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A New Approach for Computing WZ Factorization

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Abstract

Linear systems arise frequently in scientific and engineering computing. Various serial and parallel algorithms have been introduced for their solution. This paper seeks to compute the *WZ* and the *ZW* factorizations of a nonsingular matrix *A* using the right inverse of nested submatrices of *A*. We introduce two new matrix factorizations, the *QZ* and the *QW* factorizations, and compute the factorizations using our proposed approach.

Keywords: Matrix factorization, *QR* factorization, *LU* factorization, *WZ* factorization, *ZW* factorization, *QZ* factorization, *QW* factorization

MSC 2010 No.: 11D04, 15B10, 15A23

1. Introduction

Let A be an $m \times n$ real matrix, and A^R a right inverse of A (i.e., $AA^R = I$). Here we present a general formulation for computing the matrix factorizations of A, depending on the choice of right inverse of some of the submatrices of A. Using the general formulation; we present a new method for computing the matrix factorizations such as *WZ* and *ZW*. We also introduce two new matrix factorization, *QZ* and *QW*, and show how to compute the factorizations using our proposed approach.

The emergence of parallel computing caused researchers to reconsider many of the most important and common serial numerical algorithms for their usefulness and viability on parallel computers. Parallel implicit elimination (*PIE*) for the solution of linear systems was introduced by Evans (1993, 1994). A parallel approach for solving linear system of equations is provided by the *WZ* factorization. The basic idea is a factorization of A, called the Quadrant Interlocking Factorization (*QIF*) [Evans (1998)]. These *QIF* methods seem to be potentially attractive alternatives to Gaussian elimination or Cholesky factorization for parallel computation. The proposed schemes yield the solution of two elements simultaneously and are eminently suitable for parallel implementations. *WZ* factorization presents an efficient algorithm for saving problems in many fields [Asenjo et al. (1993), Bylina (2011), (2012)].

The remainder of our work is organized as follows. In Section 2, we present a general algorithm for computing a general factorization for a matrix *A* using the right inverse of the matrix *A*. In Section 3, we introduce the *WZ* factorization. In Section 4, we study the corresponding methods, related to the *WZ* and *ZW* factorizations and introduce two new factorizations the *QZ* and *QW* factorizations. Finally, we conclude in Section 5.

2. The Right Inverse of Nested Submatrices

Here, we present a recursion procedure for computing a right inverse of a matrix *A*. To do this, we give an expression that links the right inverse of matrix *A* to the right inverse of the submatrices of *A*. Choosing the submatrices leads to the computation of various new matrix factorizations.

Let $A = (a_1,...,a_m)^T \in R^{m\times n}$, where a_i^T be the *i*th row of *A*. In the sequel, unless otherwise explicitly stated, we assume that $m \le n$ and A has full rank, then using basic algebraic techniques [Rao and Mitra (1971)], we can find a matrix $Y \in R^{m\times n}$ such that $AY^T \in R^{m\times m}$ is invertible and so

$$A^R = Y^T(AY^T)^{-1} \ . \tag{2.1}$$

Assuming that $j_1,..., j_m$ *is* a permutation of the numbers $1,..,m$. Let $J_k = \{j_1,..., j_m\}$ and $B_{J_k} = \{b_{j_1},...,b_{j_k}\}^T$ denote a submatrix of the matrix $B = (b_1,...,b_m)^T \in R^{m\times n}$.

Here, we present an iterative process to establish a relationship between $A^R_{J_k}$ and $A^R{}_{J_{k-1}}$ for $k > 1$. First we recall a result from linear algebra [Khazal (2002)].

Remark 2.1.

Let

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} .$$

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Then,

$$S^{-1} = \begin{pmatrix} K & L \\ M & N \end{pmatrix},$$

where

$$\begin{cases} K = (A - BD^{-1}C)^{-1}, M = -D^{-1}CK \\ N = (D - CA^{-1}B)^{-1}, L = -A^{-1}BN \end{cases}. \tag{2.2}$$

Theorem 2.1.

Let $Y_{J_{k-1}} = \{y_{j_1}, ..., y_{j_{k-1}}\}^T \in R^{k-1 \times n}$ be such that the inverse of the square matrix $A_{J_{k-1}} Y_{J_{k-1}}^T \in R^{k-1 \times k-1}$ exists, and let $y_{j_k} \in R^n$ be such that

$$\beta_{j_k} = a_{j_k}^T P_k y_{j_k} \neq 0, \tag{2.3}$$

where

$$P_k = I - A_{J_{k-1}}^R A_{J_{k-1}} . \tag{2.4}$$

Let the matrix $Y_{J_k}^T$ be partitioned as $(Y_{J_{k-1}}^T, y_{j_k})$. Then,

1. $A_{J_k} Y_{J_k}^T$ is invertible and

$$A_{J_k}^R = Y_{J_k}^T (A_{J_k} Y_{J_k}^T)^{-1} = (A_{J_{k-1}}^R - \frac{1}{\beta_{j_k}} P_k y_{j_k} a_{j_k}^T A_{J_{k-1}}^R, \frac{1}{\beta_{j_k}} P_k y_{j_k}) . \tag{2.5}$$

2. The matrices P_k satisfy

$$P_{k+1} = P_k - \frac{1}{\beta_{j_k}} P_k y_{j_k} a_{j_k}^T P_k . \tag{2.6}$$

Proof:

Since

$$A_{J_K} = \begin{pmatrix} A_{J_{k-1}} \\ a_{j_k}^T \end{pmatrix} \quad \text{and} \quad Y_{J_k}^T = (Y_{J_{k-1}}^T, y_{j_k}), \tag{2.7}$$

$$A_{J_k}Y_{J_k}^T = \begin{pmatrix} A_{J_{k-1}}Y_{J_{k-1}}^T & A_{J_{k-1}}y_{j_k} \\ a_{j_k}^T Y_{J_{k-1}}^T & a_{j_k}^T y_{j_k} \end{pmatrix}. \quad (2.8)$$

Using the formula (2.2) for the inverse of a matrix, we deduce the results given in the theorem.

Remark 2.2.

From the algebraic definition of P_k given by

$$P_k = I - Y_{J_{k-1}}^T (A_{J_{k-1}} Y_{J_{k-1}}^T)^{-1} A_{J_{k-1}} \ . \quad (2.9)$$

It follows that P_k is an idempotent matrix. It is an oblique projection matrix unless $y_{j_k} = a_{j_k}$ for $i = 1,...,k-1$. In that case, we have

$$P_k = P_k^T = P_k^2 \quad (2.10)$$

The following properties of P_k are easily verified:

$$P_i = P_i P_k = P_k P_i \text{; for } k < i\text{; and; } P_i Y_{J_{i-1}}^T = A_{J_{i-1}} P_i = 0\,, \quad (2.11)$$

and

$$\{0\} \subseteq N(P_1) \subseteq N(P_2) \subseteq ... \subseteq N(P_m)\,, \quad (2.12)$$

where, $N(B)$ denotes the null space of B.

Theorem 2.2.

Let $s_{j_k} = P_k y_{j_k}$ and $t_{j_k} = P_k^T a_{j_k}$, for $k = 1,...,m$. Then, we have:

1. $Span(s_{j_1},...,s_{j_k}) = Span = (y_{j_1},...,y_{j_k})$.
2. $Span(t_{j_1},...,t_{j_k}) = Span = (a_{j_1},...,a_{j_k})$.
3. $a_{j_i}^T s_{j_k} = 0$ *for i < k.*
4. $t_{j_i}^T y_{j_k} = 0$ *for i > k.*
5. $t_{j_i}^T s_{j_k} = \begin{cases} \beta_{j_i}, i = k \\ 0, otherwise \end{cases} \quad (2.13)$

Proof:

1: By the definition of s_{j_k} and from (2.9) we have

$$s_{j_k} = y_{j_k} - Y_{J_{k-1}}^T (A_{J_{k-1}} Y_{J_{k-1}}^T)^{-1} A_{J_{k-1}} y_{j_k} = y_{j_k} - Y_{J_{k-1}}^T d_k . \tag{2.14}$$

Hence, $s_{j_k} \in Span(y_{j_1},...,y_{j_k})$. Let S_{J_k} be the matrix whose columns are $s_{j_1},...,s_{j_k}$; we have $S_{J_k} = Y_{J_k}^T U_k$. Consequently

$$Span(s_{j_1},...,s_{j_k}) = Span = (y_{j_1},...,y_{j_k}) . \tag{2.15}$$

2: The proof is similar to the preceding one.

3, 4: Parts 3 and 4 follow from (2.11).

5: Part 5 follows from part 3, part 4, and (2.11).

One important result of the Theorem 2.2 is the establishment of a matrix factorization $AS = F$, where $S = (s_1,...,s_m)$. Now, we are ready to present an algorithm for computing the matrix factorization.

Algorithm 1: General Algorithm: Matrix Factorization.

Input: A full row rank matrix $A \in R^{m\times n}, m \le n$; and a permutation $j_1,..., j_m$ of the numbers $1,...,m$.

(1) Let $P_1 = I_{n,n}$ and $k = 1$.
(2) Choose y_{j_k} so that $a_{j_k}^T P_k y_{j_k} \neq 0$
(3) Compute $s_{j_k} = P_k y_{j_k}, t_{j_k} = P_k^T a_{j_k}$ and $\beta_{j_k} = t_{j_k}^T s_{j_k}$.
(4) Let $P_{k+1} = P_k - \dfrac{s_{j_k} t_{j_k}^T}{\beta_{j_k}}$.
(5) Let $k = k + 1$, **if** $k \le m$ **then go to** (2).
(6) Compute the matrix factorization $AS = F$, where $S = (s_1,...,s_m)$. **Stop**.

Remark 2.3.

Different choices of the permutation $j_1,..., j_m$ and the parameter y_{j_i} leads to various matrix factorizations. Let $A \in R^{n\times n}, j_i = i, i = 1,...,n$, and $P_1 = I_{n,n}$. Then the QR factorization is given by $y_i = a_i$ and the LU factorization is given by $y_i = e_i$, for $i = 1,\ldots,n$ [see Ballalij and Sadok (1998)].

We will show how to choose the parameters of the Algorithm 1 for computing some new matrix factorizations.

Theorem 2.3.

Let $A \in R^{n\times n}$ and $P_1 = I_{n,n}$. Consider a permutation $j_1,\ldots,j_n$ so that $e_{j_i}^T P_i a_{j_j} \neq 0$ for $i = 1,\ldots,n$ where P_i update by

$$P_{i+1} = P_i - \frac{P_i e_{j_i} a_{j_i}^T P_i}{a_{j_i}^T P_i e_{j_i}} \tag{2.16}$$

Then, the following properties are true.

(a) For $k \le i$, the j_k *th* columns of P_{i+1} are zero.

(b) For $k > i$, the j_k *th* rows of P_{i+1} are equal to the j_k th rows of P_1.

Proof:

(a) We prove, by induction, that $P_{i+1}e_{j_k} = 0$ for $k = 1,\ldots i$. For $k = 1$ we have $P_2 e_{j_1} = 0$, i.e. the j_1 th column of P_2 is zero. Now, assume that the theorem is true up to $j < i$, and prove for i. By the induction hypothesis, we have $P_i e_{j_k} = 0$, for $k < i$ and $P_{i+1}e_{j_i} = 0$, then $P_{i+1}e_{j_k} = 0$, $k = 1,\ldots,i$. Therefore, for $k \le i$ the j_k th columns of P_{i+1} are zero, proving (a).

(b) We have

$$P_{i+1} = P_i - \frac{P_i e_{j_i} a_{j_i}^T P_i}{a_{j_i}^T P_i e_{j_i}} \tag{2.17}$$

Since for $k \le i-1$, j_k *th* columns of P_i is zero by property (a) and by the update formula (2.17), the j_k th rows of P_{i+1} are equal to the j_k th rows of P_1, for k > i, proving statement (b).

Remark 2.4.

To compute the vectors s_{j_i} and t_{j_i} we do not need P_i explicitly. Let $u_{j_i} = \frac{s_{j_i}}{\beta_{j_i}}$, $i = 1,\ldots,n$, then we have

$$t_{j_i} = a_{j_i} - \sum_{k=1}^{i-1} u_{j_k}^T a_{j_i} t_{j_k},$$

$$u_{j_i} = y_{j_i} - \sum_{k=1}^{i-1} t_{j_k}^T y_{j_i} u_{j_k}.$$

3. *WZ* Factorization

Implicit matrix elimination schemes for the solution of linear systems were introduced by Evans (1993) and Evans and Hatzopoulos (1979). These schemes propose the elimination of two matrix elements simultaneously (as opposed to a single element in Gaussian Elimination) and is eminently suitable for parallel implementation [Evans and Abdullah (1994)].

Definition 3.1.

A matrix $A = (a_{i,j}) \in R^{n \times n}$ called a *W*-matrix if $a_{i,j} = 0$ for all *(i,j)* with $i > j$ and $i + j > n$ or with $i < j$ and $i + j \le n$. The matrix *A* is called a unit *W*-matrix if in addition $a_{i,i} = 1$ for $i = 1,\ldots,n$ and $a_{i,n-i+1} = 0$ for $i \ne (n-1)/2$. The transpose of a *W*-matrix is called a *Z*-matrix. Then, these matrices have the following forms:

$$W = \begin{pmatrix} \bullet & \circ & \circ & \circ & \bullet \\ \bullet & \bullet & \circ & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \circ & \bullet & \bullet \\ \bullet & \circ & \circ & \circ & \bullet \end{pmatrix}, Z = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \circ & \bullet & \bullet & \bullet & \circ \\ \circ & \circ & \bullet & \circ & \circ \\ \circ & \bullet & \bullet & \bullet & \circ \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{pmatrix} \tag{3.18}$$

Definition 3.2.

We say that a matrix *A* is factorized in the form *WZ* if

$$A = WZ \tag{3.19}$$

where the matrix *W* is a *W*-matrix and *Z* is a *Z*-matrix.

To solve a system of linear equations, the *WZ* factorization splitting procedure proposed in [Evans and Hadjidioms (1980)], is convenient for parallel computing. A detailed analysis for this factorization is given in [Evans and Hadjidioms (1980)]. The *WZ* factorization is a parallel method for solving dense linear systems (2.1), where *A* is a square $n \times n$ matrix, and *b* is an *n*-vector. The *WZ* factorization is analogous to the *LU* factorization and is suitable for parallel computers. A characterization for the existence of the *WZ* factorization is presented in [Rao

(1997)]. A backward error analysis for the *WZ* factorization is given in [Shanehchi and Evans (1982)]. A pivoting strategy for modified *WZ* factorizations is proposed in [Yalamov and Evans (1995)]. The matrices *W* and *Z* have two opposite zero quadrants. Then, we refer to *W* and *Z* as the interlocking quadrant factors of *A*. The next theorem, give a characterization for the existence of the *WZ* factorization of *A*.

Theorem 3.1. ***Factorization theorem***

Let $A \in R^{n\times n}$ be a nonsingular matrix. *A* has quadrant interlocking factorization *QIF, A=WZ* if and only if for every *k, 1* $\le k \le s$, where $s = \lfloor \frac{n}{2} \rfloor$ if *n* is even and $s = \lceil \frac{n}{2} \rceil$ iff *n* is odd ($\lfloor s \rfloor$ ($\lceil s \rceil$) denotes the greatest (least) integer less (bigger) than or equal to s) , the submatrix

$$\Delta_k = \begin{pmatrix} a_{1,1} & \cdots & a_{1,k} & a_{1,n-k+1} & \cdots & a_{1,n} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ a_{k,1} & \cdots & a_{k,k} & a_{k,n-k+1} & \cdots & a_{k,n} \\ a_{n-k+1,1} & \cdots & a_{n-k+1,k} & a_{n-k+1,n-k+1} & \cdots & a_{n-k+1,n} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ a_{n,1} & \cdots & a_{n,k} & a_{n,n-k+1} & \cdots & a_{n,n} \end{pmatrix}_{2k\times 2k} \tag{3.20}$$

of *A* is invertible.

Proof:

See Rao (1997).

Theorem 3.2.

If $A \in R^{n\times n}$ is nonsingular, then *WZ* factorization can always be carried out with pivoting. There exists a row permutation matrix *P* and the factors *W* and *Z* such that $PA = WZ$.

Proof:

See Rao (1997).

Theorem 3.3.

The *WZ* factorization exist for the symmetric positive definite or strictly diagonally dominant matrices.

Proof:

See Rao (1997).

When A is a symmetric positive definite matrix, it is possible to factor A in the form $A = LL^T$ for some lower triangular matrix L. This is known as Cholesky factorization. A variant of classical Cholesky factorization, called Cholesky QIF is given by Evans [(1998)].

4. Special Cases of the General Algorithm

Let $A \in R^{n\times n}$. It was shown in Ballalij and Sadok (1998); with parameter choices $j_i = i, i = 1,...,n$ and $P_1 = I_{n,n}$, the QR factorization via Gram-Schmith algorithms of A is given by $y_i = a_i$ and the implicit LU factorization of A via Gaussian elimination techniques is given $y_i = e_i$ using the general algorithm.

In the sequel, we shall investigate some choices of the matrix Y, and the permutation $j_1,..., j_n$ for computing WZ, ZW, QZ and QW factorizations using the general algorithm. We assume that n be an even number.

4.1. *WZ* Factorization

Let $A \in R^{n\times n}$ be a nonsingular matrix and the permutation $j_1,..., j_n$ defined by:

$$j_i = \begin{cases} i, if \cdot i \cdot is \cdot odd \\ n-i+1, if \cdot i \cdot is \cdot even \end{cases}, \tag{4.21}$$

if $e_{j_k} \in R^n$ denotes the j_k th column of the identity matrix, then the second important and easy choice for the auxiliary parameter y_{j_k} in the general algorithm is the vector e_{j_k}. First, in order to guarantee that $\beta_{j_k} \neq 0$, we assume that rows swaps in A are performed. Indeed, if $\Pi = (e_{v_{j_1}},...,e_{v_{j_n}})$ denotes a permutation matrix, then Π is such that $\Pi A = (a_{v_{j_1}},...,a_{v_{j_n}})^T$. In this case, $s_{j_k} = P_k e_{j_k}, t_{j_k} = P_k^T a_{v_{j_k}}$ and

$$\beta_{j_k} = t_{j_k}^T s_{j_k} = a_{v_{j_k}}^T P_k e_{j_k} = t_{j_k}^T e_{j_k} = a_{v_{j_k}}^T s_{j_k} \neq 0 \tag{4.22}$$

and the general algorithm computes the factorization $\Pi AS = F$.

Now, in order to guarantee that $\beta_{j_k} \neq 0$, we assume that A_{J_k} be nonsingular, for $k = 1,...,n$.

Theorem 4.1.

Let $A \in R^{n\times n}$ be an even number, j_k be defined by (4.21) and A_{J_k} be invertible, for *k=1,…,n*. Then, there exists a *WZ* factorization for *A,* obtained by the general algorithm.

Proof:

Let $y_{j_k} = e_{j_k}, k = 1,\dots,n$. Then, according to Theorem 2.3, for $i = 1,\dots.,n/2$, we have

$$P_{2i+1} = \begin{bmatrix} 0_{i,i} & R_i & 0 \\ 0 & I_{n-2i} & 0 \\ 0_{i,i} & L_i & 0 \end{bmatrix} \quad (4.23)$$

with

$$R_i, L_i \in R^{i\times n-2i},$$

and

$$P_{2i} = \begin{bmatrix} 0_{i,i} & R_i & 0 \\ 0 & I_{n-2i+1} & 0 \\ 0_{i-1,i} & L_i & 0 \end{bmatrix}, \quad (4.24)$$

where

$$R_i \in R^{i\times n-2i+1}$$

and

$$L_i \in R^{i-1\times n-2i+1}.$$

Let $s_{j_i} = P_i e_{j_i}$. According to Theorem 2.2 we obtain a *Z*-matrix $S = (s_1,\dots,s_m)$ with 1's as diagonal entries and a *W*-matrix $T = (t_1,\dots,t_m)$ so that

$$AS = T \Rightarrow A = WZ \quad (4.25)$$

where $Z = S^{-1}$ is a *Z*-matrix and $W = T$ is a *W*-matrix.

4.2. *ZW* Factorization

Now we compute the *ZW* factorization using the general algorithm. Let $A \in R^{n\times n}$ be a nonsingular matrix, n be an even number and the permutation $j_1,\dots,j_n$ defined by:

$$j_i = \begin{cases} \frac{n}{2} - i + 1, if \cdot i \cdot is \cdot odd \\ \frac{n}{2} + i, if \cdot i \cdot is \cdot even \end{cases} . \tag{4.26}$$

If $e_{j_k} \in R^n$ denotes the j_k th column of the identity matrix, then the second important and easy choice for the auxiliary parameter y_{j_k} in the general algorithm is the vector e_{j_k}. First, in order to guarantee that $\beta_{j_k} \neq 0$, we assume that A_{J_k} be nonsingular, for $k = 1,...,n$.

Theorem 4.2.

Let $A \in R^{n \times n}$, n be an even number, j_k defined by (4.26) and A_{J_k} be invertible, for *k=1,…,n.* Then, there exists a *ZW* factorization for *A,* obtained by the general algorithm.

Proof:

Let $y_{j_k} = e_{j_k}, k = 1,...,n$. Then, according to Theorem 2.3, for $i = 1,....,n/2$, we have

$$P_{2i+1} = \begin{bmatrix} I_{\frac{n}{2}-i,\frac{n}{2}-i} & 0 & 0 \\ R_i & 0_{2i} & L_i \\ 0 & 0 & I_{\frac{n}{2}-i,\frac{n}{2}-i} \end{bmatrix} \tag{4.27}$$

with

$$R_i, L_i \in R^{2i,\frac{n}{2}-i},$$

and

$$P_{2i} = \begin{bmatrix} I_{\frac{n}{2}-i+1,\frac{n}{2}-i+1} & 0 & 0 \\ R_i & 0_{2i-1} & L_i \\ 0 & 0 & I_{\frac{n}{2}-i,\frac{n}{2}-i} \end{bmatrix} \tag{4.28}$$

with $R_i \in R^{2i-1,\frac{n}{2}-i}$ and $L_i \in R^{2i-1,\frac{n}{2}-i+1}$.

Let $s_{j_i} = P_i e_{j_i}$. According to Theorem 2.2 we obtain a *W*-matrix $S = (s_1,...,s_m)$ with 1's as diagonal entries and a *Z*-matrix $T = (t_1,...,t_m)$ so that

$$AS = T \Rightarrow A = ZW\ , \tag{4.29}$$

where $W = S^{-1}$ is a *W*-matrix and $Z = T$ is a *Z*-matrix.

4.3. *QZ* Factorization

Definition 4.1.

Let $A \in R^{n\times n}$. We say that *A* is factorized in the form *QZ* if

$$A = QZ\ , \tag{4.30}$$

where the matrix Q is an orthogonal matrix, i.e., $Q^T Q = QQ^T = I_{n,n}$ [Golub and Van Loan (1983)] and *Z* is a *Z*-matrix.

Let $A \in R^{n\times n}$ be a nonsingular matrix. We show how to choose the parameters of the general algorithm for computing the *QW* factorization for *A*.

One choice for the vector y_{j_k} in the general algorithm is a_{j_k}. From Theorem 2.2, this leads to

$$s_{j_k} = t_{j_k} = P_k a_{j_k}\ . \tag{4.31}$$

Note that, in exact arithmetic $\beta_{j_k} = s_{j_k}^T s_{j_k} > 0$.

Thus, the general algorithm, by choosing $y_{j_k} = a_{j_k}$ is well defined and the next result obtained immediately.

Theorem 4.3.

Let $A \in R^{n\times n}$, j_k be defined by (4.21) and $y_{j_k} = a_{j_k}, k = 1,\ldots,n$. Then, $S = (s_1,\ldots,s_m)$ is orthogonal, *AS* is a *W*-matrix and a *QZ* factorization is recognized for A^T .

Proof:

By Theorem 2.2, we have

$$s_{j_k}^T s_{j_i} = 0 \text{ for } i \neq k \text{ and } a_{j_i}^T s_{j_k} = 0 \text{ for } i < k\ . \tag{4.32}$$

Therefore, the set of vectors $\{s_1,\ldots,s_m\}$ in R^n is orthogonal. The matrix $S = (s_1,\ldots,s_m)$ is such that

$$S^T S = D = diag(\beta_1,...,\beta_n) \quad (4.33)$$

and AS is a W-matrix. Then,

$$AS = F \Rightarrow A^T = S^{-T} F^T = QZ .$$

4.4. QW Factorization

Definition 4.2.

Let $A \in R^{n\times n}$. We say that A is factorized in the form QW if

$$A = QW , \quad (4.34)$$

where the matrix Q is an orthogonal matrix and W is a W-matrix.

Theorem 4.4.

Let $A \in R^{n\times n}$, j_k defined by (4.26) and $y_{j_i} = a_{j_i}, i = 1,...,n$. Then, $S = (s_1,...,s_m)$ is orthogonal, AS is a Z-matrix and a QW factorization is recognized for A^T .

Proof:

The proof is same as the proof of Theorem 4.3.

Now, we shall show the way of implementing $s_{j_k} = P_k a_{j_k}$ avoiding the explicit use of the P matrices during the computation. We can write the matrix P_k as

$$P_k = P_{k-1} - \frac{s_{j_{k-1}} s_{j_{k-1}}^T}{\beta_{j_{k-1}}} = I - \sum_{i=1}^{k-1} \frac{s_{j_i} s_{j_i}^T}{\beta_{j_i}}. \quad (4.35)$$

Thus,

$$s_{j_k} = P_k a_{j_k} = a_{j_k} - \sum_{i=1}^{k-1} \frac{s_{j_i}^T a_{j_k}}{s_{j_i}^T s_{j_i}} s_{j_i} . \quad (4.36)$$

The new formula is more stable numerically [Abaffy and Spedicato (1989)].

5. Conclusion

We computed the right inverse of a matrix using the right inverse of some submatrices of A. Our constructive approach allows us to choose the special submatrices and compute some new factorizations for A. We presented two new factorizations QW and QZ and show how our proposed approach computes WZ, ZW, QW and QZ factorization of a matrix. The general algorithm can be implemented for the WZ and the ZW factorizations with no more than $\frac{n^3}{3}+O(n^2)$ multiplications. The main storage for R_i and L_i is at most $\frac{n^2}{4}$. The computational cost for computing the QZ and the QW factorizations using general algorithm is no more than $\frac{3}{2}n^3+O(n^2)$ multiplications.

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