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q-Analogs of Identities Involving Harmonic Numbers and Binomial Coefficients

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Abstract

Recently, McCarthy presented two algebraic identities involving binomial coefficients and harmonic numbers, one of which generalizes an identity used to prove the Apéry number supercongruence. In 2008, Prodinger provided human proofs of identities initially obtained by Osburn and Schneider using the computer program **Sigma**. In this paper, we establish q-analogs of a fair number of the identities appearing in McCarthy (*Integers* 11 (2011): A37) and Prodinger (*Integers* 8 (2008): A10) by making use of q-partial fractions.

Keywords: partial fraction decomposition, q-identities, harmonic numbers, q-binomial coefficients

MSC 2010: 05A19, 11B65

1. Introduction

Chu (2004) showed the following binomial coefficient harmonic sum identity using the method of partial fractions:

$$\sum_{k=1}^{n}\binom{n+k}{k}^2\binom{n}{k}^2(1+2kH(n+k)+2kH(n-k)-4kH(k))=0, \tag{1}$$

where $H(n):=\sum_{i=1}^{n}\frac{1}{i}$ denotes the n-th harmonic number. This identity had previously been established using the WZ method [Ahlgren et al. (1998), Ahlgren and Ono (2000), Chu (2004), Dilcher (2008), McCarthy (2011), Osburn and Schneider (2009) and Petkovšek et al. (1996)]. It played an instrumental role in Ahlgren and Ono's proof of the Apéry number supercongruence [Ahlgren and Ono (2000)]. Recently and McCarthy (2011) extended (1) to obtain two further binomial coefficient harmonic sum identities. One of these, for instance, states that for all $m \geq n$, we have

$$\sum_{k=0}^{n}\binom{m+k}{k}\binom{n+k}{k}\binom{m}{k}\binom{n}{k}f_{n,m,k} + \sum_{k=n+1}^{m}(-1)^{k-n}\binom{m+k}{k}\binom{n+k}{k}\binom{m}{k}\binom{k-1}{n}^{-1}=(-1)^{n+m}, \tag{2}$$

where $f_{n,m,k}=1+k(H(m+k)+H(m-k)+H(n+k)+H(n-k)-4H(k))$. Note that (2) is a generalization of (1).

In this paper, we provide q-analogs for the main results in [McCarthy (2011) and Prodinger (2008)]. To do so, we make use of various q-partial fraction decompositions which generalizes prior techniques. Here, we adopt the standard notation

$$[i]_q := (1-q^i)/(1-q), \qquad [i]_q! := [1]_q[2]_q\cdots[i]_q, \qquad \begin{bmatrix} i \\ k \end{bmatrix}_q := \frac{[i]_q!}{[k]_q![i-k]_q!}$$

for the q-analog of a nonnegative integer i, the q-factorial, and the q-binomial coefficient, respectively. We will omit the base q and write $[i]$, $[i]!$, etc., since the context will be clear. If i is negative, then let $[i]=[i]!=0$, for convenience.

For a positive integer n, we further define the q-harmonic numbers of first type and second type, respectively, by

$$H_q(n):=\sum_{i=1}^{n}\frac{1}{[i]}, \quad \widetilde{H}_q(n):=\sum_{i=1}^{n}\frac{q^i}{[i]},$$

where $H_q(0)=\widetilde{H}_q(0)=0$; see, e.g., Dilcher (2008).

In particular, the q-analog of (2) that we give is as follows: Let m,n be any positive integers such that $m\ge n$. Then

$$\sum_{k=0}^{n} q^{k(m+n-k-1)} \begin{bmatrix} m+k \\ k \end{bmatrix}\begin{bmatrix} n+k \\ k \end{bmatrix}\begin{bmatrix} m \\ k \end{bmatrix}\begin{bmatrix} n \\ k \end{bmatrix} f_{n,m,k}(q) + \sum_{k=n+1}^{m} (-1)^{k-n} q^{\frac{(k-n)(k+n+1)}{2}-km} \begin{bmatrix} m+k \\ k \end{bmatrix}\begin{bmatrix} n+k \\ k \end{bmatrix}\begin{bmatrix} m \\ k \end{bmatrix}\begin{bmatrix} k-1 \\ n \end{bmatrix}^{-1} = (-1)^{n+m} q^{\binom{n+1}{2}+\binom{m+1}{2}},$$

where $f_{n,m,k}(q)=1+q^{-k}[k](\tilde{H}_q(m+k)+\tilde{H}_q(n+k)-4\tilde{H}_q(k)+H_q(m-k)+H_q(n-k))$. The above formula occurs below as Theorem 2.2. As a byproduct, setting $m=n$, we immediately have

$$\sum_{k=1}^{n} q^{k(2n-k-1)} \begin{bmatrix} n+k \\ k \end{bmatrix}^2 \begin{bmatrix} n \\ k \end{bmatrix}^2 f_{n,n,k}(q) = q^{n(n+1)} - 1,$$

which is a q-analog of (1).

The next pair of formulas, corresponding to our Theorem 3.2 and Theorem 3.4, respectively, are q-generalizations of identities of the Osburn-Schneider (2009) type that were reproven by Prodinger (2008): For all $n\ge 1$, we have

$$\sum_{k=0}^{n} (-1)^{n-k} q^{\binom{n+1-k}{2}} \begin{bmatrix} n+k \\ k \end{bmatrix}\begin{bmatrix} n \\ k \end{bmatrix} = 1 \tag{3}$$

and

$$\sum_{k=0}^{n} (-1)^{n-1-k} q^{\binom{k+1}{2}-\binom{n}{2}-k(n-1)} \begin{bmatrix} n+k \\ k \end{bmatrix}\begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{[n+k]^2} = \frac{[n-1]!^2}{[2n]!}. \tag{4}$$

The q-binomial identities (3) and (4) are extensions of [Prodinger (2008), Identity 2.1] and [Prodinger (2008), Identity 2.2], respectively. For (3), we also provide a bijective proof which makes use of a sign-changing involution and seems to be new in the case $q=1$ as well.

Our methodology may be described as follows. We aim to establish q-analogs of certain previously established combinatorial identities. First, we manifest a q-lemma, which roughly means to write a q-rational function, i.e., a product-looking q-formula, as a sum-looking q-formula, which is often the most difficult step since it involves guessing the manner in which the indeterminate q is to appear. Such an expression is frequently related to useful lemmas that lead to the identities under study but without q-theory yet. Second, we introduce partial fractions on the product-looking q-formula wherein the decomposed denominators are q-natured. The numerators are conveniently isolated upon taking appropriate limits of the involved variable. Then terms are matched and the q-lemma is confirmed. Lastly, with careful manipulations of the q-lemma, often involving taking limits once again, the desired q-identities can be obtained. We remark that while in this paper we treat the identities appearing in [McCarthy (2011)] and [Prodinger (2008)], it seems that the methodology could be applied elsewhere.

2. Identities Involving q-Binomial Coefficients and q-Harmonic Sums

In this section, we establish two identities involving q-binomial coefficients and q-harmonic sums that generalize the results of [McCarthy (2011)].

The following lemma will have as a limiting case the first of these identities.

Lemma 2.1. Let x be an indeterminate and m,n be positive integers with $m \geq n$. Then the q-generating function

$$f(x) = \frac{x\prod_{k=1}^{n}([k]-q^k x)\prod_{k=1}^{m}([k]-q^k x)}{\prod_{k=0}^{n}([k]+x)\prod_{k=0}^{m}([k]+x)} \tag{5}$$

is given by

$$\begin{aligned} f(x) = \sum_{\ell=0}^{n} \frac{q^{\ell(m+n-\ell-1)}}{[\ell]+x} \begin{bmatrix} m+\ell \\ \ell \end{bmatrix} \begin{bmatrix} n+\ell \\ \ell \end{bmatrix} \begin{bmatrix} m \\ \ell \end{bmatrix} \begin{bmatrix} n \\ \ell \end{bmatrix} \\ \cdot \left\{ \frac{-[\ell]}{[\ell]+x} + 1 + q^{-\ell}[\ell](\tilde{H}_q(m+\ell) + \tilde{H}_q(n+\ell) - 4\tilde{H}_q(\ell) + H_q(m-\ell) + H_q(n-\ell)) \right\} \\ + \sum_{\ell=n+1}^{m} \frac{(-1)^{\ell-n}}{[\ell]+x} q^{\frac{(\ell-n)(\ell+n+1)}{2}-\ell m} \begin{bmatrix} m+\ell \\ \ell \end{bmatrix} \begin{bmatrix} n+\ell \\ \ell \end{bmatrix} \begin{bmatrix} m \\ \ell \end{bmatrix} \begin{bmatrix} \ell-1 \\ n \end{bmatrix}^{-1}. \end{aligned} \tag{6}$$

Proof:

Using partial fractions, we may write (5) as

$$f(x) = \frac{A}{x} + \sum_{k=1}^{n}\left(\frac{B_k}{([k]+x)^2} + \frac{C_k}{[k]+x}\right) + \sum_{k=n+1}^{m}\frac{D_k}{[k]+x}$$

for rational functions A, B_k, C_k and D_k in q. We may isolate these coefficients by taking various limits of $f(x)$ as follows. First, we have

$$A=\lim_{x\to 0}xf(x)=\lim_{x\to 0}\frac{\prod_{k=1}^{n}([k]-q^kx)\prod_{k=1}^{m}([k]-q^kx)}{\prod_{k=1}^{n}([k]+x)\prod_{k=1}^{m}([k]+x)}=1.$$

This agrees with the $l=0$ term of expression (6). Next, if $1\le \ell\le n$, then

$$\begin{aligned}
B_\ell &= \lim_{x\to -[\ell]}([\ell]+x)^2 f(x)\\
&=\frac{-[\ell]\prod_{k=1}^{n}([k]+q^k[\ell])\prod_{k=1}^{m}([k]+q^k[\ell])}{\prod_{k=0}^{\ell-1}([k]-[\ell])\prod_{k=0}^{\ell-1}([k]-[\ell])\prod_{k=\ell+1}^{n}([k]-[\ell])\prod_{k=\ell+1}^{m}([k]-[\ell])}\\
&=\frac{-[\ell]\frac{[\ell+n]![\ell+m]!}{[\ell]![\ell]!}}{q^{2\binom{\ell}{2}+\ell(m-\ell)+\ell(n-\ell)}[\ell]![\ell]![m-\ell]![n-\ell]!}\\
&=-q^{\ell(m+n-\ell-1)}[\ell]\begin{bmatrix}m+\ell\\ \ell\end{bmatrix}\begin{bmatrix}n+\ell\\ \ell\end{bmatrix}\begin{bmatrix}m\\ \ell\end{bmatrix}\begin{bmatrix}n\\ \ell\end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
C_\ell &= \lim_{x\to -[\ell]}\frac{([\ell]+x)^2 f(x)-B_\ell}{[\ell]+x}=\lim_{x\to -[\ell]}\frac{d}{dx}(([\ell]+x)^2 f(x))\\
&=\lim_{x\to -[\ell]}\frac{d}{dx}\left(\frac{x\prod_{k=1}^{n}([k]-q^kx)\prod_{k=1}^{m}([k]-q^kx)}{\prod_{k=0}^{\ell-1}([k]+x)\prod_{k=0}^{\ell-1}([k]+x)\prod_{k=\ell+1}^{n}([k]+x)\prod_{k=\ell+1}^{m}([k]+x)}\right)\\
&=-\frac{B_\ell}{[\ell]}-B_\ell\left(\sum_{k=1}^{m}\frac{q^k}{[k+\ell]}+\sum_{k=1}^{n}\frac{q^k}{[k+\ell]}-2\sum_{k=0}^{\ell-1}\frac{q^{-k}}{[\ell-k]}+\sum_{k=\ell+1}^{m}\frac{q^{-\ell}}{[k-\ell]}+\sum_{k=\ell+1}^{n}\frac{q^{-\ell}}{[k-\ell]}\right)\\
&=-\frac{B_\ell}{[\ell]}-q^{-\ell}B_\ell\left(\widetilde{H}_q(m+\ell)+\widetilde{H}_q(n+\ell)-4\widetilde{H}_q(\ell)+H_q(m-\ell)+H_q(n-\ell)\right)\\
&=-\frac{B_\ell}{[\ell]}\left(1+q^{-\ell}[\ell](\widetilde{H}_q(m+\ell)+\widetilde{H}_q(n+\ell)-4\widetilde{H}_q(\ell)+H_q(m-\ell)+H_q(n-\ell))\right).
\end{aligned}$$

If $n+1\le \ell\le m$, then

$$\begin{aligned} D_\ell &= \lim_{x\to -[\ell]} ([\ell]+x) f(x) \\ &= \frac{-[\ell]\prod_{k=1}^{n}([k]+q^k[\ell])\prod_{k=1}^{m}([k]+q^k[\ell])}{\prod_{k=0}^{n}([k]-[\ell])\prod_{k=0}^{\ell-1}([k]-[\ell])\prod_{k=\ell+1}^{m}([k]-[\ell])} \\ &= (-1)^{\ell-n} q^{-\binom{n+1}{2}-\binom{\ell}{2}-\ell(m-\ell)} \begin{bmatrix} m+\ell \\ \ell \end{bmatrix}\begin{bmatrix} n+\ell \\ \ell \end{bmatrix}\begin{bmatrix} m \\ \ell \end{bmatrix}\begin{bmatrix} \ell-1 \\ n \end{bmatrix}^{-1}, \end{aligned}$$

which completes the proof. □

In fact, Lemma 2.1 is a q-analog of [McCarthy (2011), Theorem 4]. Now, multiplying the two different versions (5) and (6) of $f(x)$ in Lemma 2.1 by x, and letting $x \to \infty$, yields a q-analog of (1) [Ahlgren, S. and Ono (2000), Chu (2004)] and of (2) [McCarthy (2011), Theorem 2].

Theorem 2.2. Let m, n be any positive integers such that $m \geq n$. Then we have

$$\begin{aligned} &\sum_{\ell=0}^{n} q^{\ell(m+n-\ell-1)} \begin{bmatrix} m+\ell \\ \ell \end{bmatrix}\begin{bmatrix} n+\ell \\ \ell \end{bmatrix}\begin{bmatrix} m \\ \ell \end{bmatrix}\begin{bmatrix} n \\ \ell \end{bmatrix} \\ &\quad \cdot \left\{1+q^{-\ell}[\ell](\tilde{H}_q(m+\ell)+\tilde{H}_q(n+\ell)-4\tilde{H}_q(\ell)+H_q(m-\ell)+H_q(n-\ell))\right\} \\ &\quad + \sum_{\ell=n+1}^{m} (-1)^{\ell-n} q^{\frac{(\ell-n)(\ell+n+1)}{2}-\ell m} \begin{bmatrix} m+\ell \\ \ell \end{bmatrix}\begin{bmatrix} n+\ell \\ \ell \end{bmatrix}\begin{bmatrix} m \\ \ell \end{bmatrix}\begin{bmatrix} \ell-1 \\ n \end{bmatrix}^{-1} = (-1)^{n+m} q^{\binom{n+1}{2}+\binom{m+1}{2}}. \end{aligned}$$

The next lemma is a q-analog of [McCarthy (2011), Theorem 5].

Lemma 2.3. Let x be an indeterminate, p, m, n be positive integers with $p > m \geq n \geq \frac{p}{2}$, and c_1, c_2 be constants. Then, the q-generating function

$$f(x) = \frac{x\prod_{k=1}^{n}([k]-q^k x)\prod_{k=1}^{m}([k]-q^k x)}{\prod_{k=0}^{n}([k]+x)\prod_{k=0}^{m}([k]+x)} \left(c_1 \sum_{s=p-n}^{n} \frac{1}{[s]-q^s x} + c_2 \sum_{s=p-m}^{m} \frac{1}{[s]-q^s x} \right)$$

is given by

$$f(x)=\frac{u_{0,1}}{x}+\sum_{\ell=1}^{n}q^{\ell(m+n-\ell-1)}\begin{bmatrix}m+\ell\\ \ell\end{bmatrix}\begin{bmatrix}n+\ell\\ \ell\end{bmatrix}\begin{bmatrix}m\\ \ell\end{bmatrix}\begin{bmatrix}n\\ \ell\end{bmatrix}\{\frac{-[\ell]u_{\ell,1}}{([\ell]+x)^2}+\frac{u_{\ell,1}-q^{-\ell}[\ell]u_{\ell,2}}{[\ell]+x}$$
$$+\frac{q^{-\ell}u_{\ell,1}[\ell](\tilde{H}_q(m+\ell)+\tilde{H}_q(n+\ell)-4\tilde{H}_q(\ell)+H_q(m-\ell)+H_q(n-\ell))}{[\ell]+x}\}$$
$$+\sum_{\ell=n+1}^{m}(-1)^{\ell-n}q^{\frac{(\ell-n)(\ell+n+1)}{2}-\ell m}\begin{bmatrix}m+\ell\\ \ell\end{bmatrix}\begin{bmatrix}n+\ell\\ \ell\end{bmatrix}\begin{bmatrix}m\\ \ell\end{bmatrix}\begin{bmatrix}\ell-1\\ n\end{bmatrix}^{-1}\frac{u_{\ell,1}}{[\ell]+x},$$

where

$$u_{\ell,r}=c_1(H_{q;r}(\ell+n)-H_{q;r}(\ell+p-n-1))+c_2(H_{q;r}(\ell+m)-H_{q;r}(\ell+p-m-1))$$

and

$$H_{q;r}(n)=\sum_{j=1}^{n}\frac{q^{j(r-1)}}{[j]^r}$$

for a positive integer r.

Proof:

Again by the partial fraction decomposition, we may write

$$f(x)=\frac{A}{x}+\sum_{k=1}^{n}\left(\frac{B_k}{([k]+x)^2}+\frac{C_k}{[k]+x}\right)+\sum_{k=n+1}^{m}\frac{D_k}{[k]+x}$$

for rational functions A, B_k, C_k and D_k in q. By taking various limits of $f(x)$, we may isolate the coefficients above as follows. First, we have

$$A=\lim_{x\to 0}xf(x)=c_1\sum_{s=p-n}^{n}\lim_{x\to 0}\frac{\prod_{k=1}^{n}([k]-q^kx)\prod_{k=1}^{m}([k]-q^kx)}{([s]-q^sx)\prod_{k=1}^{n}([k]+x)\prod_{k=1}^{m}([k]+x)}$$
$$+c_2\sum_{s=p-m}^{m}\lim_{x\to 0}\frac{\prod_{k=1}^{n}([k]-q^kx)\prod_{k=1}^{m}([k]-q^kx)}{([s]-q^sx)\prod_{k=1}^{n}([k]+x)\prod_{k=1}^{m}([k]+x)}$$
$$=u_{0,1}.$$

If $1\le\ell\le n$, then

$$B_\ell = \lim_{x\to -[\ell]} ([\ell]+x)^2 f(x)$$

$$= \frac{-[\ell]\prod_{k=1}^{n}([k]+q^k[\ell])\prod_{k=1}^{m}([k]+q^k[\ell])}{\prod_{k=0}^{\ell-1}([k]-[\ell])\prod_{k=0}^{\ell-1}([k]-[\ell])\prod_{k=\ell+1}^{n}([k]-[\ell])\prod_{k=\ell+1}^{m}([k]-[\ell])} u_{\ell,1}$$

$$= \frac{-u_{\ell,1}[\ell]\frac{[\ell+n]![\ell+m]!}{[\ell]![\ell]!}}{q^{2\binom{\ell}{2}+\ell(m-\ell)+\ell(n-\ell)}[\ell]![\ell]![m-\ell]![n-\ell]!}$$

$$= -u_{\ell,1}q^{\ell(m+n-\ell-1)}[\ell]\begin{bmatrix} m+\ell \\ \ell \end{bmatrix}\begin{bmatrix} n+\ell \\ \ell \end{bmatrix}\begin{bmatrix} m \\ \ell \end{bmatrix}\begin{bmatrix} n \\ \ell \end{bmatrix}$$

and

$$C_\ell = \lim_{x\to -[\ell]} \frac{([\ell]+x)^2 f(x) - B_\ell}{[\ell]+x} = \lim_{x\to -[\ell]} \frac{d}{dx}(([\ell]+x)^2 f(x))$$

$$= \lim_{x\to -[\ell]} \frac{d}{dx}\left(\frac{x\prod_{k=1}^{n}([k]-q^k x)\prod_{k=1}^{m}([k]-q^k x)\left(c_1\sum_{s=p-n}^{n}\frac{1}{[s]-q^s x} + c_2\sum_{s=p-m}^{m}\frac{1}{[s]-q^s x}\right)}{\prod_{k=0}^{\ell-1}([k]+x)\prod_{k=0}^{\ell-1}([k]+x)\prod_{k=\ell+1}^{n}([k]+x)\prod_{k=\ell+1}^{m}([k]+x)} \right)$$

$$= \left(-\frac{B_\ell}{[\ell]} - B_\ell(\sum_{k=1}^{m}\frac{q^k}{[k+\ell]} + \sum_{k=1}^{n}\frac{q^k}{[k+\ell]} - 2\sum_{k=0}^{\ell-1}\frac{q^{-k}}{[\ell-k]} + \sum_{k=\ell+1}^{m}\frac{q^{-\ell}}{[k-\ell]} + \sum_{k=\ell+1}^{n}\frac{q^{-\ell}}{[k-\ell]}) \right) u_{\ell,1}$$

$$+ q^{-\ell}B_\ell\left(c_1\sum_{s=p+\ell-n}^{n+\ell}\frac{q^s}{[s]^2} + c_2\sum_{s=p+\ell-m}^{m+\ell}\frac{q^s}{[s]^2} \right)$$

$$= \left(-\frac{B_\ell}{[\ell]} - q^{-\ell}B_\ell(\tilde{H}_q(m+\ell) + \tilde{H}_q(n+\ell) - 4\tilde{H}_q(\ell) + H_q(m-\ell) + H_q(n-\ell)) \right) u_{\ell,1} + q^{-\ell}u_{\ell,2}B_\ell$$

$$= -\frac{B_\ell(u_{\ell,1} - q^{-\ell}[\ell]u_{\ell,2} + q^{-\ell}u_{\ell,1}[\ell](\tilde{H}_q(m+\ell) + \tilde{H}_q(n+\ell) - 4\tilde{H}_q(\ell) + H_q(m-\ell) + H_q(n-\ell)))}{[\ell]}.$$

If $n+1 \le \ell \le m$, then

$$\begin{aligned} D_\ell &= \lim_{x\to-[\ell]}([\ell]+x)f(x) \\ &= \frac{-u_{\ell,1}[\ell]\prod_{k=1}^{n}([k]+q^k[\ell])\prod_{k=1}^{m}([k]+q^k[\ell])}{\prod_{k=0}^{n}([k]-[\ell])\prod_{k=0}^{\ell-1}([k]-[\ell])\prod_{k=\ell+1}^{m}([k]-[\ell])} \\ &= (-1)^{\ell-n}u_{\ell,1}q^{-\binom{n+1}{2}-\binom{\ell}{2}-\ell(m-\ell)}\begin{bmatrix} m+\ell \\ \ell \end{bmatrix}\begin{bmatrix} n+\ell \\ \ell \end{bmatrix}\begin{bmatrix} m \\ \ell \end{bmatrix}\begin{bmatrix} \ell-1 \\ n \end{bmatrix}^{-1}, \end{aligned}$$

completing the proof. □

Lemma 2.3 has the following corollary which is a q-analog of [McCarthy (2011), Theorem 3]. As before, we multiply the two expressions of $f(x)$ in Lemma 2.3 by x and then let $x\to\infty$ to obtain the following result.

Theorem 2.4. Let p,m,n be positive integers such that $p>m\geq n\geq \frac{p}{2}$. Then

$$\begin{aligned} u_{0,1} + \sum_{\ell=1}^{n} q^{\ell(m+n-\ell-1)}[\ell]\begin{bmatrix} m+\ell \\ \ell \end{bmatrix}\begin{bmatrix} n+\ell \\ \ell \end{bmatrix}\begin{bmatrix} m \\ \ell \end{bmatrix}\begin{bmatrix} n \\ \ell \end{bmatrix}\{u_{\ell,1} - q^{-\ell}[\ell]u_{\ell,2} \\ + q^{-\ell}u_{\ell,1}[\ell](\tilde{H}_q(m+\ell)+\tilde{H}_q(n+\ell)-4\tilde{H}_q(\ell)+H_q(m-\ell)+H_q(n-\ell))\} \\ + \sum_{\ell=n+1}^{m}(-1)^{\ell-n}u_{\ell,1}q^{\frac{(\ell-n)(\ell+n+1)}{2}-\ell m}\begin{bmatrix} m+\ell \\ \ell \end{bmatrix}\begin{bmatrix} n+\ell \\ \ell \end{bmatrix}\begin{bmatrix} m \\ \ell \end{bmatrix}\begin{bmatrix} \ell-1 \\ n \end{bmatrix}^{-1} = 0, \end{aligned}$$

where $u_{\ell,r}$ is as before.

3. q-Analogs of Identities by Osburn and Schneider

The methodology of the prior section may be used to establish q-generalizations of other identities occurring in the literature. In this section, we focus on some identities discovered by Osburn and Schneider (2009) (done by the computer program **Sigma**) and reproved by Prodinger (2008) (done by human mathematics).

Lemma 3.1. Let x be an indeterminate and let n be any positive integer. Then the generating function

$$f(x) = \frac{\prod_{k=1}^{n}([k]+x)}{x\prod_{k=1}^{n}([k]-q^k x)}$$

is given by

$$f(x) = \sum_{\ell=0}^{n} \frac{(-1)^{\ell-1} q^{\binom{\ell+1}{2}-\ell n} \begin{bmatrix} n+\ell \\ \ell \end{bmatrix} \begin{bmatrix} n \\ \ell \end{bmatrix}}{[\ell]-q^{\ell}x}.$$

Proof:

Use partial fractions and write $f(x) = \frac{A}{x} + \sum_{k=1}^{n} \frac{B_k}{([k]-q^k x)}$ for rational functions A and B_k in q. It is easy to observe the fact that $A=1$ by taking the limit $\lim_{x\to 0} xf(x)$. To find B_ℓ, note that for $1 \le \ell \le n$, we have

$$B_\ell = \lim_{x\to q^{-\ell}[l]} ([\ell]-q^\ell x) f(x) = (-1)^{\ell-1} q^{\binom{\ell+1}{2}-\ell n} \begin{bmatrix} n+\ell \\ \ell \end{bmatrix} \begin{bmatrix} n \\ \ell \end{bmatrix}. \square$$

Multiplying the generating function $f(x)$ in the previous lemma by x, and letting $x \to \infty$, yields a q-analog of [Prodinger (2008), Identity 2.1].

Theorem 3.2. For all $n \ge 0$,

$$\sum_{\ell=0}^{n} (-1)^{n-\ell} q^{\binom{n+1-\ell}{2}} \begin{bmatrix} n+\ell \\ \ell \end{bmatrix} \begin{bmatrix} n \\ \ell \end{bmatrix} = 1.$$

Although Theorem 3.2 is obtained as a corollary of Lemma 3.1, it is possible to provide a bijective proof of this result, which seems to be new in the $q=1$ case as well.

Combinatorial Proof of Theorem 3.2:

Proof:

If $n \ge 1$ and $0 \le k \le n$, then let $\mathcal{A}_{n,k}$ consist of the set of words in $\{1,2,3\}$ of length $n+k$ having exactly k 1's, $n-k$ 2's, and k 3's. Let $\mathcal{A}_n = \cup_{k=0}^{n} \mathcal{A}_{n,k}$. We first consider three statistics on the set $\mathcal{A}_n$ as follows. Given $\alpha \in \mathcal{A}_n$, let α' denote the binary word of the same length obtained by replacing each 1 by 0 and each 2 or 3 by 1. We define the statistic $perm^*$ on $\mathcal{A}_n$ by setting $perm^*(\alpha) = perm(\alpha')$, where $perm(w)$ records the number of *permanences* in the word $w = w_1 w_2 \cdots$, i.e., the number of ordered pairs (i, j) with $1 \le i < j \le n$ and $w_i < w_j$.

We now consider the second and third statistics on $\mathcal{A}_n$. If $\alpha \in \mathcal{A}_n$, then let $\alpha'' = a_1 a_2 \cdots a_n$ denote the word of length n on $\{2,3\}$ obtained by writing, in *reverse* order, the subsequence of

α consisting of all the 2's and 3's. Let $sum^*(\alpha) = sum(\alpha'')$, where $sum(\alpha'')$ denotes the sum of the positions within α'' corresponding to 2, i.e.,

$$sum(\alpha'') = \sum_{i:\, a_i=2} i.$$

Finally, define the statistic tot on $\mathcal{A}_n$ by setting $tot(\alpha) = perm^*(\alpha) + sum^*(\alpha)$ for all α. For example, if $n=5$, $k=2$, and $\alpha = 2132213 \in \mathcal{A}_{5,2}$, then $\alpha' = 1011101$, $\alpha'' = 32232$, and

$$tot(\alpha) = perm^*(\alpha) + sum^*(\alpha) = perm(\alpha') + sum(\alpha'') = (4+1) + (2+3+5) = 15.$$

Then $q^{\binom{n+1-k}{2}} \begin{bmatrix} n+k \\ n \end{bmatrix} \begin{bmatrix} n \\ n-k \end{bmatrix}$ gives the distribution of the statistic tot on $\mathcal{A}_{n,k}$ for $0 \le k \le n$. To see this, first note that $\begin{bmatrix} n+k \\ n \end{bmatrix}$ accounts for the distribution of the $perm$ statistic on the set of binary words α' that arise (see, e.g., [Stanley (1997), Prop. 1.3.17]). The remaining factor $q^{\binom{n+1-k}{2}} \begin{bmatrix} n \\ n-k \end{bmatrix}$ then accounts for the distribution of the sum statistic on the words α'', which follows from replacing x with qx in the q-binomial theorem (see, e.g., Stanley (1997), Page 162)

$$(1+x)(1+qx)\cdots(1+q^{n-1}x) = \sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} x^k.$$

For any member π of $\mathcal{A}_{n,k}$ ($0 \le k \le n$), define its sign by $\mathrm{sgn}(\pi) = (-1)^{n-k}$. Then the sum in Theorem 3.2 above gives the total signed weight of all the members of $\mathcal{A}_n$, i.e., it gives

$$\sum_{\pi \in \mathcal{A}_n} \mathrm{sgn}(\pi) q^{tot(\pi)}.$$

To complete the proof, it suffices to identify a sign-reversing, tot-preserving involution of $\mathcal{A}_n$ off of a set whose signed weight is 1. Let $c = 3^n 1^n \in \mathcal{A}_n$; note that c has signed weight 1 since it belongs to $\mathcal{A}_{n,n}$ and since $perm^*(c)$ and $sum^*(c)$ are both zero. If $\pi \in \mathcal{A}_n - \{c\}$, then it may be expressed in the form

$$\pi = \gamma x 3^r 1^s, \qquad (7)$$

where γ is some word, $r, s \ge 0$, and $x = 2$ or $x = 13$ (i.e., a 1 followed by a 3). To see this, consider the right-most occurrence of the letter 2 or the subword 13, which exists since we are excluding the element c. Switch options with respect to x in (7) above, which pairs off words $\pi = \gamma 23^r 1^s$ and $\rho = \gamma 13^{r+1} 1^s$ for various γ, r, and s. The resulting mapping is an involution of $\mathcal{A}_n - \{c\}$ which changes the sign since the number k of 1's changes by one. It also preserves *tot* since $perm^*(\rho) - perm^*(\pi) = sum^*(\pi) - sum^*(\rho)$ $(= r+1)$, as required, which completes the proof. □

Lemma 3.3. Let x be an indeterminate and let n be any positive integer. Then the generating function

$$f(x) = \frac{\prod_{k=1}^{n-1}([k]+x)}{([n]+x)\prod_{k=0}^{n}([k]-q^k x)}$$

is given by

$$f(x) \quad = \sum_{\ell=0}^{n} \frac{(-1)^\ell q^{\binom{\ell+1}{2} - \ell(n-2)} \begin{bmatrix} n+\ell \\ \ell \end{bmatrix} \begin{bmatrix} n \\ \ell \end{bmatrix}}{[n+\ell]^2([\ell]-q^\ell x)} + \frac{(-1)^{n-1} q^{\binom{n}{2}} [n-1]!^2}{[2n]!([n]+x)}.$$

Proof:

Using the partial fraction decomposition, we may write $f(x) = \frac{A}{[n]+x} + \sum_{k=0}^{n} \frac{B_k}{([k]-q^k x)}$ for rational functions A and B_k in q. We may isolate these coefficients by taking various limits of $f(x)$. We have

$$A \quad = \lim_{x \to -[n]} ([n]+x) f(x) = \frac{(-1)^{n-1} q^{\binom{n}{2}} [n-1]!^2}{[2n]!}$$

and, for $0 \le \ell \le n$,

$$B_\ell \quad = \lim_{x \to q^{-\ell}[l]} ([\ell]-q^\ell x) f(x) = (-1)^\ell q^{\binom{\ell+1}{2} - \ell(n-2)} \begin{bmatrix} n+\ell \\ \ell \end{bmatrix} \begin{bmatrix} n \\ \ell \end{bmatrix} \frac{1}{[n+\ell]^2},$$

which gives the result required. □

Multiplying the generating function $f(x)$ in Lemma 3.3 by x, and letting $x \to \infty$, gives a q-analog of Prodinger (2008), Identity 2.2.

Theorem 3.4. For all $n \geq 1$,

$$\sum_{\ell=0}^{n}(-1)^{n-1-\ell} q^{\binom{\ell+1}{2}-\binom{n}{2}-\ell(n-1)} \begin{bmatrix} n+\ell \\ \ell \end{bmatrix} \begin{bmatrix} n \\ \ell \end{bmatrix} \frac{1}{[n+\ell]^2} = \frac{[n-1]!^2}{[2n]!}.$$

Arguments similar to the prior ones yield this section's third and final lemma whose proof we omit.

Lemma 3.5. Let x be an indeterminate and let n, j be positive integers. Then the generating function

$$f(x) = \frac{\prod_{k=1}^{n}([k]+x)}{[j]([j]+x)\prod_{k=1}^{n}([k]-q^k x)}$$

is given by

$$f(x) = \sum_{\ell=1}^{n} \frac{(-1)^{\ell-1} q^{\binom{\ell}{2}-\ell(n-1)} \begin{bmatrix} n+\ell \\ \ell \end{bmatrix} \begin{bmatrix} n \\ \ell \end{bmatrix} [\ell]}{[j][\ell+j]([\ell]-q^{\ell}x)} + \frac{(-1)^n q^{\binom{n+1}{2}}[j-1]!^2}{[j-n-1]![n+j]!([j]+x)}.$$

In the previous lemma and work below, we take $\frac{1}{[i]!} = 0$ if i is negative. Note that the $q=1$ cases of Lemmas 3.1, 3.3 and 3.5 occur in Prodinger (2008).

Now, multiplying the generating function $f(x)$ in Lemma 3.5 by xq^j, taking the limit as $x \to \infty$, and summing over all $j \geq 1$ yields

$$\sum_{\ell=1}^{n}(-1)^{\ell} q^{\binom{\ell}{2}-\ell n} \begin{bmatrix} n+\ell \\ \ell \end{bmatrix} \begin{bmatrix} n \\ \ell \end{bmatrix} \sum_{j\geq 1} \frac{q^j[\ell]}{[j][\ell+j]} = \sum_{j\geq 1}\left(-\frac{(-1)^n q^{\binom{n+1}{2}+j}[j-1]!^2}{[j-n-1]![n+j]!} + \frac{(-1)^n q^{-\binom{n+1}{2}+j}}{[j]} \right),$$

or, equivalently,

$$\sum_{\ell=1}^{n}(-1)^{n-\ell} q^{\binom{\ell-n}{2}} \begin{bmatrix} n+\ell \\ \ell \end{bmatrix} \begin{bmatrix} n \\ \ell \end{bmatrix} H_q(\ell) = \sum_{j\geq 0}\left(\frac{q^{j+1}}{[j+1]} - \frac{q^{n^2+n+j+1}[j]!^2}{[j-n]![n+1+j]!} \right).$$

Denote the summand on the right-hand side of the last expression by $F(n,j)$. Then we have

$$F(n+1,j)-F(n,j)=\frac{q^{n^2+n+j+1}[j]!^2\,[2n+2]}{[j-n]![n+2+j]!}=G(n,j+1)-G(n,j),$$

where

$$G(n,j)=\frac{q^{(n+1)^2}[2n+2][j]!^2}{[n+1]^2[j-n-1]![n+1+j]!}.$$

Letting $S_n=\sum_{j\ge 1}F(n,j)$ gives

$$S_{n+1}-S_n=\lim_{j\to\infty}G(n,j)-G(n,0)=\frac{q^{(n+1)^2}[2n+2]}{[n+1]^2}.$$

By the initial condition $S_0=0$, we have $S_n=\sum_{j=1}^{n}\frac{q^{j^2}[2j]}{[j]^2}$, which yields the following q-analog of Prodinger (2008), Identity 2.3.

Theorem 3.6. For all $n\ge 1$,

$$\sum_{\ell=1}^{n}(-1)^{n-\ell}q^{\binom{\ell-n}{2}}\begin{bmatrix}n+\ell\\ \ell\end{bmatrix}\begin{bmatrix}n\\ \ell\end{bmatrix}H_q(\ell)=\sum_{j=1}^{n}\frac{q^{j^2}[2j]}{[j]^2}.$$

4. Conclusion

In the previous two sections, we have provided several working examples of the methodology described in the introduction. Perhaps the techniques demonstrated here can be applied to a greater number and variety of identities. For instance, it seems that the illustrated methods could be used to generalize identities involving sums of various products or identities for convolutions between binomial coefficients and other discrete rational quantities. As can be seen, finding the q-analog of a combinatorial identity can be a craft that still requires guesswork and perhaps some fortunate circumstances. Elusive are q-analogs for the following two identities shown in Prodinger (2008) which we seek:

$$\sum_{\ell=0}^{n}(-1)^{n-\ell}\binom{n+\ell}{\ell}\binom{n}{\ell}H(n+\ell)=2H(n)$$

and

$$\sum_{\ell=1}^{n}(-1)^{n-\ell}\binom{n+\ell}{\ell}\binom{n}{\ell}\ell H(\ell)=n(n+1)(2H(n)-1).$$

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