3:2,0\ldots;2,0;1,0} \left[\begin{matrix} z_1 x \\ \vdots \\ z_r x \\ -\frac{a}{b} x \end{matrix} \middle| \begin{matrix} \left(1-\nu-\sum_{j=1}^{s}\delta_j k_j; \underbrace{1,\ldots,1}_{r}, 1\right), \left(1-\mu-\sum_{j=1}^{s}\lambda_j k_j; \underbrace{1,\ldots,1}_{r}, 1\right), \left(1-\mu-\eta+\beta-\sum_{j=1}^{s}\lambda_j k_j; \underbrace{1,\ldots,1}_{r}, 1\right) \\ \left(1-\nu-\sum_{j=1}^{s}\delta_j k_j; \underbrace{1,\ldots,1}_{r}, 0\right), \left(1-\mu+\beta-\sum_{j=1}^{s}\lambda_j k_j; \underbrace{1,\ldots,1}_{r}, 1\right), \left(1-\mu-\alpha-\eta-\sum_{j=1}^{s}\lambda_j k_j; \underbrace{1,\ldots,1}_{r}, 1\right) \end{matrix} \right. \tag{32}$$

$$\left. \begin{matrix} :(1-\lambda_1,1);\ldots;(1-\lambda_r,1);- \\ :(\frac{1}{2}\pm\mu_1,1);\ldots;(\frac{1}{2}\pm\mu_1,1);(0,1) \end{matrix} \right].$$

The conditions of validity of (32) can be easily derived from those of (18).

If we put $\nu = 0$ *and* $S_{n_j}^{m_j}, r = 1$ and make suitable adjustment in the parameters in the equation (32), we arrive at the known result [see Gupta et al. (2010), p. 211, equation (31)].

Example 3.

If we reduce the H-function of one variable to the generalized Wright hypergeometric function Srivastava et al. [(1982), p.19, equation (2.6.11)] in the result given by (31), we get the following new and interesting result after little simplification:

$$\left\{ I_{0^+}^{\alpha,\beta,\eta} \left(t^{\mu-1} (b-at)^{-\nu} \prod_{j=1}^{s} S_{n_j}^{m_j} \left[c_j t^{\lambda_j} (b-at)^{-\delta_j} \right] e^{-z_1 t} {}_{P_2}\psi_{Q_2} \left[\text{-z}_2 t^{\sigma_2} (b-at)^{-\omega_2} \middle| \begin{matrix} (1-c_j,\gamma_j)_{1,P_2} \\ (0,1),(1-d_j,\delta_j)_{1,Q_2} \end{matrix} \right] \right) \right\}(x)$$

$$= b^{-\nu} x^{\mu-\beta-1} \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_s=0}^{[n_s/m_s]} \frac{(-n_1)_{m_1 k_1} \cdots (-n_s)_{m_s k_s}}{k_1! \ldots k_s!} A'_{n_1,m_1} \ldots A^{(s)}_{n_s,m_s} c_1^{k_1} \ldots c_s^{k_s} (b)^{-\sum_{j=1}^{s} \delta_j k_j} (x)^{\sum_{j=1}^{s} \lambda_j k_j}$$

$$H_{3,3:0,1;P_2,Q_2;0,1}^{0,3:1,0;1,P_2;1,0} \left[\begin{matrix} z_1 x \\ -z_2 \frac{x^{\sigma_2}}{b^{\omega_2}} \\ -\frac{a}{b} x \end{matrix} \middle| \begin{matrix} \left(1-\nu-\sum_{j=1}^{s}\delta_j k_j; 1, \omega_2, 1\right), \left(1-\mu-\sum_{j=1}^{s}\lambda_j k_j; 1, \sigma_2, 1\right), \left(1-\mu-\eta+\beta-\sum_{j=1}^{s}\lambda_j k_j; 1, \sigma_2, 1\right): \\ \left(1-\nu-\sum_{j=1}^{s}\delta_j k_j; 1, \omega_2, 0\right), \left(1-\mu+\beta-\sum_{j=1}^{s}\lambda_j k_j; 1, \sigma_2, 1\right), \left(1-\mu-\alpha-\eta-\sum_{j=1}^{s}\lambda_j k_j; 1, \sigma_2, 1\right): \end{matrix} \right. \tag{33}$$

$$\left. \begin{matrix} -;(c_j,\gamma_j)_{1,P_2};- \\ (0,1);(d_j,\delta_j)_{1,Q_2};(0,1) \end{matrix} \right].$$

The conditions of validity of the above result easily follow from (18).

If we put $\beta = -\alpha$ *and* $\nu, \omega_2 = 0$ *and* $S_{n_j}^{m_j} = 1$ and make suitable adjustment in the parameters in the equation (32), we arrive at the known result [see Kilbas (2005), p. 117, equation (11)].

If we put $\nu, \omega_2, z_1 = 0$ *and* $S_{n_j}^{m_j} = 1$ and make suitable adjustment in the parameters in the equation (33), we arrive at the known result [see Gupta et al. (2010), p. 210, equation (27)].

Example 4.

If we take $z_2, \sigma_2 = 1$ and $\omega_2 = 0$ in the equation (31) and reducing the H-function of one variable occurring therein to, generalized Mittag-Laffler function Prabhakar [(1971), p.19, equation (2.6.11)], we easily get after little simplification the following new and interesting result:

$$\left\{ I_{0^+}^{\alpha,\beta,\eta} \left(t^{\mu-1} (b-at)^{-\nu} \prod_{j=1}^{s} S_{n_j}^{m_j} \left[c_j t^{\lambda_j} (b-at)^{-\delta_j} \right] e^{-z_1 t} E_{M_2,N_2}^{\rho} [t] \right\} (x)$$

$$= \frac{b^{-\nu}}{\Gamma(\rho)} x^{\mu-\beta-1} \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_s=0}^{[n_s/m_s]} \frac{(-n_1)_{m_1 k_1} \cdots (-n_s)_{m_s k_s}}{k_1! \ldots k_s!} A'_{n_1,m_1} \ldots A^{(s)}_{n_s,m_s} c_1^{k_1} \ldots c_s^{k_s} (b)^{-\sum_{j=1}^{s} \delta_j k_j} (-x)^{\sum_{j=1}^{s} \lambda_j k_j}$$
$$H^{0,3:1,0;1,1;1,0}_{3,2:0,1;1,3;0,1} \left[\begin{matrix} z_1 x \\ x \\ -\frac{a}{b} x \end{matrix} \middle| \begin{matrix} \left(1-\nu-\sum_{j=1}^{s}\delta_j k_j;1,0,1\right), \left(1-\mu-\sum_{j=1}^{s}\lambda_j k_j;1,1,1\right), \left(1-\mu-\eta+\beta-\sum_{j=1}^{s}\lambda_j k_j;1,1,1\right): \\ \left(1-\nu-\sum_{j=1}^{s}\delta_j k_j;1,0,0\right), \left(1-\mu+\beta-\sum_{j=1}^{s}\lambda_j k_j;1,1,1\right), \left(1-\mu-\alpha-\eta-\sum_{j=1}^{s}\lambda_j k_j;1,1,1\right): \end{matrix} \right.$$
$$\left. \begin{matrix} -;(1-\rho,1);- \\ (0,1);(0,1),(1-\nu;o),(1-N_2;M_2);(0,1) \end{matrix} \right]. \tag{34}$$

The conditions of validity of the above result can be easily followed directly from those given with (18).
If we put $\beta = -\alpha$ and $\nu, \omega_2 = 0$ and $S_{n_j}^{m_j} = 1$ and make suitable adjustment in the parameters in the equation (34), we arrive at the known result [see Saxena et. al. (2009), p. 168, equation (2.1)].

If we put $\nu = 0$ and $S_{n_j}^{m_j} = 1$ and make suitable adjustment in the parameters in the equation (34), we arrive at the known result [see Gupta et al. (2010), p. 210, equation (29)].

Example 5.

If we take $\beta = -\alpha$ and $\nu, \omega_2 = 0$ and $S_{n_j}^{m_j} = 1, z_2 = \frac{1}{4}, \sigma_2 = 2$ and reduce the H-function to the Bessel function of first kind in the equation (31), we also get known result [see Kilbas and Sebastain (2008), 3, p. 873, equation (25) to (29)].

A number of other special cases of Images 1 and 2 can also be obtained but we do not mention them here on account of lack of space.

5. Conclusion

In this paper, we have obtained the Images of the generalized fractional integral operators given by Saigo. The Images have been developed in terms of the product of multivariable H-function and a general class of polynomials in a compact and elegant form with the help of Saigo operators. Most of the results obtained in this article are useful in deriving certain composition formulas involving Riemann-Liouville, Erde'lyi-Kober fractional calculus operators and multivariable H- functions. The findings of the present paper provide an extension of the results given earlier by Kilbas, Kilbas and Saigo, Kilbas and Sebastain, Saxena et al. and Gupta et al., as mentioned earlier.

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