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Generalizations of Two Statistics on Linear Tilings

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Abstract

In this paper, we study generalizations of two well-known statistics on linear square-and-domino tilings by considering only those dominos whose right half covers a multiple of k, where k is a fixed positive integer. Using the method of generating functions, we derive explicit expressions for the joint distribution polynomials of the two statistics with the statistic that records the number of squares in a tiling. In this way, we obtain two families of q-generalizations of the Fibonacci polynomials. When $k = 1$, our formulas reduce to known results concerning previous statistics. Special attention is payed to the case $k = 2$. As a byproduct of our analysis, several combinatorial identities are obtained.

Keywords: Tilings, Fibonacci numbers, Lucas numbers, polynomial generalization

MSC 2010 No.: 11B39, 05A15, 05A19.

1. Introduction

Let F_n be the Fibonacci number defined by the recurrence $F_n = F_{n-1} + F_{n-2}$ if $n \geq 2$, with initial conditions $F_0 = 0$ and $F_1 = 1$. Let L_n be the Lucas number satisfying the same recurrence, but with $L_0 = 2$ and $L_1 = 1$. See, for example, sequences A000045 and A000032 in [Sloan (2010)]. Let $G_n = G_n(t)$ be the Fibonacci polynomial defined by $G_n = tG_{n-1} + G_{n-2}$ if $n \geq 2$, with $G_0 = 0$ and $G_1 = 1$; note that $G_n(1) = F_n$ for all n. See, for example, [Benjamin and Quinn (2003) p.

141]. Finally, let $\binom{m}{j}_q$ denote the q-binomial coefficient given by $\frac{[m]_q!}{[j]_q![m-j]_q!}$ if $0 \le j \le m$, where $[m]_q! = \prod_{i=1}^{m}[i]_q$ if $m \ge 1$ denotes the q-factorial and $[i]_q = 1+q+\cdots+q^{i-1}$ if $i \ge 1$ denotes the q-integer (with $[0]_q! = 1$ and $[0]_q = 0$). We will take $\binom{m}{j}_q$ to be zero if $0 \le m < j$ or if $j < 0$. Polynomial generalizations of F_n have arisen in connection with statistics on binary words Carlitz (1974), lattice paths [Cigler (2004)], Morse code sequences [Cigler(2003)], and linear domino arrangements [Shattuck and Wagner (2005, 2007)]. Let us recall now two statistics related to domino arrangements. If $n \ge 1$, then let $\mathcal{F}_n$ denote the set of coverings of the numbers $1,2,\ldots,n$, arranged in a row by indistinguishable dominos and indistinguishable squares, where pieces do not overlap, a domino is a rectangular piece covering two numbers, and a square is a piece covering a single number. The members of $\mathcal{F}_n$ are also called (linear) *tilings* or *domino arrangements*. (If $n = 0$, then $\mathcal{F}_0$ consists of the empty tiling having length zero.)

Note that such coverings correspond uniquely to words in the alphabet $\{d, s\}$ comprising i d's and $n-2i$ s's for some i, $0 \le i \le \lfloor n/2 \rfloor$.

In what follows, we will frequently identify tilings c by such words $c_1c_2\cdots$. For example, if $n = 4$, then $\mathcal{F}_4 = \{dd, dss, sds, ssd, ssss\}$. Note that $|\mathcal{F}_n| = F_{n+1}$ for all n. Given $\pi \in \mathcal{F}_n$, let $\nu(\pi)$ denote the number of dominos in π and let $\sigma(\pi)$ denote the sum of the numbers covered by the left halves of dominos in π. For example, if $n = 16$ and $\pi = sdsdsddssds \in \mathcal{F}_{16}$ (see Figure 1 below), then $\nu(\pi) = 5$ and $\sigma(\pi) = 2+5+8+10+14 = 39$.

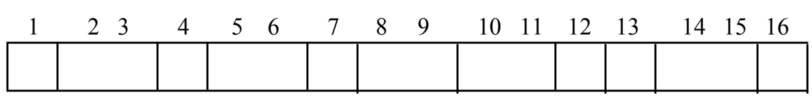

Figure 1. The tiling $\pi = sdsdsddssds \in \mathcal{F}_{16}$ has $\sigma(\pi) = 39$.

The following results concerning the distribution of the ν and σ statistics on $\mathcal{F}_n$ are well-known; see, e.g., [Shattuck and Wagner (2005)] or [Shattuck and Wagner (2007)], respectively:

$$\sum_{\pi \in \mathcal{F}_n} q^{\nu(\pi)} = \sum_{i=0}^{n} q^i \binom{n-i}{i} \tag{1}$$

and

$$\sum_{\pi\in\mathcal{F}_n} q^{\sigma(\pi)} = \sum_{i=0}^{n} q^{i^2}\binom{n-i}{i}_q. \tag{2}$$

Note that both polynomials reduce to F_{n+1} when $q=1$.

We remark that the polynomial in (2) first arose in a paper of Carlitz (1974), where he showed that it gives the distribution of the statistic $a_1+2a_2+\cdots+(n-1)a_{n-1}$ on the set of binary words $a_1a_2\cdots a_{n-1}$ with no consecutive ones. To see that this statistic is equivalent to the σ statistic on $\mathcal{F}_n$, simply append a 0 to any binary word of length $n-1$ having no two consecutive 1's and identify occurrences of 1 followed by a 0 as dominos and any remaining 0's as squares. The polynomials (2) or close variants thereof also appear in [Carlitz (1974, 1975), Cigler (2004)].

In this paper, we study generalizations of the ν and σ statistics obtained by considering only those dominos whose right half covers a multiple of k, where k is a fixed positive integer. More precisely, let ν_k record the number of dominos whose right half covers a multiple of k and let σ_k record the sum of the numbers of the form $ik-1$ covered by the left halves of dominos within a member of $\mathcal{F}_n$. The ν_k and σ_k statistics reduce to ν and σ when $k=1$. We remark that the ν_k statistic is related to a special case of the recurrence

$$Q_m = a_jQ_{m-1} + b_jQ_{m-2}, \qquad m \equiv j \pmod{k},$$

with $Q_0=0$ and $Q_1=1$, which was considered in [Petronilho (2012)] from a primarily algebraic standpoint through the use of orthogonal polynomials.

In the second and third sections, respectively, we consider the ν_k and σ_k statistics and obtain explicit formulas for their distribution on $\mathcal{F}_n$ (see Corollary 2.5 and Theorem 3.2 below), using the method of generating functions. Our formulas reduce to (1) and (2) when $k=1$ and involve q^k-binomial coefficients in the latter case. By taking ν_k and σ_k jointly with the statistic that records the number of squares within a tiling, we obtain q-generalizations of the Fibonacci polynomials G_n defined above. As a consequence of our analysis, several identities involving G_n are obtained. Special attention is payed to the case $k=2$, where some further combinatorial results may be given. Note that ν_2 records the number of dominos whose left half covers an odd number and σ_2 records the sum of the odd numbers covered by the left halves of these dominos.

2. A Generalization of the Statistic ν

Suppose k is a fixed positive integer. Given $\pi\in\mathcal{F}_n$, let $s(\pi)$ denote the number of squares of π and let $\nu_k(\pi)$ denote the number of dominos of π that cover numbers $ik-1$ and ik for some i, i.e., the number of dominos whose right half covers a multiple of k. For example, if $n=24$,

$k=3$, and $\pi = sdsdssddsdssdsdss \in \mathcal{F}_{24}$ (see Figure 2 below), then $s(\pi)=10$ and $\nu_3(\pi)=4$.

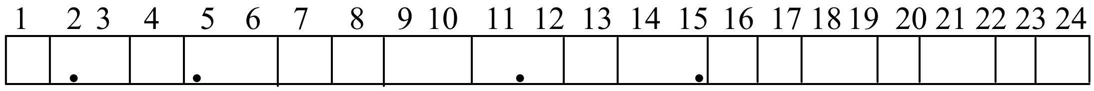

Figure 2. The tiling $\pi = sdsdssddsdssdsdss \in \mathcal{F}_{24}$ has $\nu_3(\pi)=4$.

If q and t are indeterminates, then define the distribution polynomial $a_n^{(k)}(q,t)$ by

$$a_n^{(k)}(q,t) := \sum_{\pi \in \mathcal{F}_n} q^{\nu_k(\pi)} t^{s(\pi)}, \qquad n \geq 1,$$

with $a_0^{(k)}(q,t) := 1$. For example, if $n=6$ and $k=3$, then

$$a_6^{(3)}(q,t) = t^2(t^2+1)(t^2+2) + q(t^2+1)(2t^2+1) + q^2t^2.$$

Note that $a_n^{(k)}(1,t) = G_{n+1}$ for all k and n.

In this section, we derive explicit formulas for the polynomials $a_n^{(k)}(q,t)$ and consider specifically the case $k=2$.

2.1. Preliminary Result

To establish our formulas for $a_n^{(k)}(q,t)$, we will need the following preliminary result, which was shown in (Shattuck). [See also (Petronilho (2012)] for an equivalent, though more complicated, formula involving determinants and Yayenie (2011) for the case $k=2$.) Given indeterminates $x_1, x_2, \ldots, x_k$ and $y_1, y_2, \ldots, y_k$, let p_n be the sequence defined by

$$p_0 = 0,\ p_1 = 1,\ p_n = \begin{cases} x_1 p_{n-1} + y_1 p_{n-2}, & \text{if } n \equiv 2 \pmod{k}; \\ x_2 p_{n-1} + y_2 p_{n-2}, & \text{if } n \equiv 3 \pmod{k}; \\ \vdots & \\ x_{k-1} p_{n-1} + y_{k-1} p_{n-2}, & \text{if } n \equiv 0 \pmod{k}; \\ x_k p_{n-1} + y_k p_{n-2}, & \text{if } n \equiv 1 \pmod{k}, \end{cases} \qquad (n \geq 2). \tag{3}$$

Let p_n^* be the generalized Fibonacci sequence defined by $p_0^* = 0$, $p_1^* = 1$, and $p_n^* = x_i p_{n-1}^* + y_i p_{n-2}^*$ if $n \geq 2$ and $n \equiv i \pmod{k}$. The sequence p_n then has the following Binet-like formula.

Theorem 2.1.

If $m \geq 0$ and $1 \leq r \leq k$, then

$$p_{mk+r} = \left(\frac{\alpha^m - \beta^m}{\alpha - \beta}\right) p_{k+r} + \gamma \left(\frac{\alpha^{m-1} - \beta^{m-1}}{\alpha - \beta}\right) p_r, \tag{4}$$

where α and β are the roots of the quadratic equation $x^2 - Lx - \gamma = 0$, $L = p_{k+1} + y_1 p^*_{k-1}$, and $\gamma = (-1)^{k+1} \prod_{j=1}^{k} y_j$.

2.2. General Formulae

For ease of notation, we will often suppress arguments and write a_n for $a_n^{(k)}(q,t)$. Using Theorem 2.1, one can give a Binet-like formula for a_n.

Theorem 2.2.

If $m \geq 0$ and $0 \leq r \leq k-1$, then

$$a_{mk+r} = \left(\frac{\alpha^m - \beta^m}{\alpha - \beta}\right) a_{k+r} + (-1)^{k+1} q \left(\frac{\alpha^{m-1} - \beta^{m-1}}{\alpha - \beta}\right) a_r, \tag{5}$$

where α and β are the roots of the quadratic equation

$$x^2 - (qG_{k-1}(t) + G_{k+1}(t))x + (-1)^k q = 0.$$

Proof:

Considering whether the last piece within a member of $\mathcal{F}_n$ is a square or a domino yields the recurrence

$$a_n = ta_{n-1} + qa_{n-2}, \qquad n \geq 2, \tag{6}$$

if n is divisible by k, and the recurrence

$$a_n = ta_{n-1} + a_{n-2}, \qquad n \geq 2, \tag{7}$$

if n is not, with the initial conditions $a_0 = 1$ and $a_1 = t$. By induction, recurrences (3), (6), and (7) together show that

$$a_n = p_{n+1}, \qquad n \geq 0, \tag{8}$$

where p_n denotes here the sequence defined by (3) with $x_1 = x_2 = \cdots = x_k = t$, $y_1 = y_2 = \cdots = y_{k-1} = 1$, and $y_k = q$. Thus, we have $\gamma = (-1)^{k+1}\prod_{j=1}^{k} y_j = (-1)^{k+1} q$ and

$$\begin{aligned} L &= p_{k+1} + y_1 p_{k-1}^* = a_k + p_{k-1}^* = t a_{k-1} + q a_{k-2} + p_{k-1}^* \\ &= tG_k(t) + qG_{k-1}(t) + G_{k-1}(t) = qG_{k-1}(t) + G_{k+1}(t), \end{aligned}$$

since $a_i = G_{i+1}(t)$ and $p_i^* = G_i(t)$ if $i < k$, as there is no domino whose right half covers a multiple of k . Formula (5) follows from writing $a_{mk+r} = p_{mk+r+1}$ and using (4), which completes the proof.

In determining our next formula for a_n , we will need the generating function for the sequence p_n given by (3).

Lemma 2.3.

If p_n is defined as above, then

$$\sum_{n\geq 0} p_n x^n = \frac{\sum_{r=0}^{k-1} p_r x^r + \sum_{r=0}^{k-1} (p_{k+r} - L p_r) x^{k+r}}{1 - Lx^k - \gamma x^{2k}}, \tag{9}$$

where L and γ are given in Theorem 2.1.

Proof:

From (4), we have

$$\begin{aligned} \sum_{n\geq 0} p_n x^n &= \sum_{r=0}^{k-1}\sum_{m\geq 0} p_{mk+r} x^{mk+r} \\ &= \frac{1}{\alpha-\beta}\sum_{r=0}^{k-1}\left(p_{k+r} + \frac{\gamma p_r}{\alpha}\right)\sum_{m\geq 0}\alpha^m x^{mk+r} - \frac{1}{\alpha-\beta}\sum_{r=0}^{k-1}\left(p_{k+r} + \frac{\gamma p_r}{\beta}\right)\sum_{m\geq 0}\beta^m x^{mk+r} \\ &= \frac{1}{\alpha-\beta}\left(\frac{1}{1-\alpha x^k}\sum_{r=0}^{k-1}\left(p_{k+r} + \frac{\gamma p_r}{\alpha}\right)x^r - \frac{1}{1-\beta x^k}\sum_{r=0}^{k-1}\left(p_{k+r} + \frac{\gamma p_r}{\beta}\right)x^r\right) \\ &= \frac{1}{\alpha-\beta}\left(\frac{1}{1-\alpha x^k}\sum_{r=0}^{k-1}\frac{\gamma p_r}{\alpha}x^r - \frac{1}{1-\beta x^k}\sum_{r=0}^{k-1}\frac{\gamma p_r}{\beta}x^r\right) \\ &\quad + \frac{1}{\alpha-\beta}\left(\frac{(\alpha-\beta)x^k}{(1-\alpha x^k)(1-\beta x^k)}\right)\sum_{r=0}^{k-1} p_{k+r} x^r. \end{aligned}$$

Note that

$$\frac{\gamma}{\alpha(1-\alpha x^k)}-\frac{\gamma}{\beta(1-\beta x^k)} = \gamma\frac{\beta(1-\beta x^k)-\alpha(1-\alpha)x^k}{\alpha\beta(1-\alpha x^k)(1-\beta x^k)}$$
$$=\frac{\alpha-\beta-(\alpha-\beta)(\alpha+\beta)x^k}{(1-\alpha x^k)(1-\beta x^k)}$$

since $\gamma=-\alpha\beta$. Thus, the first two sums in the last expression for $\sum_{n\geq 0}p_n x^n$ combine to give

$$\sum_{n\geq 0}p_n x^n = \frac{\sum_{r=0}^{k-1}(1-(\alpha+\beta)x^k)p_r x^r}{(1-\alpha x^k)(1-\beta x^k)}+\frac{\sum_{r=0}^{k-1}p_{k+r}x^{k+r}}{(1-\alpha x^k)(1-\beta x^k)}$$
$$=\frac{\sum_{r=0}^{k-1}p_r x^r+\sum_{r=0}^{k-1}(p_{k+r}-Lp_r)x^{k+r}}{1-Lx^k-\gamma x^{2k}}$$

since $L=\alpha+\beta$, which completes the proof. □

The generating function for the sequence a_n may be given explicitly as follows.

Theorem 2.4.

We have

$$\sum_{n\geq 0}a_n x^n = \frac{\sum_{r=0}^{k-1}G_{r+1}x^r+\sum_{r=0}^{k-1}(-1)^{r+1}G_{k-r-1}x^{k+r}}{1-(qG_{k-1}+G_{k+1})x^k+(-1)^k qx^{2k}}. \quad (10)$$

Proof:

By (8) and (9), we have

$$\sum_{n\geq 0} a_n x^n \quad = \sum_{n\geq 0} p_{n+1}x^n = \frac{\sum_{r=1}^{k-1} p_r x^{r-1} + \sum_{r=0}^{k-1}(p_{k+r} - Lp_r)x^{k+r-1}}{1 - Lx^k - \gamma x^{2k}}$$

$$= \frac{\sum_{r=1}^{k} a_{r-1}x^{r-1} + \sum_{r=1}^{k-1}(a_{k+r-1} - (qG_{k-1} + G_{k+1})a_{r-1})x^{k+r-1}}{1 - (qG_{k-1} + G_{k+1})x^k + (-1)^k qx^{2k}}$$

$$= \frac{\sum_{r=0}^{k-1} a_r x^r + \sum_{r=0}^{k-2}(a_{k+r} - (qG_{k-1} + G_{k+1})a_r)x^{k+r}}{1 - (qG_{k-1} + G_{k+1})x^k + (-1)^k qx^{2k}},$$

by $p_0 = 0$ and the expressions for L and γ given in the proof of Theorem 2.2 above. If $0 \leq r \leq k-2$, then $a_r = G_{r+1}$ and

$$a_{k+r} = qa_{k-2}a_r + a_{k-1}a_{r+1} = qG_{k-1}G_{r+1} + G_k G_{r+2},$$

the first relation upon considering whether or not the numbers $k-1$ and k are covered by a single domino within a member of $\mathcal{F}_{k+r}$. Thus,

$$\begin{aligned} a_{k+r} - (qG_{k-1} + G_{k+1})a_r \quad &= qG_{k-1}G_{r+1} + G_k G_{r+2} - (qG_{k-1} + G_{k+1})G_{r+1} \\ &= G_k G_{r+2} - G_{k+1}G_{r+1} \\ &= (-1)^{r+1} G_{k-r-1}, \end{aligned}$$

the last equality by the identity $(-1)^m G_{n-m} = G_{m+1}G_n - G_m G_{n+1}$, $0 \leq m \leq n$, which can be shown by induction (see [Benjamin and Quinn (2003), p. 30, Identity 47] for the case when $t = 1$). Substituting this into the last expression above for $\sum_{n\geq 0} a_n x^n$, and noting $G_0 = 0$, completes the proof.

□

Corollary 2.5.

If $m \geq 0$ and $0 \leq s \leq k-1$, then

$$\begin{aligned} a_{mk+s} \quad &= G_{s+1} \sum_{j=0}^{\frac{m}{2}} (-1)^{(k-1)j} q^j \binom{m-j}{j} (qG_{k-1} + G_{k+1})^{m-2j} \\ &+ (-1)^{s+1} G_{k-s-1} \sum_{j=0}^{\frac{m-1}{2}} (-1)^{(k-1)j} q^j \binom{m-1-j}{j} (qG_{k-1} + G_{k+1})^{m-1-2j}. \end{aligned} \tag{11}$$

Proof:

By (10), we have

$$\sum_{n\geq 0} a_n x^n = \frac{\sum_{r=0}^{k-1} G_{r+1}x^r + \sum_{r=0}^{k-1}(-1)^{r+1}G_{k-r-1}x^{k+r}}{1-x^k(qG_{k-1}+G_{k+1}-(-1)^k qx^k)}$$
$$= \left(\sum_{r=0}^{k-1} G_{r+1}x^r + \sum_{r=0}^{k-1}(-1)^{r+1}G_{k-r-1}x^{k+r}\right)\sum_{j\geq 0} x^{jk}(qG_{k-1}+G_{k+1}-(-1)^k qx^k)^j$$
$$= \left(\sum_{r=0}^{k-1} G_{r+1}x^r + \sum_{r=0}^{k-1}(-1)^{r+1}G_{k-r-1}x^{k+r}\right)\sum_{j\geq 0}\sum_{i=0}^{j}\binom{j}{i}(qG_{k-1}+G_{k+1})^{j-i}(-1)^{i(k-1)}q^i x^{ik+jk}.$$

Since each power of x in the infinite double sum on the right side of the last expression is a multiple of k for all i and j, only one term from each of the two finite sums on the left contributes towards the coefficient of x^{mk+s}, namely, the $r=s$ term. Thus, the coefficient to x^{mk+s} in the last expression is given by

$$G_{s+1}\sum_{j=0}^{m}((-1)^{k-1}q)^{m-j}\binom{j}{m-j}(qG_{k-1}+G_{k+1})^{2j-m}$$
$$+\quad (-1)^{s+1}G_{k-s-1}\sum_{j=0}^{m-1}((-1)^{k-1}q)^{m-j-1}\binom{j}{m-j-1}(qG_{k-1}+G_{k+1})^{2j+1-m}.$$

Replacing j by $m-j$ in the first sum and j by $m-1-j$ in the second gives (11).

□

Taking $k=1$ in (11) implies

$$a_n^{(1)}(q,t) = \sum_{j=0}^{n} q^j t^{n-2j}\binom{n-j}{j}, \qquad n\geq 0, \tag{12}$$

which is well-known (see, e.g., [Benjamin and Quinn (2003), Shattuck and C. Wagner (2005)]. Taking $k=2$ in (11) implies

$$a_n^{(2)}(q,t) = \begin{cases} Q(m)-Q(m-1), & \text{if } n=2m; \\ tQ(m), & \text{if } n=2m+1, \end{cases} \tag{13}$$

where

$$Q(m) = \sum_{j=0}^{\frac{m}{2}}(-1)^j q^j\binom{m-j}{j}(t^2+q+1)^{m-2j}.$$

Taking $k=3$ in (11) implies

$$a_n^{(3)}(q,t)=\begin{cases} R(m)-tR(m-1), & \text{if } n=3m; \\ tR(m)+R(m-1), & \text{if } n=3m+1; \\ (t^2+1)R(m), & \text{if } n=3m+2, \end{cases} \tag{14}$$

where

$$R(m)=\sum_{j=0}^{\frac{m}{2}} q^j \binom{m-j}{j} (t^3+(2+q)t)^{m-2j}.$$

Let $H_n = H_n(t)$ denote the Lucas polynomial defined by the recurrence $H_n = tH_{n-1} + H_{n-2}$ if $n \geq 2$, with $H_0 = 2$ and $H_1 = t$, or, equivalently, by $H_n = G_{n+1} + G_{n-1}$ if $n \geq 1$.

Corollary 2.6.

If $m \geq 0$ and $0 \leq s \leq k-1$, then

$$\begin{aligned} G_{mk+s+1} &= G_{s+1} \sum_{j=0}^{\frac{m}{2}} (-1)^{(k-1)j} \binom{m-j}{j} H_k^{m-2j} \\ &+ (-1)^{s+1} G_{k-s-1} \sum_{j=0}^{\frac{m-1}{2}} (-1)^{(k-1)j} \binom{m-1-j}{j} H_k^{m-1-2j}. \end{aligned} \tag{15}$$

In particular, we have

$$G_{mk} = G_k \sum_{j=0}^{\frac{m-1}{2}} (-1)^{(k-1)j} \binom{m-1-j}{j} H_k^{m-1-2j}, \qquad m \geq 0. \tag{16}$$

Proof:

Taking $q=1$ in (11) and noting $a_n^{(k)}(1,t) = G_{n+1}(t)$ for all k gives (15). Furthermore, if $s = k-1$ in (15), then the second sum drops out since $G_0 = 0$, which yields (16).

We were unable to find formulas (15) or (16) in the literature, though the $t=1$ case of (16) is similar in form to Identities V82 and V83 in (Benjamin and Quinn (2003), p. 145).

2.3. The Case $k = 2$.

We consider further the case when $k=2$. Note that $a_n^{(2)}(q,t)$ is the joint distribution polynomial on $\mathcal{F}_n$ for the statistics recording the number of squares and the number of dominos whose right half covers an even number. The next result follows from taking $k=2$ in formula (10), though we provide another derivation here. Let $a_n^{(2)} = a_n^{(2)}(q,t)$ and $a(x;q,t) = \sum_{n\geq 0} a_n^{(2)}(q,t)x^n$, which we'll often denote by $a(x)$.

Proposition 2.7.

We have

$$a(x;q,t) = \frac{1+tx-x^2}{1-(1+q+t^2)x^2+qx^4}. \tag{17}$$

Proof:

Considering whether or not a tiling ends in a square yields the recurrences

$$a_{2n}^{(2)} = ta_{2n-1}^{(2)} + qa_{2n-2}^{(2)}, \qquad n\geq 1,$$

and

$$a_{2n+1}^{(2)} = ta_{2n}^{(2)} + a_{2n-1}^{(2)}, \qquad n\geq 1,$$

with $a_0^{(2)} = 1$ and $a_1^{(2)} = t$. Multiplying the first recurrence by x^{2n} and the second by x^{2n+1}, summing both over $n\geq 1$, and adding the two equations that result implies

$$a(x)-tx-1 = tx(a(x)-1) + x^2\left(\frac{a(x)-a(-x)}{2}\right) + qx^2\left(\frac{a(x)+a(-x)}{2}\right),$$

which may be rewritten as

$$(2-2tx-(1+q)x^2)a(x) = 2+x^2(q-1)a(-x). \tag{18}$$

Replacing x with $-x$ in (18) gives

$$(2+2tx-(1+q)x^2)a(-x) = 2+x^2(q-1)a(x), \tag{19}$$

and solving the system of equations (18) and (19) in $a(x)$ and $a(-x)$ yields

$$a(x) = \frac{1+tx-x^2}{1-(1+q+t^2)x^2+qx^4},$$

as desired. □

We next consider some particular values of the polynomials $a_n^{(2)}(q,t)$.

Proposition 2.8

If $n \geq 1$, then

$$a_n^{(2)}(0,t) = \begin{cases} t^2(1+t^2)^{m-1}, & \text{if } n = 2m; \\ t(1+t^2)^m, & \text{if } n = 2m+1. \end{cases} \tag{20}$$

Proof:

We provide both algebraic and combinatorial proofs. Taking $q = 0$ in (17) implies

$$\begin{aligned} a(x;0,t) &= \frac{1+tx-x^2}{1-(1+t^2)x^2} = (1+tx-x^2)\sum_{m\geq 0}(1+t^2)^m x^{2m} \\ &= 1+\sum_{m\geq 1}((1+t^2)^m-(1+t^2)^{m-1})x^{2m} + \sum_{m\geq 0} t(1+t^2)^m x^{2m+1} \\ &= 1+\sum_{m\geq 1} t^2(1+t^2)^{m-1}x^{2m} + \sum_{m\geq 0} t(1+t^2)^m x^{2m+1}, \end{aligned}$$

from which (20) follows.

For a combinatorial proof, first let $n = 2m$, where $m \geq 1$. Then members π of $\mathcal{F}_n$ having zero ν_2 value are of the form

$$\pi = (sd^{a_1}s)(sd^{a_2}s)\cdots(sd^{a_\ell}s)$$

for some ℓ, where $a_i \geq 0$ for each $i \in [\ell] = \{1,2,\ldots,\ell\}$.

Note that the sequence $(a_1+1, a_2+1, \ldots, a_\ell+1)$ is a composition of m.

Thus, the polynomial $a_n^{(2)}(0,t)$ may be viewed as the weighted sum of compositions of m, where the weight of a composition having exactly ℓ parts is $t^{2\ell}$. Since there are $\binom{m-1}{\ell-1}$ compositions of m having ℓ parts, we have

$$a_n^{(2)}(0,t) = \sum_{\ell=1}^{m}\binom{m-1}{\ell-1}t^{2\ell} = \sum_{\ell=0}^{m-1}\binom{m-1}{\ell}t^{2\ell+2} = t^2(1+t^2)^{m-1},$$

which gives the even case.

If $n = 2m+1$, then the weighted sum of tilings πs, where $\pi \in \mathcal{F}_n$ has zero ν_2 value, is given by $t^2(1+t^2)^m$, by the even case. Dividing this by t (to account for the square that was added at the end) gives the odd case and completes the proof.

Proposition 2.9.

If $n \geq 1$, then

$$a_n^{(2)}(-1,t) = \begin{cases} G_{m+1}(t^2) - G_m(t^2), & \text{if } n = 2m; \\ tG_{m+1}(t^2), & \text{if } n = 2m+1. \end{cases} \tag{21}$$

Proof:

We provide both algebraic and combinatorial proofs of this result. Taking $q = -1$ in (17) and replacing x with x^2 and t with t^2 in

$$\sum_{m\geq 0} G_{m+1}(t)x^m = \frac{1}{1-tx-x^2}$$

implies

$$\begin{aligned} a(x;-1,t) &= \frac{1+tx-x^2}{1-t^2x^2-x^4} = (1+tx-x^2)\sum_{m\geq 0} G_{m+1}(t^2)x^{2m} \\ &= \sum_{m\geq 0} tG_{m+1}(t^2)x^{2m+1} + \sum_{m\geq 0}(G_{m+1}(t^2) - G_m(t^2))x^{2m}, \end{aligned}$$

which gives the result.

We provide a bijective proof of (21) in the case when $t = 1$, the general case being similar, and show

$$a_n^{(2)}(-1,1) = \begin{cases} F_{m-1}, & \text{if } n = 2m; \\ F_{m+1}, & \text{if } n = 2m+1. \end{cases} \tag{22}$$

To do so, define the sign of $\lambda \in \mathcal{F}_n$ by $sgn(\lambda) = (-1)^{\nu_2(\lambda)}$, and let $\mathcal{F}_n^e$ and $\mathcal{F}_n^o$ denote the subsets of $\mathcal{F}_n$ whose members have positive and negative sign, respectively. Then $a_n^{(2)}(-1,1) = |\mathcal{F}_n^e| - |\mathcal{F}_n^o|$ and it suffices to identify a subset $\mathcal{F}_n^*$ of $\mathcal{F}_n^e$ having cardinality F_{m-1} or F_{m+1}, along with a sign-changing involution of $\mathcal{F}_n - \mathcal{F}_n^*$.

Let $n = 2m$ and $\mathcal{F}_n' \subseteq \mathcal{F}_n$ consist of those coverings $\lambda = \lambda_1\lambda_2\cdots$ such that $\lambda_{2i-1} = \lambda_{2i}$ for all i. If

$\lambda \in \mathcal{F}_n - \mathcal{F}'_n$, then let i_o denote the smallest index i such that $\lambda_{2i-1} \neq \lambda_{2i}$, i.e., $\lambda_{2i-1}\lambda_{2i} = ds$ or sd. Let $f(\lambda)$ denote the covering that is obtained from λ by exchanging the positions of the $(2i_o - 1)$-st and $(2i_o)$-th pieces of λ, leaving all other pieces undisturbed. Then the mapping f is seen to be a sign-changing involution of $\mathcal{F}_n - \mathcal{F}'_n$.

We now define an involution of $\mathcal{F}'_n$. Let $\mathcal{F}^*_n \subseteq \mathcal{F}'_n$ consist of those members containing an even number of pieces and ending in a domino. Note that $\mathcal{F}^*_n \subseteq \mathcal{F}^e_n$ and that $|\mathcal{F}^*_n| = F_{m-1}$ since members of $\mathcal{F}^*_n$ are synonymous with members of $\mathcal{F}_m$ ending in a domino, upon halving. Observe further that if $\lambda \in \mathcal{F}'_n - \mathcal{F}^*_n$ has an odd number of pieces, then λ ends in a domino since n is even, while if λ has an even number of pieces, it must end in two squares. If $\lambda \in \mathcal{F}'_n - \mathcal{F}^*_n$, then let $g(\lambda)$ be obtained from λ by either changing the final domino to two squares or changing the final two squares to a domino. Then g is seen to be a sign-changing involution of $\mathcal{F}'_n - \mathcal{F}^*_n$. Combining the two mappings f and g yields a sign-changing involution of $\mathcal{F}_n - \mathcal{F}^*_n$, as desired.

If $n = 2m+1$, then apply the mapping f defined above to $\mathcal{F}_n$. Note that the set of survivors has cardinality F_{m+1}, upon halving, since they are of the form $\lambda = \lambda_1\lambda_2 \cdots \lambda_{2\ell}\lambda_{2\ell+1}$ for some ℓ, with $\lambda_{2i-1} = \lambda_{2i}$ for each $i \in [\ell]$ and $\lambda_{2\ell+1} = s$. This completes the proof of (22).

Let $t_n(\nu_2)$ denote the sum of the ν_2 values taken over all of the members of $\mathcal{F}_n$.

Proposition 2.10 .

If $n \geq 1$, then

$$t_n(\nu_2) = \begin{cases} \dfrac{nL_n + 4F_n}{10}, & \text{if } n \text{ is even;} \\[2ex] \dfrac{(n-1)L_n + 2F_{n-1}}{10}, & \text{if } n \text{ is odd.} \end{cases} \tag{23}$$

Proof:

To find $t_n(\nu_2)$, we consider the contribution of the dominos that cover the numbers $2i-1$ and $2i$ for some i fixed within all of the members of $\mathcal{F}_n$. Let $n = 2m+1$.

Note that there are $F_{2i-1}F_{2m+2-2i}$ dominos that cover the numbers $2i-1$ and $2i$ within all of the members of $\mathcal{F}_n$.

Summing over all i, we have

$$t_n(\nu_2)=\sum_{i=1}^{m}F_{2i-1}F_{2m-2i+2}=\sum_{i=0}^{m-1}F_{2i+1}F_{2m-2i}.$$

To simplify this sum, we recall the Binet formulas $F_n=\frac{\alpha^n-\beta^n}{\sqrt{5}}$ and $L_n=\alpha^n+\beta^n$, $n\geq 0$, where α and β denote the positive and negative roots, respectively, of the equation $x^2-x-1=0$. Then for m even, we have

$$\begin{aligned}
5\sum_{i=0}^{m-1}F_{2i+1}F_{2m-2i} &= \sum_{i=0}^{m-1}(\alpha^{2i+1}-\beta^{2i+1})(\alpha^{2m-2i}-\beta^{2m-2i})\\
&= \sum_{i=0}^{m-1}(\alpha^{2m+1}+\beta^{2m+1})+\sum_{i=0}^{\frac{m}{2}-1}(\alpha^{2m-4i-1}+\beta^{2m-4i-1})\\
&\quad -\sum_{i=\frac{m}{2}}^{m-1}(\alpha^{4i-2m+1}+\beta^{4i-2m+1})\\
&= \sum_{i=0}^{m-1}L_{2m+1}+\sum_{i=0}^{\frac{m}{2}-1}L_{2m-4i-1}-\sum_{i=0}^{\frac{m}{2}-1}L_{2m-4i-3}\\
&= mL_{2m+1}+\sum_{i=0}^{\frac{m}{2}-1}L_{2m-4i-2}=mL_{2m+1}+\sum_{i=0}^{\frac{m}{2}-1}(F_{2m-4i-1}+F_{2m-4i-3})\\
&= mL_{2m+1}+F_mF_{m+1}+F_{m-1}F_m=mL_{2m+1}+F_mL_m\\
&= mL_{2m+1}+F_{2m},
\end{aligned}$$

by Identities 28, 26 and 33 in Benjamin and Quinn (2003) and since $L_m=F_{m+1}+F_{m-1}$. Substituting $m=\frac{n-1}{2}$ gives the second formula when $n\equiv 1\,(\mathrm{mod}\ 4)$. A similar calculation gives the same formula when $n\equiv 3\,(\mathrm{mod}\ 4)$.

If $n=2m$ and m is odd, then similar reasoning shows that

$$\begin{aligned}
5t_n(\nu_2) &= 5\sum_{i=0}^{m-1} F_{2i+1}F_{2m-2i-1} \\
&= \sum_{i=0}^{m-1}(\alpha^{2m}+\beta^{2m}) + \sum_{i=0}^{\frac{m-1}{2}}(\alpha^{2m-4i-2}+\beta^{2m-4i-2}) + \sum_{i=\frac{m+1}{2}}^{m-1}(\alpha^{4i-2m+2}+\beta^{4i-2m+2}) \\
&= mL_{2m} + 2 + 2\sum_{i=0}^{\frac{m-3}{2}} L_{2m-4i-2} \\
&= mL_{2m} + 2\sum_{i=0}^{\frac{m-1}{2}} F_{2m-4i-1} + 2\sum_{i=0}^{\frac{m-3}{2}} F_{2m-4i-3} \\
&= mL_{2m} + 2F_mF_{m+1} + 2F_{m-1}F_m = mL_{2m} + 2F_{2m},
\end{aligned}$$

and the first formula in (23) follows when $n \equiv 2 \pmod 4$, upon replacing m with $\frac{n}{2}$. A similar calculation gives the same formula when $n \equiv 0 \pmod 4$.

We close this section with a general formula for $a_n^{(2)}(q,t)$.

Theorem 2.11.

If $n \geq 0$, then

$$a_n^{(2)}(q,t) = \begin{cases} q^m + \displaystyle\sum_{i=0}^{m-1}\sum_{j=0}^{i}\binom{m-i+j}{j}\binom{m-j-1}{i-j}q^j t^{2m-2i}, & \text{if } n = 2m; \\ \displaystyle\sum_{i=0}^{m}\sum_{j=0}^{i}\binom{m-i+j}{j}\binom{m-j}{i-j}q^j t^{2m-2i+1}, & \text{if } n = 2m+1. \end{cases} \tag{24}$$

***Proof*:**

We will refer to a domino whose left half covers an odd (resp., even) number as *odd-positioned* (resp., *even-positioned*). First suppose $n = 2m$ is even. If $\lambda \in \mathcal{F}_n$ contains no squares, then it consists of m odd-positioned dominos, whence the q^m term. So suppose that λ contains i dominos, where $0 \leq i \leq m-1$, and that j of the dominos are odd-positioned. There are $2m-2i$ squares and $m-i+1$ possible positions to insert each of the j odd-positioned dominos relative to the squares, whence there are $\binom{m-i+j}{j}$ choices concerning their placement. There are $m-i$ possible positions to insert each of the $i-j$ even-positioned dominos, whence there are

$\binom{m-j-1}{i-j}$ choices concerning their placement. Thus, there are $\binom{m-i+j}{j}\binom{m-j-1}{i-j}$ members of $\mathcal{F}_n$ containing i dominos, j of which are odd-positioned. Summing over all i and j gives the even case of (24). A similar argument applies to the odd case. □

Remark: Setting $q=0$ in (24) gives (20). Comparing the odd cases of (24) and (13) and replacing t with $\sqrt{t}$ gives the following polynomial identity in q and t:

$$\sum_{i=0}^{m}\sum_{j=0}^{i} q^j t^{m-i}\binom{m-i+j}{j}\binom{m-j}{i-j} = \sum_{j=0}^{\frac{m}{2}}(-1)^j q^j \binom{m-j}{j}(q+t+1)^{m-2j}, \qquad m \geq 0. \tag{25}$$

A similar identity can be obtained by comparing the even cases of (24) and (13). Setting $q=-1$ in (24), comparing with (21), and replacing t with $\sqrt{t}$ gives a pair of formulas for $G_m(t)$.

3. A Generalization of The Statistic σ

Suppose k is a fixed positive integer. Given $\pi \in \mathcal{F}_n$, let $s(\pi)$ denote the number of squares of π and let $\sigma_k(\pi)$ denote the sum of the numbers of the form $ik-1$ that are covered by the left half of a domino. For example, if $n=24$, $k=4$, and $\pi = ssdsddsdssddssdd \in \mathcal{F}_{24}$ (see Figure 3 below), then $s(\pi)=8$ and $\sigma_4(\pi)=3+11+15+23=52$. If q and t are indeterminates, then define the distribution polynomial $b_n^{(k)}(q,t)$ by

$$b_n^{(k)}(q,t) := \sum_{\pi \in \mathcal{F}_n} q^{\sigma_k(\pi)} t^{s(\pi)}, \qquad n \geq 1,$$

with $b_0^{(k)}(q,t) := 1$. For example, if $n=6$ and $k=3$, then

$$b_6^{(3)}(q,t) = t^2(t^2+1)(t^2+2) + q^2t^2(t^2+1) + q^5(t^2+1)^2 + q^7t^2.$$

Note that $b_n^{(k)}(1,t) = G_{n+1}$ for all k and n.

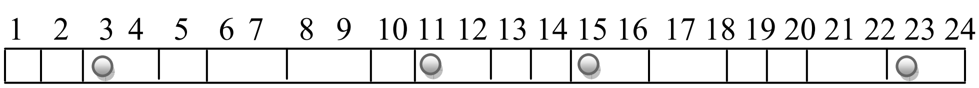

Figure 3. The tiling $\pi = ssdsddsdssddssdd \in \mathcal{F}_{24}$ has $\sigma_4(\pi) = 52$.

In what follows, we will often suppress arguments and write b_n for $b_n^{(k)}(q,t)$. Considering whether the last piece within a member of $\mathcal{F}_n$ is a square or a domino yields the recurrence

$$b_n = tb_{n-1} + q^{n-1}b_{n-2}, \qquad n \geq 2,$$

if n is divisible by k, and the recurrence

$$b_n = tb_{n-1} + b_{n-2}, \qquad n \geq 2,$$

if n is not, with initial conditions $b_0 = 1$ and $b_1 = t$. In [4], Carlitz studied the polynomials $b_{n-1}^{(1)}(q,t)$ from an algebraic point of view. See also the related paper by Cigler (2003).

In this section, we will derive explicit formulas for the polynomials $b_n^{(k)}(q,t)$ and their generating function, with specific consideration of the case $k = 2$. Note that $\sigma_2(\pi)$ records the sum of the odd numbers covered by left halves of dominos in π.

3.1. General Formulas

We first establish an explicit formula for the generating function of the sequence b_n.

Theorem 3.1.

We have

$$\sum_{n\geq 0} b_n x^n = \sum_{r=0}^{k-3} G_{r+1}x^r + q^{k-1}c_{k-1}(q^k x^k)\sum_{r=0}^{k-3} G_{r+1}x^{k+r} + c_k(x^k)\sum_{r=0}^{k-3} G_{r+2}x^{k+r} + x^{k-2}c_{k-1}(x^k) + x^{k-1}c_k(x^k), \tag{26}$$

where

$$c_{k-1}(x) = \sum_{j\geq 0} \frac{x^j q^{k\binom{j+1}{2}-j} \prod_{i=0}^{j}(G_{k-1} + (-1)^{k-1}xq^{ik})}{\prod_{i=0}^{j}(1 - xq^{ik}G_{k+1})}$$

and

$$c_k(x) = G_k \sum_{j\geq 0} \frac{x^j q^{k\binom{j+1}{2}-j} \prod_{i=1}^{j}(G_{k-1} + (-1)^{k-1}xq^{ik})}{\prod_{i=0}^{j}(1 - xq^{ik}G_{k+1})}.$$

Proof:

It is more convenient to first consider the generating function for the numbers $b_n' := b_{n-1}^{(k)}(q,t)$.

Then the sequence b'_n has initial values $b'_0 = 0$ and $b'_1 = 1$ and satisfies the recurrences

$$b'_{mk+r} = tb'_{mk+r-1} + b'_{mk+r-2}, \qquad 2 \le r \le k \qquad \text{and} \qquad m \ge 0, \tag{27}$$

with

$$b'_{mk+1} = tb'_{mk} + q^{mk-1}b'_{mk-1}, \qquad m \ge 1. \tag{28}$$

Let

$$c_r(x) = \sum_{m\ge 0} b'_{mk+r}x^m,$$

where $r \in [k]$. Then multiplying the recurrences (27) and (28) by x^m, and summing the first over $m \ge 0$ and the second over $m \ge 1$, gives

$$\begin{aligned} c_r(x) &= tc_{r-1}(x) + c_{r-2}(x), \quad r = 3,4,\ldots,k, \\ c_2(x) &= tc_1(x) + xc_k(x), \\ c_1(x) &= 1 + xtc_k(x) + xq^{k-1}c_{k-1}(q^k x). \end{aligned}$$

By induction on r, we obtain

$$c_r(x) = G_r + G_{r+1}xc_k(x) + G_r xq^{k-1}c_{k-1}(q^k x), \qquad 1 \le r \le k. \tag{29}$$

Taking $r = k$ and $r = k-1$ in (29) gives

$$c_k(x) = \frac{G_k}{1 - xG_{k+1}} + \frac{xq^{k-1}G_k}{1 - xG_{k+1}} c_{k-1}(q^k x)$$

and

$$\begin{aligned} c_{k-1}(x) &= G_{k-1} + G_k xc_k(x) + G_{k-1}xq^{k-1}c_{k-1}(q^k x) \\ &= G_{k-1} + G_k x\left(\frac{G_k}{1 - xG_{k+1}} + \frac{xq^{k-1}G_k}{1 - xG_{k+1}} c_{k-1}(q^k x)\right) + G_{k-1}xq^{k-1}c_{k-1}(q^k x) \\ &= \frac{G_{k-1} + x(G_k^2 - G_{k-1}G_{k+1})}{1 - xG_{k+1}} + \frac{x^2q^{k-1}(G_k^2 - G_{k-1}G_{k+1}) + xq^{k-1}G_{k-1}}{1 - xG_{k+1}} c_{k-1}(q^k x) \\ &= \frac{G_{k-1} + x(-1)^{k-1}}{1 - xG_{k+1}} + \frac{xq^{k-1}(G_{k-1} + x(-1)^{k-1})}{1 - xG_{k+1}} c_{k-1}(q^k x), \end{aligned}$$

where we have used the identity $G_k^2 - G_{k-1}G_{k+1} = (-1)^{k-1}$ [see, e.g., (Benjamin and Quinn (2003), Identity 246].

Iterating the last recurrence gives

$$\begin{aligned} c_{k-1}(x) &= \sum_{j\geq 0} \frac{G_{k-1}+(-1)^{k-1}xq^{kj}}{1-xq^{kj}G_{k+1}} \prod_{i=1}^{j} \frac{xq^{ik-1}(G_{k-1}+(-1)^{k-1}xq^{(i-1)k})}{1-xq^{(i-1)k}G_{k+1}} \\ &= \sum_{j\geq 0} \frac{x^j q^{k\binom{j+1}{2}-j} \prod_{i=0}^{j}(G_{k-1}+(-1)^{k-1}xq^{ik})}{\prod_{i=0}^{j}(1-xq^{ik}G_{k+1})}, \end{aligned}$$

which implies

$$\begin{aligned} c_k(x) &= \frac{G_k}{1-xG_{k+1}} + G_k \sum_{j\geq 1} \frac{x^j q^{k\binom{j}{2}+kj-j} \prod_{i=0}^{j-1}(G_{k-1}+(-1)^{k-1}xq^{(i+1)k})}{\prod_{i=0}^{j}(1-xq^{ik}G_{k+1})} \\ &= G_k \sum_{j\geq 0} \frac{x^j q^{k\binom{j+1}{2}-j} \prod_{i=1}^{j}(G_{k-1}+(-1)^{k-1}xq^{ik})}{\prod_{i=0}^{j}(1-xq^{ik}G_{k+1})}. \end{aligned}$$

Then, by (29), we have

$$\begin{aligned} \sum_{n\geq 0} b'_n x^n &= \sum_{r=1}^{k}\sum_{m\geq 0} b'_{mk+r} x^{mk+r} = \sum_{r=1}^{k} x^r c_r(x^k) \\ &= \sum_{r=1}^{k-2} x^r (G_r + G_{r+1}x^k c_k(x^k) + G_r x^k q^{k-1} c_{k-1}(q^k x^k)) + x^{k-1}c_{k-1}(x^k) + x^k c_k(x^k) \\ &= \sum_{r=1}^{k-2} G_r x^r + c_k(x^k)\sum_{r=1}^{k-2} G_{r+1}x^{k+r} + q^{k-1}c_{k-1}(q^k x^k)\sum_{r=1}^{k-2} G_r x^{k+r} \\ &\quad + x^{k-1}c_{k-1}(x^k) + x^k c_k(x^k), \end{aligned}$$

where $c_{k-1}(x)$ and $c_k(x)$ are as given. Formula (26) now follows upon noting

$$\sum_{n\geq 0} b_n^{(k)}(q,t)x^n = \sum_{n\geq 0} b'_{n+1}x^n = \frac{1}{x}\sum_{n\geq 0} b'_n x^n.$$

One can find explicit expressions for the b_n using Theorem 3.1 and the following formulas that involve the q-binomial coefficient [see, e.g., (Andrews (1976) or Stanley (1997)]:

$$\frac{x^j}{\prod_{i=0}^{j}(1-xq^i)}=\sum_{a\geq j}\binom{a}{j}_q x^a \tag{30}$$

and

$$\prod_{i=1}^{j}(y+xq^i)=\sum_{a=0}^{j}q^{\binom{a+1}{2}}\binom{j}{a}_q x^a y^{j-a}, \tag{31}$$

where j is a non-negative integer.

Theorem 3.2.

The following formulas hold for b_n. If $m\geq 0$, then

$$b_{mk+k-1}=G_k\sum_{j=0}^{m}q^{k\binom{j+1}{2}-j}\sum_{a=0}^{m}(-1)^{(k-1)a}q^{k\binom{a+1}{2}}G_{k-1}^{j-a}G_{k+1}^{m-a-j}\binom{j}{a}_{q^k}\binom{m-a}{j}_{q^k}. \tag{32}$$

If $m\geq 0$, then

$$b_{mk+k-2}=\sum_{j=0}^{m}q^{k\binom{j+1}{2}-j}\sum_{a=0}^{m}(-1)^{(k-1)a}q^{k\binom{a}{2}}G_{k-1}^{j-a+1}G_{k+1}^{m-a-j}\binom{j+1}{a}_{q^k}\binom{m-a}{j}_{q^k}. \tag{33}$$

If $m\geq 1$ and $0\leq r\leq k-3$, then

$$\begin{aligned}b_{mk+r}&=G_kG_{r+2}\sum_{j=0}^{m-1}q^{k\binom{j+1}{2}-j}\sum_{a=0}^{m-1}(-1)^{(k-1)a}q^{k\binom{a+1}{2}}G_{k-1}^{j-a}G_{k+1}^{m-a-j-1}\binom{j}{a}_{q^k}\binom{m-a-1}{j}_{q^k}\\&+q^{km-1}G_{r+1}\sum_{j=0}^{m-1}q^{k\binom{j+1}{2}-j}\sum_{a=0}^{m-1}(-1)^{(k-1)a}q^{k\binom{a}{2}}G_{k-1}^{j-a+1}G_{k+1}^{m-a-j-1}\binom{j+1}{a}_{q^k}\binom{m-a-1}{j}_{q^k},\end{aligned} \tag{34}$$

with $b_r=G_{r+1}$.

Proof:

Let $n=mk+k-1$, where $m\geq 0$. Then the coefficient of x^n on the right-hand side of (26) is given by

$$[x^m](c_k(x)) = G_k \sum_{j\geq 0}[x^m]\left(\frac{x^j q^{k\binom{j+1}{2}-j}\prod_{i=1}^{j}(G_{k-1}+(-1)^{k-1}xq^{ik})}{\prod_{i=0}^{j}(1-xq^{ik}G_{k+1})}\right).$$

By (30) and (31), we have for each $j \geq 0$,

$$\frac{(xG_{k+1})^j}{\prod_{i=0}^{j}(1-xq^{ki}G_{k+1})} = \sum_{a\geq j}\binom{a}{j}_{q^k} x^a G_{k+1}^a$$

and

$$\prod_{i=1}^{j}(G_{k-1}+(-1)^{k-1}xq^{ik}) = \sum_{a=0}^{j}(-1)^{(k-1)a} q^{k\binom{a+1}{2}}\binom{j}{a}_{q^k} x^a G_{k-1}^{j-a},$$

so that coefficient of x^n is given by

$$G_k \sum_{j=0}^{m}\frac{q^{k\binom{j+1}{2}-j}}{G_{k+1}^j}\sum_{a=0}^{m}(-1)^{(k-1)a} q^{k\binom{a+1}{2}} G_{k-1}^{j-a}\binom{j}{a}_{q^k}\cdot\binom{m-a}{j}_{q^k} G_{k+1}^{m-a},$$

which yields (32). Similar proofs apply to formulas (33) and (34), in the latter case, upon extracting the coefficient of x^n from two separate terms in (26).

When $k=1$ in Theorem 3.2, the inner sum in (32) reduces to a single term since $G_0 = 0$ and gives

$$b_n^{(1)}(q,t) = \sum_{j=0}^{n} q^{j^2}\binom{n-j}{j}_q t^{n-2j}, \qquad n \geq 0, \tag{35}$$

which is well-known [see, e.g., Shattuck and Wagner (2005)] When $k=2$ in Theorem 3.2, we get for all $m \geq 0$ the formulas

$$b_{2m}^{(2)}(q,t) = \sum_{j=0}^{m} q^{j^2}\sum_{a=0}^{m}(-1)^a q^{a^2-a}(t^2+1)^{m-a-j}\binom{j+1}{a}_{q^2}\binom{m-a}{j}_{q^2} \tag{36}$$

and

$$b_{2m+1}^{(2)}(q,t)=t\sum_{j=0}^{m}q^{j^2}\sum_{a=0}^{m}(-1)^a q^{a^2+a}(t^2+1)^{m-a-j}\binom{j}{a}_{q^2}\binom{m-a}{j}_{q^2}. \tag{37}$$

The polynomials $b_n^{(2)}(q,t)$ are considered in more detail below.

3.2. The Case $k=2$

Let us write $b_n^{(2)}$ for $b_n^{(2)}(q,t)$. The even and odd terms of the sequence $b_n^{(2)}$ satisfy the following two-term recurrences.

Proposition 3.3.

If $m\geq 2$, then

$$b_{2m}^{(2)}=(q^{2m-1}+t^2+1)b_{2m-2}^{(2)}-q^{2m-3}b_{2m-4}^{(2)}, \tag{38}$$

with $b_0^{(2)}=1$ and $b_2^{(2)}=t^2+q$, and

$$b_{2m+1}^{(2)}=(q^{2m-1}+t^2+1)b_{2m-1}^{(2)}-q^{2m-1}b_{2m-3}^{(2)}, \tag{39}$$

with $b_1^{(2)}=t$ and $b_3^{(2)}=t^3+(1+q)t$.

Proof:

To show (39), first note that the total weight of all the members of $\mathcal{F}_{2m+1}$ ending in d or ss is $b_{2m-1}^{(2)}$ and $t^2b_{2m-1}^{(2)}$, respectively. The weight of all members of $\mathcal{F}_{2m+1}$ ending in ds is $q^{2m-1}(b_{2m-1}^{(2)}-b_{2m-3}^{(2)})$. To see this, we insert a d just before the final s in any $\lambda\in\mathcal{F}_{2m-1}$ ending in s. By subtraction, the total weight of all tilings λ that end in s is $b_{2m-1}^{(2)}-b_{2m-3}^{(2)}$, and the inserted d contributes $2m-1$ towards the ν_2 value since it covers the numbers $2m-1$ and $2m$. For (38), note that by similar reasoning, the total weight of all members of $\mathcal{F}_{2m}$ ending in d, ss, and ds is $q^{2m-1}b_{2m-2}^{(2)}$, $t^2b_{2m-2}^{(2)}$, and $b_{2m-2}^{(2)}-q^{2m-3}b_{2m-4}^{(2)}$, respectively. □

We were unable to find, in general, two-term recurrences comparable to (38) and (39) for the sequences $b_{mk+r}^{(k)}(q,t)$, where k and r are fixed and $m\geq 0$. Let $b(x;q,t)=\sum_{n\geq 0}b_n^{(2)}(q,t)x^n$. Using (38) and (39), it is possible to determine explicit formulas for the generating functions $\sum_{m\geq 0}b_{2m}^{(2)}x^m$ and $\sum_{m\geq 0}b_{2m+1}^{(2)}x^m$ and thus for $b(x;q,t)$, upon proceeding in a manner analogous to the proof of Theorem 3.1 above. The following formula results, which may also be obtained by taking $k=2$ in Theorem 3.1.

Proposition 3.4.

We have

$$b(x;q,t)=\sum_{j\geq 0}\frac{x^{2j}q^{j^2}(1+tx-x^2)\prod_{i=1}^{j}(1-q^{2i}x^2)}{\prod_{i=0}^{j}(1-(t^2+1)q^{2i}x^2)}. \tag{40}$$

Taking $q=0$ and $q=-1$ in (40) shows that $b_n^{(2)}(0,t)$ and $b_n^{(2)}(-1,t)$ are the same as $a_n^{(2)}(0,t)$ and $a_n^{(2)}(-1,t)$ and are thus given by Propositions 2.8 and 2.9, respectively. This is easily seen directly since a member of $\mathcal{F}_n$ has zero σ_2 value if and only if it has zero ν_2 value and since the parity of the σ_2 and ν_2 values is the same for all members of $\mathcal{F}_n$. Comparing with (36) and (37) when $q=-1$, and replacing t with $\sqrt{t}$, then gives a pair of formulas for the Fibonacci polynomials.

Corollary 3.5.

If $m\geq 0$, then

$$G_{m+1}(t)-G_m(t)=\sum_{j=0}^{m}\sum_{a=0}^{m}(-1)^{a+j}(t+1)^{m-a-j}\binom{j+1}{a}\binom{m-a}{j} \tag{41}$$

and

$$G_{m+1}(t)=\sum_{j=0}^{m}\sum_{a=0}^{m}(-1)^{a+j}(t+1)^{m-a-j}\binom{j}{a}\binom{m-a}{j}. \tag{42}$$

Taking $q=1$ in (36) and (37), and noting $b_n^{(2)}(1,t)=G_{n+1}(t)$, gives another pair of formulas.

Corollary 3.6.

If $m\geq 0$, then

$$G_{2m+1}(t)=\sum_{j=0}^{m}\sum_{a=0}^{m}(-1)^{a}(t^2+1)^{m-a-j}\binom{j+1}{a}\binom{m-a}{j} \tag{43}$$

and

$$G_{2m+2}(t)=t\sum_{j=0}^{m}\sum_{a=0}^{m}(-1)^{a}(t^2+1)^{m-a-j}\binom{j}{a}\binom{m-a}{j}. \tag{44}$$

Let $t_n(\sigma_2)$ denote the sum of the σ_2 values taken over all of the members of $\mathcal{F}_n$. We conclude with the following explicit formula for $t_n(\sigma_2)$.

Proposition 3.7.

If $n \geq 0$, then

$$t_n(\sigma_2) = (-1)^n \frac{(2n+1)F_{n+2} - (2n+3)F_{n+1}}{8} + \frac{(6n^2+2n+15)F_{n+1} - (2n^2+2n+5)F_{n+2}}{40}. \qquad (45)$$

Proof:

To find $t_n(\sigma_2)$, first note that $\sum_{n\geq 0} t_n(\sigma_2)x^n = \frac{d}{dq} b(x;q,1)|_{q=1}$. By Proposition 3.4 and partial fractions, we have

$$\begin{aligned}\frac{d}{dq} b(x;q,1)|_{q=1} &= (1+x-x^2)\sum_{n\geq 0} \frac{n^2x^{2n}(1-x^2)^n}{(1-2x^2)^{n+1}} - \frac{x^2(1+x-x^2)}{1-x^2}\sum_{n\geq 0}\frac{n(n+1)x^{2n}(1-x^2)^n}{(1-2x^2)^{n+1}} \\ &+ \frac{x^2(1+x-x^2)}{1-2x^2}\sum_{n\geq 0}\frac{2n(n+1)x^{2n}(1-x^2)^n}{(1-2x^2)^{n+1}} \\ &= \frac{1}{x}\left(\frac{7+x}{8(1+x-x^2)} - \frac{3+4x}{4(1+x-x^2)^2} - \frac{9-7x}{8(1-x-x^2)} + \frac{3-3x}{(1-x-x^2)^2} - \frac{4-7x}{2(1-x-x^2)^3}\right).\end{aligned}$$

Note that

$$\begin{aligned}[x^n]\frac{1}{1-x-x^2} &= F_{n+1}, \\ [x^n]\frac{1}{(1-x-x^2)^2} &= \frac{(n+1)F_{n+2} + 2(n+2)F_{n+1}}{5}, \\ [x^n]\frac{1}{(1-x-x^2)^3} &= \frac{(5n+16)(n+1)F_{n+2} + (5n+17)(n+2)F_{n+1}}{50};\end{aligned}$$

see sequences A000045, A001629, and A001628, respectively, in Sloane (2010). Thus, the coefficient of x^n in $\frac{d}{dq} b(x;q,1)|_{q=1}$ is given by

$$(-1)^n \frac{(2n+1)F_{n+2} - (2n+3)F_{n+1}}{8} + \frac{(6n^2+2n+15)F_{n+1} - (2n^2+2n+5)F_{n+2}}{40},$$

which completes the proof. □

4. Conclusion

In this paper, we have studied two statistics on square-and-domino tilings that generalize previous ones by considering only those dominos whose right half covers a multiple of k, where k is a fixed positive integer. We have derived explicit formulas for all k for the joint distribution polynomials of the two statistics with the statistic that records the number of squares in a tiling. This yields two infinite families of q-generalizations of the Fibonacci polynomials. When $k=1$, our formulas reduce to prior results. Upon noting some special cases, several combinatorial identities were obtained as a consequence. Finally, it seems that other statistics on square-and-domino tilings could possibly be generalized. Perhaps one could also modify statistics on permutations and set partitions by introducing additional requirements concerning the positions, $\bmod k$, of various elements.

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