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On Lattice Structure of the Probability Functions on L^*

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Abstract:

In this paper, the set of all probability functions on L^* is studied, where L^* is the lattice of both-valued fuzzy sets or intuitionistic fuzzy sets. It is shown that the set of all probability functions on L^* endowed with two appropriate operations has a monoid structure which is also a distributive complete lattice. Also the lattice structure of the set of all probability functions on L^* induced by an appropriate function on [0, 1] to itself is studied. Some lattice (dual) isomorphisms are discussed that suggests probabilities on L^* could be considered in the framework of theories modeling imprecision.

Keywords: Probability, lattice, monoid, complete lattice, fuzzy set, intuitionistic fuzzy set

MSC 2010: 06B23, 06D30

1. Introduction

Deschrijver and Kerre (2003) have shown that the underlying structure of both interval-valued fuzzy sets and intuitionistic fuzzy sets is an L^*-fuzzy set with respect to the lattice L^*, in the sense of Goguen (1967). Deschrijver and Kerre (2007) also discussed the position of intuitionistic fuzzy set theory in the framework of theories modeling imprecision, where an overview of interrelationships that exists between intuitionistic fuzzy set theory and other theories modeling imprecision is described. In this direction, the study of intuitionistic balanced operators is studied by Saeb and Mashinchi (2008) which reveals an extension to intuitionistic fuzzy set theory. A complete study of this topic is reported by Saeb (2009).

A probability p on L^* has been studied by K. Lendelova and Riecan (2006). They found the representation for a probability p on L^* with respect to the Lukasiewicz connectives. Recently, Saeb and Mashinchi (2007) followed this trend and extended the notion of a probability on a balanced lattice, which is introduced by Homenda (2006). This topic is also considered from different points of view by M. Rencova (2010), Riecan (2006) and Riecan and Petrovicov (2010).

The study of algebraic structures of e-implications and pseudo-e-implications on the lattice L^* are considered by Khaledi et al. (2005) and (2007). Inspired by the research on the study of algebraic structures of implications on L^*, and the direction of the study of probabilities on the lattice L^*, in this paper, the set of all probability functions on L^* is considered and it is shown this set endowed with two appropriate operations has a monoid structure which is also a distributive complete lattice with De-Morgan algebra. Then, several other related lattice structures are provided. The results of this paper suggest that probabilities on L^* can be considered as the representation of modeling imprecision when viewed from the perspective of Deschrijver et al. (2007). Kaburlasos and Ritter (2007) demonstrated that lattice theory may suggest viable alternatives in practical clustering, classification, pattern analysis and regression applications as worthily noted by Ajmal and Jain (2009) in their recent research. The lattice structures studied in this paper are therefore very useful apparatus in applications as explained by Ajmal, Naseem et al. (2009) that the system of lattice algebra plays a significant role in information theory and can be used within the numerous subfields of computational intelligence. These quotations stress that the results reported in this paper have their potential values both from the theoretical and application points of view in information processing.

The organization of this paper is as follows. Following this introduction some preliminaries are discussed in Section 2. Here the structure of the lattice L^* and the definition of probability on L^* are reviewed. In Section 3, the algebraic structure of the set **POL**, of all probabilities on L^*, is studied. In Section 4, we induce a probability function on L^* by a function $f:[0,1]\rightarrow[0,1]$. Then we study the distributive complete lattice structure of the set $\mathbf{POL}_f$ of all induced probabilities on the lattice L^*. This is done based on appropriate lattice operations on L^*, when f is a fixed strictly increasing function. Also the lattice structure of the set $\mathbf{POL}_f \times \mathbf{POL}_g$, is studied, where the fixed functions f and g are strictly increasing. It is proved that this structure is a distributive complete lattice which is isomorphic to $L^\square$, where $L^\square$ is the set $[0,1]^2$ considered as a super lattice of L^*. More sub lattices of the lattices $L^\square$ and L^* are obtained.

2. Preliminaries

In this section, we review some known definitions and results which will be used later, for more details see Birkhoff (1940), Deschrijver (2004) and Lendelova et al. (2006).

Definition 2.1. Let $L^\square = \{(x,y) | (x,y) \in [0,1]^2\}$ and assume

$$X = (x_1, y_1),\ Y = (x_2, y_2) \in L^{\square}.$$

Define

$$X \wedge_{L^{\square}} Y = (Min\{x_1, x_2\}, Max\{y_1, y_2\})$$

$$X \vee_{L^{\square}} Y = (Max\{x_1, x_2\}, Min\{y_1, y_2\})$$

$$X \leq_{L^{\square}} Y \Leftrightarrow x_1 \leq x_2 \text{ and } y_1 \geq y_2.$$

Assume, $0_{L^{\square}} = (0,1)$, $1_{L^{\square}} = (1,0)$ and set

$$D = \{(x,y) | (x,y) \in [0,1]^2 \ and \ x + y = 1\},$$

then, we have the following.

Theorem 2.2. $(\boldsymbol{L}^{\square}, \leq_{\boldsymbol{L}^{\square}})$ is a complete lattice with D as its sub lattice.

Definition 2.3. Let

$$L^* = \{(x,y) | (x,y) \in [0,1]^2 \ and \ x + y \leq 1\},$$

and assume

$$X = (x_1, y_1),\ Y = (x_2, y_2) \in L^*.$$

Define

$$X \wedge_{L^*} Y = (Min\{x_1, x_2\}, Max\{y_1, y_2\})$$

$$X \vee_{L^*} Y = (Max\{x_1, x_2\}, Min\{y_1, y_2\})$$
$$X \leq_{L^*} Y \Leftrightarrow x_1 \leq x_2 \text{ and } y_1 \geq y_2.$$

Assume, $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$, then we have the following.

Lemma 2.4. $(L^*, \leq_{L^*})$ is a complete lattice.

Definition 2.5. Define the binary operations $\oplus$ and $\otimes$ on L^* as follows

$$X \oplus Y = (Min\{x_1 + x_2, 1\}, Max\{y_1 + y_2 - 1, 0\})$$

$$X \otimes Y = (Max\{x_1 + x_2 - 1, 0\}, Min\{y_1 + y_2, 1\}),$$

where,

$$X = (x_1, y_1)$$

and

$$Y = (x_2, y_2).$$

Definition 2.6. A probability on L^* is any function $p : L^* \to [0,1]$ satisfying the following properties:

1) $p(0,1) = 0$, $p(1,0) = 1$

2) $p(X \oplus Y) + p(X \otimes Y) = p(X) + p(Y)$ for each $X, Y \in L^*$

3) If $X_n \uparrow X$, then $p(X_n) \uparrow p(X)$ for each $X, X_n \in L^*$, $n \in N$,

where N is the set of natural numbers.

Remark 2.7. The notation $X_n \uparrow X$, means that $\{X_n\}$ is an increasing sequence in L^* and $X = \vee_{n \in N} X_n$.

Theorem 2.8. Let $p : L^* \to [0,1]$ be a probability on L^* . Then there exists $\alpha_p \in [0,1]$ such that *p* has the following form:

$$p(x, y) = \alpha_p x + (1 - \alpha_p)(1 - y), \text{ for all } (x, y) \in L^* .$$

Moreover, α_p is unique.

Proof:

We only prove the uniqueness of α_p, since the rest of the proof is given by Lendelova et al. (2006) . Suppose on the contrary that the statement is not true. So, there exist $\alpha_p^{(1)}, \alpha_p^{(2)} \in [0,1]$, where $\alpha_p^{(1)} \neq \alpha_p^{(2)}$. Also, for all $(x, y) \in L^*$

$$p(x, y) = \alpha_p^{(1)} x + (1 - \alpha_p^{(1)})(1 - y)$$

and

$$p(x,y)=\alpha_p^{(2)}x+\left(1-\alpha_p^{(2)}\right)(1-y).$$

Let

$$(x_0,y_0)\in L^*\setminus D.$$

So, we have:

$$\alpha_p^{(1)}x_0+\left(1-\alpha_p^{(1)}\right)(1-y_0)=\alpha_p^{(2)}x_0+\left(1-\alpha_p^{(2)}\right)(1-y_0).$$

Hence,

$$\left(\alpha_p^{(1)}-\alpha_p^{(2)}\right)x_0=\left(\alpha_p^{(1)}-\alpha_p^{(2)}\right)(1-y_0).$$

But, $\alpha_p^{(1)}\neq\alpha_p^{(2)}$, therefore, $x_0=1-y_0$. This is a contradiction. Hence α_p is unique. ■

The following Lemma is an immediate consequence of a well-known result for sequences in R, the fact that the limit in a cartesian product of two metric spaces (here $[0,1]^2$) is equal to the pair of limits of the components, and the fact that the limit and the supremum of a sequence in a closed subset (here L^*) of a metric space is still in that subset.

Lemma 2.9. Let $\{X_n=(x_n,y_n)\}$ be an increasing sequence in L^*. Then,

$$X=(x,y)=\vee_{n\in N}X_n \quad \text{if and only if} \quad X=\left(\lim_{n\to\infty}x_n,\lim_{n\to\infty}y_n\right).$$

Theorem 2.10. Let $\alpha\in[0,1]$, $X=(x_1,y_1)\in L^*$ and

$$p:L^*\to[0,1],$$

be defined by

$$p(X)=\alpha x_1+(1-\alpha)(1-y_1).$$

Then, p is a probability on L^*.

Proof:

Obviously p is well-defined. We show that p satisfies the conditions of Definition 2.6.

1) $p(0,1)=0$, $p(1,0)=1$

2) Let $X=(x_1,y_1), \mathrm{Y}=(x_2,y_2)\in L^*$. We have:

$$X\oplus Y=\left(Min\{x_1+x_2,1\}, Max\{y_1+y_2-1,0\}\right)$$

$$X\otimes Y=\left(Max\{x_1+x_2-1,0\}, Min\{y_1+y_2,1\}\right).$$

Consider the following four cases:

(a) $x_1+x_2\leq 1$ and $y_1+y_2\leq 1$

In this case we have:

$$p(X\oplus Y)=\alpha(x_1+x_2)+(1-\alpha)(1-0),$$

and

$$p(X\otimes Y)=\alpha(0)+(1-\alpha)(1-(y_1+y_2)).$$

So,

$$\begin{aligned}p(X\oplus Y)+p(X\otimes Y)&=\alpha x_1+\alpha x_2+(1-\alpha)+(1-\alpha)(1-y_1)-(1-\alpha)y_2\\&=\alpha x_1+(1-\alpha)(1-y_1)+\alpha x_2+(1-\alpha)(1-y_2)\\&=p(X)+p(Y).\end{aligned}$$

(b) $x_1+x_2\leq 1$ and $y_1+y_2>1$.

In this case we have:

$$p(X\oplus Y)=\alpha(x_1+x_2)+(1-\alpha)(1-(y_1+y_2-1)),$$

and

$$p(X\otimes Y)=\alpha(0)+(1-\alpha)(1-1).$$

So,

$$p(X \oplus Y) + p(X \otimes Y) = \alpha(x_1 + x_2) + (1-\alpha)(1 - y_1 + 1 - y_2)$$
$$= \alpha x_1 + (1-\alpha)(1-y_1) + \alpha x_2 + (1-\alpha)(1-y_2)$$
$$= p(X) + p(Y).$$

(c) $x_1 + x_2 > 1$ and $y_1 + y_2 \leq 1$

$$p(X \oplus Y) = \alpha + (1-\alpha)(1-0) = 1$$

$$p(X \otimes Y) = \alpha(x_1 + x_2 - 1) + (1-\alpha)(1 - y_1 - y_2).$$

So,

$$p(X \oplus Y) + p(X \otimes Y) = 1 + \alpha x_1 + \alpha x_2 - \alpha + (1-\alpha)(1-y_1) - (1-\alpha)y_2$$
$$= \alpha x_1 + (1-\alpha)(1-y_1) + \alpha x_2 + (1-\alpha)(1-y_2)$$
$$= p(X) + p(Y).$$

(d) $x_1 + x_2 > 1$ and $y_1 + y_2 > 1$

In this case we have: $x_1 + x_2 + y_1 + y_2 > 2$. Therefore, $(x_1 + y_1) + (x_2 + y_2) > 2$. Hence, $x_1 + y_1 > 1$ or $x_2 + y_2 > 1$. And, $X \notin L^*$ or $Y \notin L^*$. So case (d) does not occur.

Therefore, in all cases the condition 2 of Definition 2.6 does hold.

3) Let $X_n \uparrow X$, then we have:

$$p(X_n) = \alpha x_n + (1-\alpha)(1-y_n).$$

$\{p(X_n)\}$ is an increasing sequence. Let $X_n \leq_{L^*} X_{n+1}$, so $x_n \leq x_{n+1}$ and $y_n \geq y_{n+1}$.Therefore, $\alpha x_n + (1-\alpha)(1-y_n) \leq \alpha x_{n+1} + (1-\alpha)(1-y_{n+1})$.

Hence,

$$p(X_n) \leq p(X_{n+1})$$.

Therefore,

$$\lim_{n\to\infty} p(X_n) = \lim_{n\to\infty}\left(\alpha x_n + (1-\alpha)(1-y_n)\right)$$

$$= \alpha \lim_{n\to\infty} x_n + (1-\alpha)\left(1-\lim_{n\to\infty} y_n\right)$$

$$= \alpha x + (1-\alpha)(1-y) \qquad \text{(By Lemma 2.9)}$$

$$= p(X) .\blacksquare$$

3. Algebraic Structure of the Set of Probabilities on L^*

In this section we assume that $X = (x_1, y_1)$ and $Y = (x_2, y_2)$ are elements of L^*, unless clearly stated otherwise.

Notation 3.1. Set

POL$= \{p \mid p \text{ is a probability on } L^*\}$.

Definition 3.2. Define $\wedge$ and $\vee$ on **POL** as follows:

$$p \wedge q : L^* \to [0,1]$$

$$X \mapsto Min\{\alpha_p, \alpha_q\}x_1 + (1 - Min\{\alpha_p, \alpha_q\})(1-y_1),$$

and

$$p \vee q : L^* \to [0,1]$$

$$X \mapsto Max\{\alpha_p, \alpha_q\}x_1 + (1 - Max\{\alpha_p, \alpha_q\})(1-y_1).$$

Lemma 3.3. The operations $\wedge$ and $\vee$ defined in Definition 3.2 are well-defined, closed and associative on **POL**.

Proof:

It is clear that $\wedge$ and $\vee$ are well-defined and closed on **POL**. We show that $\wedge$ and $\vee$ are associative on **POL**. Let $p, q, r \in$ **POL** and $X \in L^*$.

$$
\begin{aligned}
((p \wedge q) \wedge r)(X) &= Min\{\alpha_{p\wedge q}, \alpha_r\}\, x_1 + (1 - Min\{\alpha_{p\wedge q}, \alpha_r\})(1 - y_1) \\
&= Min\{Min\{\alpha_p, \alpha_q\}, \alpha_r\}\, x_1 + (1 - Min\{Min\{\alpha_p, \alpha_q\}, \alpha_r\})(1 - y_1) \\
&= Min\{\alpha_p, Min\{\alpha_q, \alpha_r\}\}\, x_1 + (1 - Min\{\alpha_p, Min\{\alpha_q, \alpha_r\}\})(1 - y_1) \\
&= Min\{\alpha_p, \alpha_{q\wedge r}\}\, x_1 + (1 - Min\{\alpha_p, \alpha_{q\wedge r}\})(1 - y_1) \\
&= (p \wedge (q \wedge r))(X).
\end{aligned}
$$

Similarly, we can show that, $((p \vee q) \vee r)(X) = (p \vee (q \vee r))(X)$.■

The following Lemma follows immediately from Theorem 2.10 by putting $\alpha = 1$ and $\alpha = 0$ to obtain $\mathbf{1}(X)$ and $\mathbf{0}(X)$ respectively.

Lemma 3.4. Define the mappings $\mathbf{1}, \mathbf{0} : L^* \to [0,1]$ in the following:

$1(X) = x_1$ and $\mathbf{0}(X) = 1 - y_1$. Then, $\mathbf{1}, \mathbf{0} \in \mathbf{POL}$.

Lemma 3.5. Let $p \in \mathbf{POL}$. Then:

(1) $p \wedge \mathbf{0} = \mathbf{0} \wedge p = \mathbf{0}$

(2) $p \wedge \mathbf{1} = \mathbf{1} \wedge p = p$

(3) $p \vee \mathbf{0} = \mathbf{0} \vee p = p$

(4) $p \vee \mathbf{1} = \mathbf{1} \vee p = \mathbf{1}$.

Proof:

We shall only prove (1), the other parts are similar. By commutative property of $\wedge$, we have $p \wedge \mathbf{0} = \mathbf{0} \wedge p$. Also,

$$
\begin{aligned}
(p \wedge \mathbf{0})(X) &= Min\{\alpha_p, \alpha_0\} x_1 + (1 - Min\{\alpha_p, \alpha_0\})(1 - y_1) \\
&= Min\{\alpha_p, 0\} x_1 + (1 - Min\{\alpha_p, 0\})(1 - y_1) \\
&= 1 - y_1 \\
&= 0(X).
\end{aligned}
$$
■

Theorem 3.6. (**POL**,∧) and (**POL**,∨) are monoids.

Proof:

It follows from Lemmas 3.3-3.5.■

Definition 3.7. The ordering relation $\leq$ on **POL** is defined as follows. For $p, q \in \mathbf{POL}$:

$$p \leq q \iff \alpha_p \leq \alpha_q .$$

Definition 3.8. Define $\Phi : [0,1] \rightarrow \mathbf{POL}$ by $\Phi(a) = p_a$, where

$$p_a(X) = ax_1 + (1-a)(1-y_1) \text{ for all } X \in L^* .$$

The following reveals the relation between the lattices [0,1] and **POL.**

Lemma 3.9. Consider Φ in Definition 3.8, then:

(1) Φ is a bijection.

(2) $\Phi(Min\{a,b\}) = \Phi(a) \wedge \Phi(b)$.

(3) $\Phi(Max\{a,b\}) = \Phi(a) \vee \Phi(b)$.

(4) $a \leq b$ if and only if $\Phi(a) \leq \Phi(b)$.

Proof:

Straightforward. ■

Corollary 3.10 $(\text{POL}, \wedge, \vee, \leq)$ is a distributive complete lattice.

Proof:

Lemma 3.9 shows that Φ is an isomorphism between the lattices [0,1] and **POL**. Hence $(\text{POL}, \wedge, \vee, \leq)$ is a distributive complete lattice. ■

Definition 3.11. Let $p \in \mathbf{POL}$. We define p' in the following:

$$p' : L^* \to [0,1]$$

$$p'(X) = 1 - p(X'),$$

where

$$X' = (y_1, x_1).$$

Lemma 3.12. Let $p \in \mathbf{POL}$. Then:

(i) $p' \in \mathbf{POL}$,

(ii) $\alpha_{p'} = 1 - \alpha_p$,

(iii) (a) $(p')' = p$,

(b) $\forall\ p, q \in \mathbf{POL} \quad (p \leq q \Rightarrow q' \leq p')$,

(iv) (De-Morgan properties):

$$(i)\ \forall\ p, q \in \mathbf{POL}\left((p \wedge q)' = p' \vee q'\right)$$

$$(ii)\ \forall\ p, q \in \mathbf{POL}\left((p \vee q)' = p' \wedge q'\right),$$

(v) $\mathbf{0}' = \mathbf{1}$ and $\mathbf{1}' = \mathbf{0}$.

Proof:

(i)

$$(1)\ p'(0,1) = 1 - p(1,0) = 1 - 1 = 0$$

$$p'(1,0) = 1 - p(0,1) = 1 - 0 = 1$$

(2) We show that:

$$(X \oplus Y)' = Y' \otimes X' \text{ and } (X \otimes Y)' = Y' \oplus X'.$$

$$(X \oplus Y)' = (Max\{y_1 + y_2 - 1,0\}, Min\{x_1 + x_2, 1\}).$$

On the other hand:

$$Y' \otimes X' = (Max\{y_1 + y_2 - 1,0\}, Min\{x_1 + x_2, 1\}).$$

Similarly we can show that:

$$(X \otimes Y)' = Y' \oplus X'.$$

$$\begin{aligned} p'(X \oplus Y) + p'(X \otimes Y) &= 1\text{-}p\left((X \oplus Y)'\right) + 1 - p\left((X \otimes Y)'\right) \\ &= 1\text{-}p(Y' \otimes X') + 1 - p(Y' \oplus X') \\ &= 2\text{-}\left(p(Y' \otimes X') + p(Y' \oplus X')\right) \\ &= 2\text{-}\left(p(Y') + p(X')\right) \\ &= 2\text{-}\left(1\text{-}p'(Y) + 1 - p'(X)\right) \\ &= p'(X) + p'(Y). \end{aligned}$$

(3) Let $X_n \uparrow X$. Then:

$$\lim_{n\to\infty} p'(X_n) = \lim_{n\to\infty}\left(1 - p\left(X_n{}'\right)\right) = (1 - p(X')) = p'(X).$$

(ii)

$$\begin{aligned} p'(X) &= \alpha_{p'} x_1 + (1-\alpha_{p'})(1-y_1) \\ &= 1 - p(X') \\ &= 1 - \alpha_p y_1 - 1 + x_1 + \alpha_p - \alpha_p x_1 \\ &= (1 - \alpha_p) x_1 + \alpha_p (1-y_1) \end{aligned}$$

So, $\alpha_{p'} = 1 - \alpha_p$.

(iii)

(a)

$$\begin{aligned} (p')'(X) &= 1 - p'(X') \\ &= 1 - \left(1 - p\left((X')'\right)\right) \\ &= p(X) \end{aligned}$$

(b) Let $p \le q$, then $\alpha_p \le \alpha_q$. So, $1-\alpha_q \le 1-\alpha_p$. Hence, $\alpha_{q'} \le \alpha_{p'}$. Hence, $q' \le p'$.

(iv)

$$\begin{aligned} (p \wedge q)'(X) &= 1 - (p \wedge q)(X') \\ &= 1 - \left(Min\{\alpha_p, \alpha_q\} y_1 + (1 - Min\{\alpha_p, \alpha_q\})(1-x_1)\right) \\ &= 1 - Min\{\alpha_p, \alpha_q\} y_1 - 1 + x_1 + Min\{\alpha_p, \alpha_q\} - Min\{\alpha_p, \alpha_q\} x_1 \\ &= Min\{\alpha_p, \alpha_q\}(1-y_1) + (1 - Min\{\alpha_p, \alpha_q\}) x_1 \end{aligned}$$

On the other hand:

$$\begin{aligned} (p' \vee q')(X) &= Max\{\alpha_{p'}, \alpha_{q'}\} x_1 + \left(1 - Max\{\alpha_{p'}, \alpha_{q'}\}(1-y_1)\right) \\ &= Max\{1-\alpha_p, 1-\alpha_q\} x_1 + \left(1 - Max\{1-\alpha_p, 1-\alpha_q\}\right)(1-y_1) \\ &= \left(1 - Min\{\alpha_p, \alpha_q\}\right) x_1 + Min\{\alpha_p, \alpha_q\}(1-y_1). \end{aligned}$$

Similarly, we can show that:

$$(p \vee q)' = p' \wedge q'.$$

(v)

(i)

$$\begin{aligned}\mathbf{0}'(X) &= 1 - \mathbf{0}(X') \\ &= 1 - (1 - x_1) \\ &= x_1 = \mathbf{1}(X)\end{aligned}$$

(ii)

$$\begin{aligned}\mathbf{1}'(X) &= 1 - \mathbf{1}(X') \\ &= 1 - y_1 \\ &= \mathbf{0}(X)\ .\ \blacksquare\end{aligned}$$

Theorem 3.13. $(\mathbf{POL}, \vee, \wedge, ', \mathbf{0}, \mathbf{1})$ is De-Morgan algebra.

Proof:

The proof follows from Lemma 3.12.■

4. f − Probability on L^*

In this section we induce a probability function on L^* by an appropriate function $f:[0,1]\to[0,1]$. Then we study the distributive complete lattice structure of the set of all induced probabilities on L^* based on appropriate lattice operations, when f is a fixed strictly increasing function.

Definition 4.1. Let $f:[0,1]\to[0,1]$ be any function, $p:L^*\to[0,1]$ be a probability on L^* and α_p be the unique real number obtained in Theorem 2.8. Define the induced $p_f : L^*\to[0,1]$ by f in the following:

$$p_f(X) = f(\alpha_p)x_1 + (1 - f(\alpha_p))(1 - y_1).$$

Lemma 4.2. The induced p_f in Definition 4.1 is a probability on L^* .

Proof:

It is straightforward.■

In the following we will consider a class of probabilities on L^* induced by Sugeno negation.

Example 4.3. Let $p : L^* \to [0,1]$ be a probability function on L^*. Consider the Sugeno negation $N_\lambda : [0,1] \to [0,1]$, where

$$N_\lambda(x) = \frac{1-x}{1+\lambda x}, \lambda \in (1,\infty).$$

Then,

$$p_{N_\lambda}(X) = \left(\frac{1-\alpha_p}{1+\lambda\alpha_p}\right)x_1 + \left(\frac{(1+\lambda)\alpha_p}{1+\lambda\alpha_p}\right)(1-y_1), \lambda \in (1,\infty)$$

is a probability on L^*, where α_p is the unique real number obtained in Theorem 2.8.

Lemma 4.4. Let $p : L^* \to [0,1]$ be an arbitrary probability function on L^* and $f : [0,1] \to [0,1]$ be any function. Then p_f in Definition 4.1 is onto.

Proof:

Let $y \in [0,1]$. Define $X = (y, 1-y)$. It is clear that $X \in L^*$ and $p_f(X) = y$.■

Remark 4.5. Let $p : L^* \to [0,1]$ be an arbitrary probability function on L^* and $f : [0,1] \to [0,1]$ be any function. Then p_f in Definition 4.1 is not 1-1.

Define $X = (1 - f(\alpha_p), f(\alpha_p))$, where α_p is the unique real number obtained in Theorem 2.8 and $Y = (0,0)$. It is clear that $X, Y \in L^*$ and $X \neq Y$. Also, $p_f(X) = p_f(Y) = 1 - f(\alpha_p)$.

Notation 4.6. Let $f : [0,1] \to [0,1]$ be any function. Set:

$$\mathbf{POL}_f = \{p_f | p \in \mathbf{POL}\}.$$

Definition 4.7. Let $\mathbf{POL}_f$ be as in Notation 4.6. Define the operations $\wedge$ and $\vee$ on $\mathbf{POL}_f$ as follows:

$$p_f \wedge q_f : L^* \to [0,1]$$

$$X \mapsto Min\{f(\alpha_p), f(\alpha_q)\}x_1 + (1 - Min\{f(\alpha_p), f(\alpha_q)\})(1 - y_1),$$

and

$$p_f \vee q_f : L^* \to [0,1]$$

$$X \mapsto Max\{f(\alpha_p), f(\alpha_q)\}x_1 + (1 - Max\{f(\alpha_p), f(\alpha_q)\})(1 - y_1).$$

Definition 4.8. For a fixed $f:[0,1] \to [0,1]$, define the ordering relation $\leq_f$ on $\mathbf{POL}_f$ as follows:

$$p_f \leq_f q_f \iff f(\alpha_p) \leq f(\alpha_q), \text{ for all } p_f, q_f \in \mathbf{POL}_f.$$

Lemma 4.9. Let $f:[0,1] \to [0,1]$ be a strictly increasing (strictly decreasing) function and $p, q \in \mathbf{POL}$. Then:

(1) $(p \wedge q)_f = p_f \wedge q_f \qquad ((p \wedge q)_f = p_f \vee q_f)$

(2) $(p \vee q)_f = p_f \vee q_f \qquad ((p \vee q)_f = p_f \wedge q_f)$.

Proof:

(1) Let $X \in L^*$ and $f:[0,1] \to [0,1]$ be a strictly increasing function, then:

$$\begin{aligned}(p \wedge q)_f(X) &= f(\alpha_{p\wedge q})x_1 + (1 - f(\alpha_{p\wedge q}))(1 - y_1)\\ &= f[Min\{\alpha_p, \alpha_q\}]x_1 + (1 - f[Min\{\alpha_p, \alpha_q\}])(1 - y_1)\\ &= Min\{f(\alpha_p), f(\alpha_q)\}x_1 + (1 - Min\{f(\alpha_p), f(\alpha_q)\})(1 - y_1)\\ &= (p_f \wedge q_f)(X).\end{aligned}$$

Let $X \in L^*$ and $f:[0,1] \to [0,1]$ be a strictly decreasing function then:

$$\begin{aligned}(p \wedge q)_f(X) &= f(\alpha_{p\wedge q})x_1 + (1 - f(\alpha_{p\wedge q}))(1 - y_1)\\ &= f[Min\{\alpha_p, \alpha_q\}]x_1 + (1 - f[Min\{\alpha_p, \alpha_q\}])(1 - y_1)\\ &= Max\{f(\alpha_p), f(\alpha_q)\}x_1 + (1 - Max\{f(\alpha_p), f(\alpha_q)\})(1 - y_1)\end{aligned}$$

$$= (p_f \vee q_f)(X).$$

(2) The proof is similar to part (1). ■

Definition 4.10. Let $f:[0,1]\to[0,1]$ be a function. Define $\psi:\mathbf{POL}\to\mathbf{POL}_f$ as

$$\psi(p) = p_f .$$

Lemma 4.11. Let $f:[0,1]\to[0,1]$ be a function, $p,q\in\mathbf{POL}$ and ψ be as in Definition 4.10, then:

(1) ψ is well-defined.
If $f:[0,1]\to[0,1]$ is strictly increasing (strictly decreasing) function then:

(2) ψ is a bijection,

(3) $\psi(p\wedge q) = \psi(p)\wedge\psi(q)$ ($\psi(p\wedge q) = \psi(p)\vee\psi(q)$),

(4) $\psi(p\vee q) = \psi(p)\vee\psi(q)$ ($\psi(p\vee q) = \psi(p)\wedge\psi(q)$),

(5) $p\le q$ if and only if $\psi(p)\le_f\psi(q)$
($p\le q$ if and only if $\psi(q)\le_f\psi(p)$).

Proof:

It is similar to the proof of Lemma 4.9. ■

Now the following fact is immediate.

Corollary 4.12. Let $f:[0,1]\to[0,1]$ be a function.

(1) If f is a strictly increasing function, then $(\mathbf{POL}_f, \wedge, \vee, \le_f)$ is a distributive complete lattice.

(2) If f is a strictly decreasing function, then ψ in Definition 4.10 is a dual isomorphism.

Example 4.13.

(1) Consider $f:[0,1] \to [0,1]$, where $f(x) = x^2$. Then f is a strictly increasing function and $\mathbf{POL}_f = \{p_f | p \in \mathbf{POL}\}$, where

$$p_f(X) = \alpha_p^2 x_1 + (1 - \alpha_p^2)(1 - y_1).$$

Hence, $(\mathbf{POL}_f, \wedge, \vee, \leq_f)$ is a distributive complete lattice.

(2) Consider $f:[0,1] \to [0,1]$, where $f(x) = 1 - x^2$. Then f is a strictly decreasing function and $\psi(p) = p_f$, where

$$p_f(X) = (1 - \alpha_p^2)x_1 + \alpha_p^2(1 - y_1).$$

Hence, ψ is a dual isomorphism.

Remark 4.14. As Birkhoff, G. (1940) mentioned, the product of two lattices is also a lattice. Let $f, g : [0,1] \to [0,1]$ be two strictly increasing functions, then $\mathbf{POL}_f \times \mathbf{POL}_g$ is also a distributive complete lattice.

Lemma 4.15. Let $f : [0,1] \to [0,1]$ be onto. Then $\mathbf{POL} = \mathbf{POL}_f$.

Proof:

It is clear that $\mathbf{POL}_f \subseteq \mathbf{POL}$. Let $p \in \mathbf{POL}$. We show that $p \in \mathbf{POL}_f$. Since $p \in \mathbf{POL}$, we have:

$$p(X) = \alpha_p x_1 + (1 - \alpha_p)(1 - y_1) \text{ and } \alpha_p \in [0,1].$$

f is onto, so there exist $\beta \in [0,1]$ such that $f(\beta) = \alpha_p$. Define

$$q(X) = \beta x_1 + (1 - \beta)(1 - y_1).$$

By Theorem 2.10, $q \in \mathbf{POL}$ and $q_f(X) = f(\beta)x_1 + (1 - f(\beta))(1 - y_1)$. Therefore

$$q_f(X) = \alpha_p x_1 + (1 - \alpha_p)(1 - y_1) = p(X).$$

Hence, $p \in \mathbf{POL}_f$. Therefore, $\mathbf{POL} = \mathbf{POL}_f$. ■

Corollary 4.16. Let N_λ be the Sugeno negation as in Example 4.3, then $\mathbf{POL} = \mathbf{POL}_{N_\lambda}$.

Definition 4.17. Let $f, g : [0,1] \to [0,1]$ be two functions. Define

$$\theta : L^\circ \longrightarrow \mathbf{POL}_f \times \mathbf{POL}_g \text{ by } \theta(\alpha, \beta) = (p_f, q_g),$$

where, for all $X \in L^*$

$$p_f(X) = f(\alpha)x_1 + (1 - f(\alpha))(1 - y_1),$$

and

$$q_g(X) = g(\beta)x_1 + (1 - g(\beta))(1 - y_1).$$

Definition 4.18. Let $f, g : [0,1] \to [0,1]$ be two functions and assume

$(p_f, q_g), (r_f, s_g) \in \mathbf{POL}_f \times \mathbf{POL}_\mathbf{g}$. Define

(1) $(p_f, q_g) \wedge_{\mathbf{L}^\circ} (r_f, s_g) = (p_f \wedge r_f, q_g \vee s_g)$,

(2) $(p_f, q_g) \vee_{\mathbf{L}^\circ} (r_f, s_g) = (p_f \vee r_f, q_g \wedge s_g)$,

(3) $(p_f, q_g) \leq_{\mathbf{L}^\circ} (r_f, s_g)$ if and only if $p_f \leq_f r_f$ and $q_g \ {}_g\!\geq s_g$,

Theorem 4.19. Let $f, g : [0,1] \to [0,1]$ be two functions and θ be as in Definition 4.17. Then:

(1) θ is well-defined.

If $f, g : [0,1] \to [0,1]$ be two strictly increasing (strictly decreasing) functions then:

(2) θ is a bijection.

(3) $\theta\big((\alpha, \beta) \wedge_{L^\circ} (\gamma, \sigma)\big) = \theta(\alpha, \beta) \wedge_{\mathbf{L}^\circ} \theta(\gamma, \sigma)$

$(\theta\big((\alpha, \beta) \wedge_{L^\circ} (\gamma, \sigma)\big) = \theta(\alpha, \beta) \vee_{\mathbf{L}^\circ} \theta(\gamma, \sigma))$

(4) $\theta\big((\alpha, \beta) \vee_{L^\circ} (\gamma, \sigma)\big) = \theta(\alpha, \beta) \vee_{\mathbf{L}^\circ} \theta(\gamma, \sigma)$

$(\theta\big((\alpha, \beta) \vee_{L^\circ} (\gamma, \sigma)\big) = \theta(\alpha, \beta) \wedge_{\mathbf{L}^\circ} \theta(\gamma, \sigma))$

(5) $(\alpha,\beta) \leq_{L^{\square}} (\gamma,\sigma)$ if and only if $\theta(\alpha,\beta) \leq_{\mathbf{L}^{\square}} \theta(\gamma,\sigma)$.

$((\alpha,\beta) \leq_{L^{\square}} (\gamma,\sigma)$ if and only if $\theta(\gamma,\sigma) \leq_{\mathbf{L}^{\square}} \theta(\alpha,\beta)$.)

Proof:

Let $(\alpha,\beta),(\gamma,\sigma) \in L^{\square}$ and $X \in L^*$. Define p_f, q_g, r_f and s_g as follows:

$$p_f(X) = f(\alpha)x_1 + (1 - f(\alpha))(1 - y_1),$$

$$q_g(X) = g(\beta)x_1 + (1 - g(\beta))(1 - y_1),$$

$$r_f(X) = f(\gamma)x_1 + (1 - f(\gamma))(1 - y_1),$$

and

$$s_g(X) = g(\sigma)x_1 + (1 - g(\sigma))(1 - y_1).$$

It is clear that (p_f, q_g) and $(r_f, s_g) \in \mathbf{POL}_f \times \mathbf{POL}_g$.

(1) Let $(\alpha,\beta) = (\gamma,\sigma)$, therefore $\alpha = \gamma$ and $\beta = \sigma$. Hence $f(\alpha) = f(\gamma)$ and $g(\beta) = g(\sigma)$. So $p_f = r_f$ and $q_g = s_g$. Therefore, $\theta(\alpha,\beta) = \theta(\gamma,\sigma)$.

We only prove the Lemma in the case that *f*, *g* are strictly increasing functions. The proof of the case that *f*, *g* are strictly decreasing functions is similar.

(2) Let $\theta(\alpha,\beta) = \theta(\gamma,\sigma)$. Hence $(p_f, q_g) = (r_f, s_g)$. Therefore, $p_f = r_f$ and $q_g = s_g$. So $f(\alpha) = f(\gamma)$ and $g(\beta) = g(\sigma)$. Hence, $\alpha = \gamma$ and $\beta = \sigma$. Therefore, $(\alpha,\beta) = (\gamma,\sigma)$.

Let $(u_f, v_g) \in \mathbf{POL}_f \times \mathbf{POL}_g$, such that:

$$u_f(X) = f(\alpha_u)x_1 + (1 - f(\alpha_u))(1 - y_1)$$

and

$$v_g(X) = g(\alpha_v)x_1 + (1 - g(\alpha_v))(1 - y_1).$$

It is clear that $(\alpha_u, \alpha_v) \in L^{\square}$ and $\theta(\alpha_u, \alpha_v) = (u_f, v_g)$.

(3)
$$\theta((\alpha,\beta) \wedge_{L^{\circ}} (\gamma,\sigma)) = \theta(Min\{\alpha,\gamma\}, Max\{\beta,\sigma\}) = (m_f, n_g),$$

where,

$$m_f(X) = f(Min\{\alpha,\gamma\})x_1 + \big(1 - f(Min\{\alpha,\gamma\})\big)(1 - y_1)$$

and

$$n_g(X) = g(Max\{\beta,\sigma\})x_1 + (1 - g(Max\{\beta,\sigma\}))(1 - y_1).$$

Therefore,

$$m_f(X) = Min\{f(\alpha), f(\gamma)\}x_1 + (1 - Min\{f(\alpha), f(\gamma)\})(1 - y_1),$$

and

$$n_g(X) = Max\{g(\beta), g(\sigma)\}x_1 + \big(1 - Max\{g(\beta), g(\sigma)\}\big)(1 - y_1).$$

So, $m_f(X) = (p_f \wedge r_f)(X)$ and $n_g(X) = (q_g \vee s_g)(X)$.

Hence,

$$\theta((\alpha,\beta) \wedge_L (\gamma,\sigma)) = (p_f \wedge r_f, q_g \vee s_g)$$

$$= (p_f, q_g) \wedge_{\mathbf{L}^{\circ}} (r_f, s_g)$$

$$= \theta(\alpha,\beta) \wedge_{\mathbf{L}^{\circ}} \theta(\gamma,\sigma).$$

(4) It is similar to the part (3).

(5) $(\Rightarrow)$

Let $(\alpha,\beta) \leq_{L^{\circ}} (\gamma,\sigma)$. Hence, $\alpha \leq \gamma$ and $\beta \geq \sigma$. So, $f(\alpha) \leq f(\gamma)$ and $g(\beta) \geq g(\sigma)$. Therefore, $p_f \leq_f r_f$ and $q_g \ {}_g\!\geq s_g$. Hence, $(p_f, q_g) \leq_{\mathbf{L}^{\circ}} (r_f, s_g)$.
$(\Leftarrow)$
It is similar to the above part. ■

Corollary 4.20. Let $f, g : [0,1] \to [0,1]$ be two functions.

(1) If f and g are two strictly increasing functions, then $\mathbf{POL}_f \times \mathbf{POL}_g$ is a distributive complete lattice which is isomorphic to $L^{\square}$.

(2) If f and g are two strictly decreasing functions, then θ in Definition 4.17 is a dual isomorphism.

Example 4.21.

(1) Consider $f, g: [0,1] \to [0,1]$, where $f(x) = x^2, g(x) = x^3$. Then f, g are strictly increasing functions and $\mathbf{POL}_f \times \mathbf{POL}_g = \{(p_f, q_g) | p, q \in \mathbf{POL}\}$, where

$$p_f(X) = \alpha_p^2 x_1 + (1 - \alpha_p^2)(1 - y_1)$$

and

$$q_g(X) = \alpha_q^3 x_1 + (1 - \alpha_q^3)(1 - y_1).$$

Hence, $\mathbf{POL}_f \times \mathbf{POL}_g$ is a distributive complete lattice which is isomorphic to $L^{\square}$.

(2) Consider $f, g: [0,1] \to [0,1]$, where $f(x) = 1 - x^2, g(x) = 1 - x^3$. Then f, g are strictly decreasing functions and $\theta(\alpha, \beta) = (p_f, q_g)$, where

$$p_f(X) = (1 - \alpha^2)x_1 + \alpha^2(1 - y_1)$$

and

$$q_g(X) = (1 - \beta^3)x_1 + \beta^3(1 - y_1).$$

Then θ is a dual isomorphism.

Notation 4.22. Let $f, g : [0,1] \to [0,1]$ be two functions. Define

$\mathbf{L}_{\mathbf{POL}_f \times \mathbf{POL}_g} \subseteq \mathbf{POL}_f \times \mathbf{POL}_g$ as follows:

$$\mathbf{L}_{\mathbf{POL}_f \times \mathbf{POL}_g} = \{(p_f, q_g) | p, q \in \mathbf{POL} \text{ and } \alpha_p + \alpha_q \le 1\}.$$

Definition 4.23. Let $f, g : [0,1] \to [0,1]$ be two functions. Define $\Gamma : L^* \to \mathbf{L}_{\mathbf{POL}_f \times \mathbf{POL}_g}$ by $\Gamma(\alpha, \beta) = (p_f, q_g)$, where

$$p_f(X) = f(\alpha)x_1 + (1 - f(\alpha))(1 - y_1)$$

and

$$q_g(X) = g(\beta)x_1 + (1 - g(\beta))(1 - y_1), \text{ for all } X \in L^*.$$

Definition 4.24. Let $f, g : [0,1] \to [0,1]$ be two functions and assume

$$(p_f, q_g), (r_f, s_g) \in \mathbf{L}_{\mathbf{POL}_f \times \mathbf{POL}_g}.$$

Define:

(1) $(p_f, q_g) \wedge_{\mathbf{L}^*} (r_f, s_g) = (p_f \wedge r_f, q_g \vee s_g)$

(2) $(p_f, q_g) \vee_{\mathbf{L}^*} (r_f, s_g) = (p_f \vee r_f, q_g \wedge s_g)$

(3) $(p_f, q_g) \leq_{\mathbf{L}^*} (r_f, s_g)$ if and only if $p_f \leq_f r_f$ and $q_g \ {}_g\!\geq s_g$.

Theorem 4.25. Let $f, g : [0,1] \to [0,1]$ be two functions and Γ be as in Definition 4.23. Then:

(1) Γ is well-defined.

If $f, g : [0,1] \to [0,1]$ are strictly increasing (strictly decreasing) functions then:

(2) Γ is a bijection.

(3) $\Gamma((\alpha, \beta) \wedge_{L^*} (\gamma, \sigma)) = \Gamma(\alpha, \beta) \wedge_{\mathbf{L}^*} \Gamma(\gamma, \sigma)$

$(\Gamma((\alpha, \beta) \wedge_{L^*} (\gamma, \sigma)) = \Gamma(\alpha, \beta) \vee_{\mathbf{L}^*} \Gamma(\gamma, \sigma))$

(4) $\Gamma((\alpha, \beta) \vee_{L^*} (\gamma, \sigma)) = \Gamma(\alpha, \beta) \vee_{\mathbf{L}^*} \Gamma(\gamma, \sigma)$

$(\Gamma((\alpha, \beta) \vee_{L^*} (\gamma, \sigma)) = \Gamma(\alpha, \beta) \wedge_{\mathbf{L}^*} \Gamma(\gamma, \sigma))$

(5) $(\alpha,\beta) \leq_{L^*} (\gamma,\sigma)$ if and only if $\Gamma(\alpha,\beta) \leq_{\mathbf{L}^*} \Gamma(\gamma,\sigma)$.

($(\alpha,\beta) \leq_{L^*} (\gamma,\sigma)$ if and only if $\Gamma(\gamma,\sigma) \leq_{\mathbf{L}^*} \Gamma(\alpha,\beta)$.)

Proof:

It is similar to the proof of Theorem 4.19. ■

Corollary 4.26 Let $f, g : [0,1] \to [0,1]$ be two functions.

(1) If f and g are two strictly increasing functions, then $\mathbf{L}_{\mathbf{POL}_f \times \mathbf{POL}_g}$ is a distributive complete lattice which is isomorphic to L^*.

(2) If f and g are two strictly decreasing functions, then Γ in Definition 4.23 is a dual isomorphism.

Example 4.27.

(1) Consider f, g in Example 4.21 part (1). Then

$$\mathbf{L}_{\mathbf{POL}_f \times \mathbf{POL}_g} = \{(p_f, q_g) \mid p, q \in \mathbf{POL} \text{ and } \alpha_p + \alpha_q \leq 1\},$$

where

$$p_f(X) = \alpha_p^2 x_1 + (1 - \alpha_p^2)(1 - y_1)$$

and

$$q_g(X) = \alpha_q^3 x_1 + (1 - \alpha_q^3)(1 - y_1).$$

Hence, $\mathbf{L}_{\mathbf{POL}_f \times \mathbf{POL}_g}$ is a distributive complete lattice which is isomorphic to L^*.

(2) Consider f, g in Example 4.21 part (2). Then $\Gamma(\alpha,\beta) = (p_f, q_g)$, where

$$p_f(X) = (1 - \alpha^2)x_1 + \alpha^2(1 - y_1)$$

and

$$q_g(X) = (1 - \beta^3)x_1 + \beta^3(1 - y_1).$$

Then, Γ is a dual isomorphism.

Definition 4.28. Let $f, g : [0,1] \to [0,1]$ be two functions. Define $\mathbf{D}_{\mathbf{POL}_f \times \mathbf{POL}_g} = \{(p_f, q_g) \mid p, q \in \mathbf{POL} \text{ and } \alpha_p + \alpha_q = 1\}$.

Lemma 4.29. Let $f, g : [0,1] \to [0,1]$ be two functions. Then

$$\mathbf{D}_{\mathbf{POL}_f \times \mathbf{POL}_g} = \{(p_f, p'_g) \mid p \in \mathbf{POL}\}.$$

Proof:

Let $(p_f, q_g) \in \mathbf{D}_{\mathbf{POL}_f \times \mathbf{POL}_g}$. So, $\alpha_p + \alpha_q = 1$. Hence, $\alpha_q = 1 - \alpha_p$. By Lemma 3.12, we have $q = p'$. Therefore, $q_g = p'_g$. Hence, $(p_f, q_g) = (p_f, p'_g)$. So, $(p_f, q_g) \in \{(p_f, p'_g) \mid p \in \mathbf{POL}\}$.

Let $(p_f, p'_g) \in \{(p_f, p'_g) \mid p \in \mathbf{POL}\}$. By Lemma 3.12, $\alpha_{p'} = 1 - \alpha_p$, so $\alpha_p + \alpha_{p'} = \alpha_p + 1 - \alpha_p = 1$. Therefore, $(p_f, p'_g) \in \mathbf{D}_{\mathbf{POL}_f \times \mathbf{POL}_g}$. ■

Definition 4.30. Let $f, g : [0,1] \to [0,1]$ be two functions.

Define $\Psi : D \to \mathbf{D}_{\mathbf{POL}_f \times \mathbf{POL}_g}$, by $\Psi(\alpha, 1-\alpha) = (p_f, p'_g)$, where

$$p(X) = \alpha x_1 + (1-\alpha)(1-y_1) \text{ for all } X \in L^*.$$

Theorem 4.31. Let $f, g : [0,1] \to [0,1]$ be two functions and Ψ be as in Definition 4.30. Then:

(1) Ψ is well-defined.

If $f, g : [0,1] \to [0,1]$ are two strictly increasing (strictly decreasing) functions, then:

(2) Ψ is a bijection.

(3) $\Psi((\alpha, 1-\alpha) \wedge_{L^*} (\beta, 1-\beta)) = \Psi(\alpha, 1-\alpha) \wedge_{\mathbf{L}^*} \Psi(\beta, 1-\beta)$

$(\Psi((\alpha, 1-\alpha) \wedge_{L^*} (\beta, 1-\beta)) = \Psi(\alpha, 1-\alpha) \vee_{\mathbf{L}^*} \Psi(\beta, 1-\beta))$

(4) $\Psi((\alpha, 1-\alpha) \vee_{L^*} (\beta, 1-\beta)) = \Psi(\alpha, 1-\alpha) \vee_{\mathbf{L}^*} \Psi(\beta, 1-\beta)$

$(\Psi((\alpha,1-\alpha)\vee_{L^*}(\beta,1-\beta)) = \Psi(\alpha,1-\alpha)\wedge_{\mathbf{L}^*}\Psi(\beta,1-\beta))$

(5) $(\alpha,1-\alpha)\leq_{L^*}(\beta,1-\beta)$ if and only if $\Psi(\alpha,1-\alpha)\leq_{\mathbf{L}^*}\Psi(\beta,1-\beta)$.

$((\alpha,1-\alpha)\leq_{L^*}(\beta,1-\beta)$ if and only if $\Psi(\beta,1-\beta)\leq_{\mathbf{L}^*}\Psi(\alpha,1-\alpha)$.)

Proof:

The proof is similar to the proof of Theorem 4.19. ■

Corollary 4.32 Let $f, g : [0,1]\to[0,1]$ be two functions.

(1) If f and g are two strictly increasing functions, then $\mathbf{D}_{\mathbf{POL}_f\times\mathbf{POL}_g}$ is a distributive complete lattice which is isomorphic to D.

(2) If f and g are two strictly decreasing functions, then Ψ in Definition 4.30 is a dual isomorphism.

Example 4.33.

(1) Consider f, g in Example 4.21 part (1). Then

$$\mathrm{D}_{\mathrm{POL}_f\times\mathrm{POL}_g} = \{(p_f, p'_g) \mid p\in\mathbf{POL}\},$$

where

$$p_f(X) = \alpha_p^2 x_1 + (1-\alpha_p^2)(1-y_1)$$

and

$$p'_g(X) = (1-\alpha_p)^3 x_1 + (1-(1-\alpha_p)^3)(1-y_1).$$

Hence, $\mathbf{D}_{\mathbf{POL}_f\times\mathbf{POL}_g}$ is a distributive complete lattice which is isomorphic to D.

(2) Consider f, g in Example 4.21 part (2).Then $\Psi(\alpha,1-\alpha) = (p_f, p'_g)$, where

$$p_f(X) = (1-\alpha^2)x_1 + \alpha^2(1-y_1)$$

and

$$p'_g(X) = (1-(1-\alpha)^3)x_1 + (1-\alpha)^3(1-y_1).$$

Ψ is a dual isomorphism.

Remark 4.34. Note that the results given in Corollary 4.26 show that a probability on the lattice $\mathbf{L}_{\mathbf{POL}_f \times \mathbf{POL}_g}$ (as an isomorphism of L^*) could be viewed as a representation of modeling imprecision, if it is seen from the perspective of Figure 1 in the paper of Ajmal, Naseem et al. (2009) , where it is proved that different models of imprecision such as grey sets, vague sets, intuitionistic [0,1]-fuzzy sets, L^*- fuzzy sets, intuitionistic fuzzy sets and interval-valued fuzzy sets are equivalent up to isomorphism.

5. Conclusion

In this paper, **POL**, the set of all probability functions on L^* is studied. It is shown that this set is sufficiently large by constructing many examples using Sugeno's negation. Two operations are defined to endow this set as monoid structure which is a distributive complete lattice and also De-Morgan algebra. Then, $\mathbf{POL}_f$, the set of all f −probabilities on L^*, induced by a fixed strictly increasing function on [0,1] to itself is studied and it is proved that this set is a distributive complete lattice when endowed with appropriate lattice operations. It is shown that the product lattice $\mathbf{POL}_f \times \mathbf{POL}_g$, when f and g are strictly increasing functions, is a distributive complete lattice isomorphic to $L^{\square}$, where $L^{\square}$ is the set $[0,1]^2$ considered as a super lattice of L^*. Then more sub lattices of $L^{\square}$ and L^* are obtained. Some lattices (dual) isomorphism studied in this paper actually reveal that probabilities on the lattice L^* could be considered as a representation of modeling imprecision as explained in Remark 4.34.

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