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Commutativity Results in Non Unital Real Topological Algebras

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Abstract

We give conditions entailing commutativity in certain non unital real topological algebras. Several other results of complex algebras are also examined for real ones.

Keywords: Commutativity, Le Page inequality, Banach algebra, topological algebra, non-unital algebra, circle operation

MSC 2010: 46J05, 46K05

1. Introduction

Le Page (1967) has shown that a complex unital Banach algebra E is necessarily commutative whenever it satisfies the following condition (Le Page's inequality)

(LP) $\|xy\| \leq \|yx\|$, with $x, y \in E$.

A counter-example has been given in the non-unital case by Cheikh and Oudadess (1988). The latter is a little bit ameliorated in Section 2 (Counter-Example 2.2). Niestegge (1984) came with an inequality, appropriate in the non-unital case (see Section 2); that is,

(N) $\|x + xy\| \leq \|x + yx\|$, for any $x, y \in E$

Tsertos (1986) displayed yet another one in that context

(T) $\|x + y + xy\| \leq \|x + y + yx\|$, with $x \in G^q(E)$ and $y \in E$,

where $G^q(E)$ is the set of q-invertible elements in E.

On the other hand, the quaternion's H is an example of a non commutative real unital Banach algebra satisfying (LP). This is due to the fact that the principal ingredient in the proof of Le Page is the theorem of Liouville on bounded holomorphic functions. Also none of (N) or (T) works in the non-unital real case (Counter-Examples 3.1 and 3.2). Following an idea in Oudadess (Med. J. Math. to appear), we have strengthened (T) and (N), to give an inequality which implies commutativity in the latter case (Section 3). Our inequality, adequately expressed, in general topological algebras, via a neighborhood basis, produces analogous results (Section 4).

This work is the outcome of an idea of the first author and another one of the second.

2. Preliminaries

In a unital algebra E (real or complex) the set of invertible elements is denoted by $G(E)$, and its complement by Sing(E), i.e., the set of singular elements of E. For a complex algebra, the spectrum of an element x is $Sp_E(x) = \{z \in C: x - ze \notin G(E)\}$. If the algebra is real, $Sp_E(x)$ stands for the spectrum of x in the complexification E_C of E. The spectral radius of x is $\rho(x) = \sup\{|z|: z \in Sp_C(x)\}$.

A topological algebra is an algebra E over K (R or C) endowed with a topological vector space T for which multiplication is separately continuous. If the map $(x, y) \rightarrow xy$ is continuous (in both variables), then E is said to be with continuous multiplication. We say that a unital topological algebra is a Q-algebra if the set $G(E)$ of its invertible elements is open.

Recall that, given a non-unital algebra E (real or complex), the circle operation, on E, is defined by $x \circ y = x+y+xy$. An element x is said to be quasi-invertible (q-invertible) if there is an $x' \in E$ such that $x \circ x' = 0$ and $x' \circ x = 0$. The set of q-invertible elements is denoted by $G^q(E)$. A non-unital topological algebra is a Q-algebra if the set $G^q(E)$ is open.

2.1. Complex Banach Case

In this section, attention is confined to the non-unital complex case. It is known that the Le Page inequality (LP) does not imply commutativity. The following counter- example has been given.

Counter-Example 2.1. Cheikh/Oudadess (1988). Let e_1, e_2 be two symbols such that $e_1^2 = 0$, $e_2^2 = 0$ and $e_i = 0\ e_j = 0$ for $i, j \in \{1, 2\}$. Consider the complex algebra $E_C(e_1, e_2)$ spanned by the two elements e_1 and e_2. It is the vector space with $\{e_1, e_2, e_1e_2, e_2e_1\}$ as a basis. A norm is given, on E, by

$$||x|| = \sum_{1 \leq i \leq 4} |x_i|,\ if\ x = \sum_{1 \leq i \leq 4} x_i$$

Then, E is a non commutative algebra such that $|xy| = |yx|$, for all x, y in E.

Actually one can exhibit algebra of dimension 3, as follows.

Counter-Example 2.2. Oudadess (Med. J. Math.) Consider any anticommutative algebra, i.e., one has $xy = -yx$, for all x, y in E. So $||xy|| = ||yx||$, for all x, y in E. Any square is clearly zero. Thus, due to anticommutativity, $xyx = 0$ for all x, y in E. Now take two different arbitrary elements e_1 and e_2 such that $e_1e_2 \neq 0$. Then the algebra E spanned by the two elements e_1 and e_2 has $\{e_1, e_2, e_1e_2\}$ as a basis.

Remark 2.3. Tsertos (1986), certainly guided by the central role played by the exponential function in the proof of Le Page's result (1967), considered the circle-exponent function, in the non-unital case. He then put the appropriate inequality; that is,

$$||x \circ y|| \leq ||y \circ x||$$

or yet

$$\text{(T)} \quad ||x + y + xy|| \leq ||x + y + yx||,$$

with $x \in G^q(E)$ and $y \in E$, where $G^q(E)$ is the set of quasi-invertible elements in E, thus pointing out the importance of $G^q(E)$.

Remark 2.4. One observes that Niestegge's inequality

$$\text{(N)} \quad ||x + xy|| \leq ||x + yx||, \text{ with } x, y \in E$$

is actually equivalent to

$$\text{(N}'\text{)} \quad ||x + xy|| \leq ||x + yx||, \text{ with } x \in G^q(E) \text{ and } y \in E$$

This brings it closer to Tsertos inequality (T). On the other hand, according to the proof of *Le* Page, inequality (*LP*) is equivalent to

$$\text{(LP}'\text{)} \quad ||xy \leq ||yx||, \text{ with } x \in G^q(E) \text{ and } y \in E.$$

Now, as a matter of fact, the inequalities (LP′), (N′) and (T) are actually equivalent in the unital case. To check that (LP′) implies (N′), take $e + y$ instead of y, and for the converse, take $e - y$ instead of y. To see that (T) implies (N′), let x be a quasi invertible element with x' its quasiinverse, and then take $x' + y$ instead of y. To show that (N′) implies (T), let x be a quasi-invertible element. Then e + x is invertible, so one has

$$||x + y + xy|| = ||x + (e + x)y||$$

$$= ||(e + x)[(e + x)^{-1}x + y]||$$

$$\leq ||[(e + x)^{-1}x + y](e + x)||$$

$$= ||x + y + yx||.$$

Remark 2.5. One seeks to understand why

(LP) $||xy|| \leq ||yx||$, for any $x, y \in E$

is sufficient in the unital complex case. Does the "proximity near zero" of ab and ba imply the same near every point x $\in E$? That is,

$$||x - ab|| \leq ||x - ba|| \text{ , for, any } x, a, b \in E.$$

But, as observed above, it is sufficient to consider $a \in G(E)$. Then,

$$||x - ab|| = ||a(a^{-1}x - b)|| \leq ||(a^{-1}\mathrm{x} \leq b)a|| = ||a^{-1}xa \leq ba|| \text{ .}$$

So one wonders if

$$||a^{-1}xa - ba|| \leq ||x - ba|| \text{ .}$$

Now to point out the role of the complex variable, recall that what is really used is the "proximity near zero" of $e^{\lambda a}b$ and $be^{\lambda a}$. Thus one has to prove that

$$||e^{-\lambda a} xe^{\lambda a} - be^{\lambda a} || \leq ||x - be^{\lambda a} ||.$$

In fact, one gets something more; indeed, take the following holomorphic function

$$g(\lambda) = (e^{-\lambda a} xe^{\lambda a} - be^{\lambda a}) - (x - be^{\lambda a}) = e^{-\lambda a} xe^{\lambda a} - x$$

Then, one has

$$||g(\lambda)|| \leq ||x|| + e^{-\lambda a}xe^{\lambda a} \leq ||x|| + ||x\, e^{-\lambda a}e^{\lambda a}|| = 2||x||,$$

which proves the boundedness of g. Hence,

$$e^{-\lambda a} xe^{\lambda a} - be^{\lambda a} = x - be^{\lambda a}$$

Remark 2.6. According to an interpretation of Prof. Mallios, the results of Le Page, Niestegge and Tsertos can be viewed as local which have become global ones.

3. Real Banach Case

Due to Remark 2.4, the inequalities of Niestegge and Tsertos work also in unital complex algebras. But none of them is sufficient in the real case. Indeed, we have the next counter-example.

Counter-Example 3.1.

The quaternions H is an example of non commutative real unital Banach algebra satisfying (N) and (T). Indeed for (N), one has

$$||x + xy|| = ||x(e + y)|| = ||(e + y)x|| = ||x + yx||$$

For (T), it is clearly satisfied if $x = -e$. If not, then $x + e$ is invertible. So, one has

$$||x + y + xy|| = ||x + (e + x)y|| = ||(e + x)[(e + x)^{-1}x + y]|| = ||[(e + x)^{-1} x + y](e + x)|| = ||x + y+ yx|| .$$

Now to obtain a non-unital counter-example, take any normed real vector space F. Make of it algebra by the trivial multiplication. Then the standard normed product algebra $H \times F$ is not unital, satisfies both (N) and (T), but it is not commutative.

Counter-Example 3.2. Consider any real anticommutative algebra, i.e., one has $xy = -yx$, for all x, y in E. So $||xy|| = ||yx||$, for all x, y in E. Any square is clearly zero. Thus, due to anticommutativity, $xyx = 0$ for all x, y in E. Now take two arbitrary different elements e_1 and e_2 such that $e_1e_2 \neq 0$. Then the algebra E spanned by the two elements e_1 and e_2 has $\{e_1, e_2, e_1e_2\}$ as a basis.

Remark 3.3. Let e_1, e_2 be two symbols such that $e_1^2 = 0$, $e_2^2 = 0$ and $e_i = 0$ $e_j = 0$ for $i, j \in \{1, 2\}$. Consider the real algebra $E_R(e_1, e_2)$ spanned by the two elements e_1 and e_2. It is the vector space with $\{e_1, e_2, e_1e_2, e_2e_1\}$ as a basis. A norm is given, on E, by

$$||x|| = \sum_{1 \leq i \leq 4} | x_i|, \text{ if } x = \sum_{1 \leq i \leq 4} x_i$$

Then E is a non commutative algebra such that $||xy|| = ||yx||$, for all x, y in E. It does not satisfy the inequality (N). Indeed, if

$$x = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_1e_2 + \alpha_4 e_2e_1 \text{ and } y = \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_1e_2 + \beta_4 e_2e_1,$$

then

$$xy = \alpha_1\beta_2 e_1e_2 + \alpha_2\beta_1 e_2e_1 \text{ and } yx = \alpha_2\beta_1 e_1e_2 + \alpha_1\beta_2 e_2e_1.$$

So inequality (N) yields

$$|\alpha_3 + \alpha_1\beta_2| + |\alpha_4 + \alpha_2\beta_1| \leq |\beta_3 + \alpha_2\beta_1| + |\beta_4 + \alpha_1\beta_2|.$$

But plenty of counter-examples can be given to the latter inequality. Take for instance, to begin with, $\alpha_2 = 0$ and $\beta_2 = 0$.

Remark 3.4. The origin of (T) and (N) cannot be directly found in (LP) itself, but in their use in the proof of Le Page's result. Indeed it is applied with $e^{\lambda x}\, y\, e^{-\lambda x}$, hence with $e^{\lambda x} \in G(E)$ and $ye^{-\lambda x} \in E$.

Remark 3.5. Due to Counter-Example 3.1, the inequalities (T) and (N) are not sufficient to imply commutativity in the non-unital case. They have to be reinforced. We exhibit an inequality which englobes both of them. Moreover, the following result is general. The expected commutativity results will easily ensue as corollaries.

Proposition 3.6. Let $(E, \|\ \|)$ and $(B, \|\ \|)$ be unital real normed algebras. Suppose $f : E \to R$ and $g : B \to R$ are continuous functions such that $f(0) = 0$, and $g(x) = 0$ implies $x = 0$. If $T: E \to B$ is a continuous algebra morphism such that

(C) $g\,[T(a - b \circ c)] \leq f(a - c \circ b)$; for, any $a, b \in G^q(E)$, and, $c \in E$,

then

$$T(xy) = T(yx), \text{ with } x, y \in E.$$

Proof :

Since T, f and g are continuous, the inequality is satisfied by the completion of $(E, \|\ \|)$. So without loss of generality, one can consider that E is a Banach algebra. Now take a q-invertible x $\in E$ and a q-invertible y $\in E$. One then has

$$g[T(x - y' \circ x \circ y)] \leq f(x - x \circ y \circ y') = f(x - x \circ 0) = f(x - x) = 0.$$

Whence $T(xy) = T(yx)$. If x or y is not q-invertible, consider the q-invertible elements $(\|x\| + 1)^{-1}\, x$ and $(\|y\| + 1)^{-1}\, y$ which, by the preceding, satisfy the latter equality.

Remark 3.7. Mallios has observed that the roles of b and c can be interverted in the inequality (C). Indeed, it can be replaced by the following one

(C′) $g\,[T(a - b \circ c)] \leq f(a - c \circ b)$; for any $a, b \in G^q(E)$ and $c \in E$.

Remark 3.8. If $(E, \|\ \|)$ is complete, then the maps T, f and g need not be continuous.

In the following corollaries, we restrict ourselves to Banach algebras. The statements in normed non complete ones are obtained by adding the needed continuity hypotheses.

Corollary 3.9. Let $(E, \| \|)$ be a unital real Banach algebra, endowed with two vector space norms $\| \|_1$ and $\| \|_2$. If there is a $k > 0$ such that

$\|a - b \circ c\|_1 \leq k\|a - c \circ b\|_2$; for any $a, b \in G^q(E)$ and $c \in E$,

then E is commutative.

An expected and rather more classical statement is the following.

Corollary 3.10. Let $(E, \| \|)$ be a unital real Banach algebra. If there is a $k > 0$ such that

(C) $\|a - b \circ c\| \leq k\|a - c \circ b\|$; with $a, b \in G^q(E)$ and $c \in E$,

then E is commutative.

Remark 3.11. Also, Le Page considered, for a given element a, the condition

$\|(a + \lambda)x\| \leq \|x(a + \lambda)\|,\ x \in E, \lambda \in C.$

He showed that a lies in the center of E. This is not true in the real case: take e.g. $E = H$. But the previous condition is strong enough to imply that an element a, satisfying it, is necessarily in the center. Moreover, this fact does not need here a particular proof (see the proof of Proposition 3.1).

Remark 3.12. Taking $a = 0$, one gets

$\|b \circ c\| \leq \|c \circ b\|,\ b, c \in E,$

that is, Tsertos inequality, which is satisfied by the quaternions H (cf. Counter- Example 3.1). But the latter does not satisfy the inequality in the previous proposition. Indeed, one has

$i - k \circ j = 2i - j - k$, so $\|i - k \circ j\| = 4$.

But,

$i - j \circ k = -j - k$, so $\|i - j \circ k\| = 2$.

Remark 3.13. Concerning the previous remark, one notices that in the unital case

$\|bc\| \leq \|cb\|$, with b, $c \in E$,

is equivalent to

$\|x + xy\| \leq \|x + yx\|$, with $x, y \in E$.

So the latter condition is not sufficient to imply the commutativity in the real case. But it is in the complex one.

Remark 3.14. It is tempting to ask whether in the inequality

$$\|a - b \circ c\| \leq \|a - c \circ b\|, \text{ with } a, b \in G^q(E) \text{ and } c \in E,$$

one can consider only $a = e$. Again the quaternion's H is a counter-example. Indeed,

$$\|e - bc\| = \|b\| \, \|b^{-1} - c\| = \|b^{-1} - c\| \, \|b\| = \|e - cb\| .$$

In H one has $\| \; \| = r(\;)$, which does not imply commutativity. The analogue of another criterion of Le Page is the following. Notice that we do not use the spectral radius given by the complexification.

Corollary 3.15. Let $(E, \| \; \|)$ be a real normed algebra. If E satisfies

$$\|a - b \circ c\| \leq r(a - c \circ b); \text{ with } a, b \in G^q(E) \text{ and } c \in E,$$

then it is commutative. By standard arguments, the condition in the following corollary implies that of the previous one.

Corollary 3.16. Let $(E, \| \; \|)$ be a unital real Banach algebra. Suppose that there exist a continuous norm $\| \; \|_1$ and $k > 0$ such that

$$\|a - b \circ c\|_1^2 \leq k\|(a - c \circ b)^2\|_1, \text{ with } a, b \in G^q(E) \text{ and } c \in E;$$

then E is commutative.

In unital complex Banach algebras, it is known that submultiplicativity - or equiv- alently subadditivity - of the spectral radius implies the commutativity modulo the (Jacobson) radical, cf. Hirshfeld/Zelazko (1968). Again, the quaternion's H is a counter-example in the real case. The analogue of that in the latter case is the following result. It is an immediate consequence of the proposition above. Recall that for an element x in E, r(x) is defined by $r(x) = \lim \|x^n\|1/n$. It is not granted here that $r(x) = \rho(x)$, the spectral radius of x. Thus, one has

Corollary 3.17. Let $(E, \| \; \|)$ be a real Banach algebra. If

$$(a - b \circ c) \leq r(a - c \circ b); \; a, b; c \in E, \; r(x) = 0 \Rightarrow x = 0, x \in E,$$

then E is commutative.

An outcome of the approach adopted here is the determination of all bilinear mappings on a given normed algebra, which reduce to commutative circle multiplications.

Proposition 3.18. Let $(E, \|\ \|)$ be a non-unital real normed algebra, and h : $E \times E \to E$ a continuous bilinear mapping. Suppose that $f : E \to R_+$ and g: $B \to R_+$ are continuous functions such that $f(0) = 0$, and $g(x) = 0$ implies $x = 0$. If $g[a - h(b, c)] \leq f(a - c \circ b)$; $a, b, c \in E$, then,

$h(x, y) = y \circ x$; with $x, y \in E$.

That is, E is commutative.

Proof :

Taking x, y q-invertibles in E, one has

$$g[x - h(y' \circ x, y)] \leq f(x - y \circ y' \circ -x) = f(x - x) = 0.$$

Whence, $x = h(y' \circ x, y)$. Putting $y \circ x$ instead of x, in the latter equality, we get

$h(x, y) = y \circ x$, for any $a, b \in G^q(E)$.

If x is q-invertible but not y, then by the preceding

$$h\ [(x, e - (\|y\| + 1)^{-1}y)] = [e - (\|y\| + 1)^{-1}y)] \circ x.$$

One also has

$$h\ [(e - (\|x\| + 1)^{-1}x, y)] = y \circ [\mathrm{e} - (\|y\| + 1)^{-1}y)],$$

if x is not q-invertible but y is. Thus, to finish the proof, use the bilinearity of h. Use (i) and the Proposition 3.6.

Remark 3.19. If $(E, \|\ \|)$ is complete, then the maps h, f and g need not be continuous.

4. Real Topological Case

The analogue of the reinforced inequality (C) of Section 3 (Corollary 3.10), in the normed case, appears in the following result.

Proposition 4.1. Let (E, τ) be a real Hausdorff topological algebra which is also a Q-algebra. If for any $a, b \in G^q(E)$, $V \in V(0)$ and $c \in E$ with $a - b \circ c \in V$ one has $a - c \circ b \in V$, then E is commutative.

Proof:

Let $x, y \in E$. If $x, y \in G^q(E)$, then

$0 = x - x \circ y \circ y' \in V$; hence $x - y' \circ x \circ y \in V$, for every $V \in V(0)$.

Hence $x - y' \circ x \circ y = 0$. Whence $y \circ x = x \circ y$. If $y \notin G^q(E)$, there is a neighborhood U of zero such that $U \subset G^q(E)$. There is also $\lambda \in R$ such that $\lambda y \in U$. By the preceding, $\lambda y \circ x = x \circ \lambda y$. Whence, $y \circ x = x \circ y$. One argues the same way if $x \notin G^q(E)$.

Remark 4.2. The quaternions does not satisfy the condition in the previous proposition. Indeed, take i, j and k in H. One has $k \in B(j, 2^{-1} 2^{1/2}) \cup \{k = ij\}$. But, $ji = -k \cup\notin B(j, 2^{-1} 2^{-1/2}) \cup \{k = ij\}$.

Remark 4.3. So many other results, in the frame of Banach algebras, can of course be stated in the setting of this section.

5. Conclusion

Le Page (1967) initiated the study of commutativity in general Banach algebras, showing that the inequality

(LP) $||xy|| \leq ||yx||$, for any $x, y \in E$

implies commutativity in unital complex Banach algebras. the existence of a unit is essential. This fact has been pointed out in Cheikh/Oudadess (1988); see also Oudadess (2011). In the non unital case, Niestegge (1984) exhibited an inequality which entails commutativity, that is

(N) $||x + xy|| \leq ||x + yx||$, for any $x, y \in E$.

Tsertos also came with another one, employing the circle operation, that is

(T) $||x + y + xy|| \leq ||x + y + yx||$, $\forall\ (x, y) \in G^q(E) \times E$.

The H of quaternion's is a real unital Banach algebra satisfying (LP). But it is not commutative. In Oudadess (2011), (LP) was reinforced, so as to be valid for the real case. The appropriate inequality is

$||a + bc|| \leq ||a + cb||$ for any $a, b \in G^q(E)$, and $c \in E$.

In this paper, the situation is completely clarified. Putting together the idea in Oudadess (2011) and another one in Tsertos (1986), we display an inequality which works in all cases, that is;

(C) $||a - b \circ c|| \leq k||a - c \circ b||$, for any $a, b \in G^q(E)$, and $c \in E$.

Moreover, it can be expressed, in terms of neighborhoods, in general topological algebras.

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