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## Approximate Approach to the Das Model of Fractional Logistic Population Growth

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**(Dedicated to Professor J. H. He)**

### Abstract

In this article, the analytical method, Homotopy perturbation method (HPM) has been successfully implemented for solving nonlinear logistic model of fractional order. The fractional derivatives are described in the Caputo sense. Using initial value, the explicit solutions of population size for different particular cases have been derived. Numerical results show that the method is extremely efficient to solve this complicated biological model.

**Keywords:** Logistic model; Fractional Brownian motion; Caputo derivative; Homotopy perturbation method.

**MSC (2010) No.:** 26A33, 34A08, 60G22, 92D25, 74H10

### 1. Introduction

The first theoretical treatment of population dynamics was presented by Malthus (1798) viz. “Essay on the principle of population”. In 1838, Verhulst (1838) formed a mathematical model based on the “Principle of population” viz. “The logistic equation”. Nicholson (1957) highlighted the logistic model through a ‘Balancing’ phenomenon by studying the population over time. Studies reveal that the average density of population is being maintained at a constant analysis the level over a period of time, unless there is a major environmental change.

Malthusian Geometrical Law is expressed as

$$\frac{dN(t)}{dt} = r N(t) \quad (1)$$

where  $N(t)$  is the population at time  $t$  and  $r$  is the proportionality constant. But in equation (1), the author did not consider the case that the growth of the population in any environment may be stopped due to the density of the population, which was later considered in (1838) through the following model.

$$\frac{dN(t)}{dt} = r N(t) \left[ 1 - \frac{N(t)}{\pi} \right], \quad (2)$$

where  $\pi$  is the maximum that a given amount of food can support. If  $N(t) \ll \pi$ , then equation (2) becomes equation (1).

The generalization of the nonlinear logistic model is represented by

$$\frac{dN(t)}{dt} = r N(t) \left[ 1 - \left( \frac{N(t)}{\pi} \right)^\alpha \right] / \alpha, \quad (3)$$

For  $\alpha \rightarrow 0$ , the model is known as Gompertz model which can be found in the literature of Actuarial science and in the mortality analysis of elderly person.

A brief history of the origin and development of the logistic model can be found in Cramer (2003). Olson (1999) has used a general conditional logistic model to detect linkage between marker loci and common disease with samples of affected sib pairs. Mahapatra and Kant (2005) has used a multinomial logistic model to deal with estimation problems and shown that the results of multinomial logistic are more informative and robust compared to the results of binary logistic model. Recently, Manjunatha et al. (2005) have developed an integrated logistic model using supply chain management system which clearly shows a greater acceptability of logistic model in industry. This concept is successfully implemented in automobile engineering, which shows the effectiveness of the method. Multivariable logistic modeling techniques are appearing in today's medical literature with increasing frequency [Moss et al. (2003), Lin et al. (2008)].

But to the best of authors' knowledge the above type of logistic equation with fractional time derivative has not yet been considered by any researcher. In recent times, the fractional derivative operators are becoming one of the key areas for dealing with complex systems. PDE's of fractional order are frequently appearing in various applications in different branches of engineering and biological sciences.

Fractional logistic population growth model is obtained from classical equation (3) by replacing the first order time derivative by a fractional derivative of order  $\beta$  ( $0 < \beta \leq 1$ ). An important phenomenon of this evolution equation is that it generates the FBM which is Gaussian but in

general non-Markovian in nature. It is very difficult to get the exact analytical solutions of fractional order problems especially for nonlinear cases. Here, the authors have made a sincere effort to solve the fractional nonlinear problems by using a powerful mathematical tool called HPM.

The HPM is the new approach for finding the approximate analytical solution of linear and nonlinear problems. The method was first proposed by He [(1999), (2000)] and was successfully applied to solve nonlinear wave equations by He [(2005), (2005), (2005), (2005), (2006)], nonlinear partial differential equations of fractional order by He (1998), Momani and Odibat (2007), Das and Gupta (2010), Momani et al. (2008), Das et al. (2009), Odibat and Momani (2008), Yildirim (2009) etc. The basic difference of this method from the other perturbation techniques is that it does not require small parameters in the equation which overcomes the limitations of the traditional perturbation techniques.

The objective of the present article is to solve the nonlinear logistic population growth model with fractional time derivative for different fractional Brownian motions and also for standard motion for different particular cases and the result is depicted graphically.

## 2. Solution of the Problem by HPM

Consider the fractional-order Logistic model (Das Model )

$$\frac{d^\beta N(t)}{dt^\beta} = \frac{r}{\alpha} N(t) \left[ 1 - \left( \frac{N(t)}{K} \right)^\alpha \right], \quad 0 < \beta \leq 1, \quad (4)$$

with initial condition

$$N(0) = \delta. \quad (5)$$

According to the homotopy perturbation method, we construct the following homotopy of equation (4) as

$$D_t^\beta N(t) = p \frac{r}{\alpha} N(t) \left[ 1 - \left( \frac{N(t)}{K} \right)^\alpha \right]. \quad (6)$$

If the embedding parameter  $0 \leq p \leq 1$  is considered as a “small parameter”, applying the classical perturbation technique, we can assume that the solution of equation (6) can be given as a power series in  $p$ , i.e.,

$$N(t) = N_0(t) + p N_1(t) + p^2 N_2(t) + p^3 N_3(t) + \dots, \quad (7)$$

when  $p \rightarrow 1$ , equation (6) corresponds equation (4), equation (7) becomes the approximate solution of equation (4), we obtain the following set of linear differential equations

$$p^0: D_t^\beta N_0(t) = 0, \quad (8)$$

$$p^1: D_t^\beta N_1(t) = \frac{r}{\alpha} N_0(t) \left[ 1 - \left( \frac{N_0(t)}{K} \right)^\alpha \right], \quad (9)$$

$$p^2: D_t^\beta N_2(t) = \frac{r}{\alpha} N_0(t) \left[ -\alpha \frac{(N_0(t))^{\alpha-1}}{K^\alpha} N_1(t) \right] + \frac{r}{\alpha} N_1(t) \left[ 1 - \left( \frac{N_0(t)}{K} \right)^\alpha \right], \quad (10)$$

$$p^3: D_t^\beta N_3(t) = \frac{r}{\alpha} N_0(t) \left[ -\alpha \frac{(N_0(t))^{\alpha-1}}{K^\alpha} N_2(t) - \frac{\alpha(\alpha-1)}{2} \frac{(N_0(t))^{\alpha-2}}{K^\alpha} N_1(t)^2 \right] \\ + \frac{r}{\alpha} N_1(t) \left[ -\alpha \frac{(N_0(t))^{\alpha-1}}{K^\alpha} N_1(t) \right] + \frac{r}{\alpha} N_2(t) \left[ 1 - \left( \frac{N_0(t)}{K} \right)^\alpha \right], \quad (11)$$

and so on.

The method is based on applying the operator  $J_t^\beta$  (the inverse of operator  $D_t^\beta$ ) on both sides of the equations (8) – (11).

$$N_0(t) = \delta,$$

$$N_1(t) = \frac{r\delta}{\alpha} \left( 1 - \frac{\delta^\alpha}{K^\alpha} \right) \frac{t^\beta}{\Gamma(\beta+1)},$$

$$N_2(t) = \frac{r^2\delta}{\alpha^2} \left( 1 - \frac{\delta^\alpha}{K^\alpha} \right) \left( 1 - (\alpha+1) \frac{\delta^\alpha}{K^\alpha} \right) \frac{t^{2\beta}}{\Gamma(2\beta+1)},$$

$$N_3(t) = \frac{r^3\delta}{\alpha^3} \left( 1 - \frac{\delta^\alpha}{K^\alpha} \right) \left( 1 - (\alpha+1) \frac{\delta^\alpha}{K^\alpha} \right)^2 \frac{t^{3\beta}}{\Gamma(3\beta+1)} \\ - \frac{r^3\delta^{\alpha+1}(\alpha+1)}{2\alpha^2 K^\alpha} \left( 1 - \frac{\delta^\alpha}{K^\alpha} \right)^2 \frac{\Gamma(2\beta+1)t^{3\beta}}{(\Gamma(\beta+1))^2 \Gamma(3\beta+1)}.$$

Proceeding in this manner the other components of  $N_m$ ,  $m \geq 0$  of the HPM can be completely obtained, and the series solutions are thus entirely determined.

Finally, we approximate the analytical solution  $N(t)$  by the truncated series

$$N(t) = \lim_{M \rightarrow \infty} \Phi_M(t) \quad (12)$$

where  $\Phi_M(x, t) = \sum_{m=0}^{M-1} N_m(t)$ ,  $M \geq 1$ .

#### 4. Numerical Results and Discussion

In this article the numerical results of the population growth for two cases  $\alpha = 0.5 (< 1)$  and  $\alpha = 1$  for different fractional Brownian motions  $\beta = 1/4, 1/2, 3/4$  and for standard motion  $\beta = 1$  are calculated for various values of  $t$  at  $\delta = 1.25$  and  $r = 1$ . The results are presented graphically through Figures 1 and 2.

It is seen that in both cases  $N(t)$  increases with  $t$  for all  $\beta$ . Again  $N(t)$  initially decreases with the increase in  $\beta$  but afterwards the curves become opposite in nature. It is also seen from the Figures that the magnitude of  $N(t)$  increases with the increase of the power  $\alpha$ . It is also observed that as the value of fractional time derivative  $\beta$  decreases the solution assumes almost an asymptotic nature.

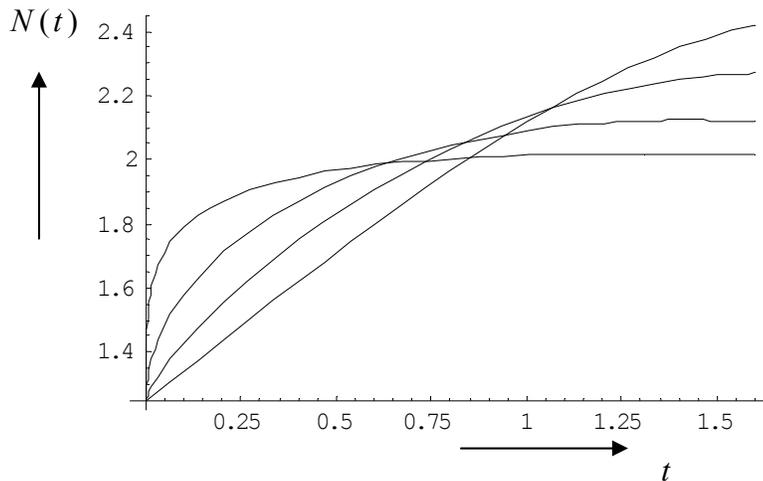


Figure 1. Plot of  $N(t)$  vs.  $t$  for different values of  $\beta$  at  $\alpha = 1/2$

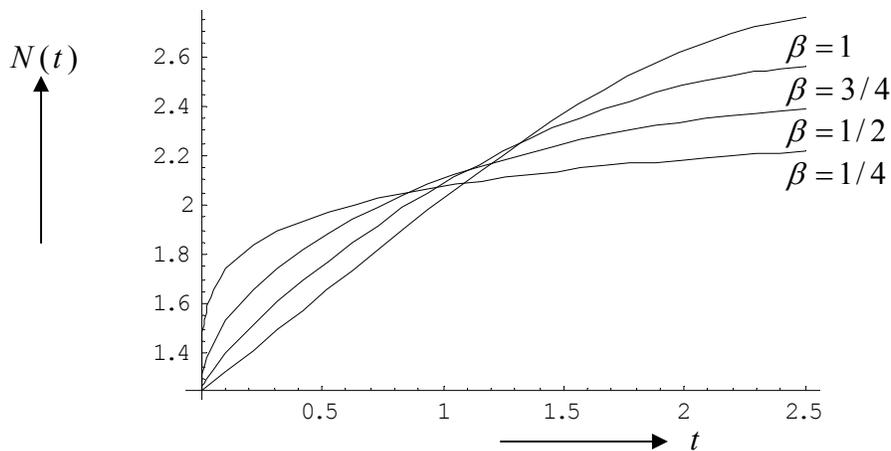


Figure 2. Plot of  $N(t)$  vs.  $t$  for different values of  $\beta$  at  $\alpha = 1$

## 5. Conclusion

The most important part of this study is to develop the idea of fractional model of historical logistic growth model, which is a first of its kind.

Another important part is to demonstrate the application of the powerful mathematical tool HPM for solving nonlinear fractional partial differential equation model. Taking fourth order term of the method, the approximate solution have been evaluated which clearly shows the simplicity, effectiveness and reliability of the proposed method.

The authors strongly believe that the article will highly be appreciated by the researchers working in the field of fractional calculus and on logistic model.

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