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Approximate Analytical Solutions for Fractional Space- and Time- Partial Differential Equations using Homotopy Analysis Method

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Abstract

This article presents the approximate analytical solutions of first order linear partial differential equations (PDEs) with fractional time- and space- derivatives. With the aid of initial values, the explicit solutions of the equations are solved making use of reliable algorithm like homotopy analysis method (HAM). The speed of convergence of the method is based on a rapidly convergent series with easily computable components. The fractional derivatives are described in Caputo sense. Numerical results show that the HAM is easy to implement and accurate when applied to space- time- fractional PDEs.

Keywords: Linear partial differential equations; Caputo derivative; Homotopy analysis method; Homogeneous/non-homogeneous equations; Fractional Brownian motion

MSC (2000) No.: 26A, 33, 35, 44, 65

1. Introduction

During the last few years it has been observed in many fields that any phenomena with strange kinetics cannot be described within the framework of classical theory using integer order derivatives. Recently, fractional differential equations have gained much attention since fractional order system response ultimately converges to the integer order system response. For high accuracy, fractional derivatives are then used to describe the dynamics of some structures. An integer order differential operator is a local operator. Whereas the fractional order differential operator is non local in the sense that it takes into account the fact that the future state not only depends upon the present state but also upon all of the history of its previous states. Because of this realistic property, the fractional order systems are becoming increasingly popular.

Another reason in support of the use of fractional order derivatives is that these are naturally related to the systems with memory that prevails for most of the physical and scientific system models. Applications and models involving fractional derivatives can be found in probability, physics, astrophysics, chemical physics [Oldham and Spanier (1974); Miller and Ross (1993); Podlubny (1999)] and various fields of engineering. Mainardi et al. (2008) provided a fundamental solution for the determination of probability density function for a general distribution of fractional time order system. Magin et al. (2008) solved the Bloch-Torrey equation after incorporating a fractional order Brownian model of diffusivity. Recently, Chen et al. (2010) have developed a fractal derivative model of anomalous diffusion and the fundamental solution of this model is compared with the existing method to establish its computational efficiency.

Finding accurate and efficient methods for solving fractional differential equations (FDEs) have been an active research undertaking. Exact solutions of most of the FDEs cannot be found easily, and this has mandated the use of both analytical and numerical methods. In 1992, a powerful analytical method for solving linear and nonlinear problems like homotopy analysis method has been developed by Liao (1992). The homotopy analysis method is a powerful mathematical tool which provides us with a simple way to ensure the convergence of the solution series, so that we can always get accurate enough approximations. In recent years, this method has been successfully employed to solve many types of problems in science and engineering [Das et al. (2010); Liao (1995), Liao (2005a, 2005b); Abbasbandy (2006); Bataineh et al. (2008); Alomari et al. (2009); Song and Zhang (2007)]. Furthermore, the homotopy analysis method logically contains the non perturbation methods such as Lyapunov's artificial small parameter method, the expansion method, Adomian's decomposition method and homotopy perturbation method [Lyapunov (1992); Jones III and Casetti (1992); Das (2009); He (2000)].

Homotopy analysis method contains an auxiliary parameter $\hbar$ which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution. The method can be also applied successfully to many linear and nonlinear problems such as application in heat radiation [Abbasbandy (2007)], solitary-wave solutions for the fifth-order KdV equation [Abbasbandy and Zakaria (2008)], vibration equation [Das and Gupta (2009)], generalized Benjamin-Bona-Mahony equation [Abbasbandy (2008)], exponentially decaying boundary layers [Liao and Magyari (2006)], hyperbolic PDEs [Das and Gupta (2010)] and many other problems.

Recently, Baitainah et al. (2008) have applied HAM to obtain the solutions of linear and nonlinear systems of first and second order PDEs, and have compared their results with the results of Wazwaz (2007) and Saha Ray (2006), who used VIM and ADM respectively. It is shown that VIM and ADM are just particular cases of HAM.

In our article, we have used HAM successfully to find the approximate analytical solutions of linear first order homogeneous/non-homogeneous PDEs with time- and space- fractional derivatives. These problems have not yet been solved by any researcher. Probability density functions $u(x,t)$ and $v(x,t)$, for different fractional Brownian motions and also for the standard motion for various particular cases are derived successfully and presented graphically.

2. Basic Idea of HAM

In this paper, we apply the HAM to the two problems to be discussed. In order to show the basic idea of HAM, consider the following differential equation:

$$N[u(x,t)] = 0, \tag{1}$$

where N is a non-linear operator, x and t are independent variables, $u(x,t)$ is the unknown function. By means of the HAM, we first construct the so-called zeroth-order deformation equation

$$(1-p)L[\phi(x,t;p) - u_0(x,t)] = \hbar H(x,t) N[\phi(x,t;p)] \tag{2}$$

where $p \in [0,1]$ is the embedding parameter, $\hbar \neq 0$, is a nonzero auxiliary parameter, $H(x,t) \neq 0$ is an auxiliary function, L is an auxiliary linear operator, $u_0(x,t)$ is the initial guess of $u(x,t)$, It is obvious that when the embedding parameter $p = 0$ and $p = 1$, Equation (2) becomes $\phi(x,t;0) = u_0(x,t)$ and $\phi(x,t;1) = u(x,t)$, respectively. Thus, as p increases from 0 to 1, the solution $\phi(x,t;p)$ varies from the initial guess $u_0(x,t)$ to the exact solution $u(x,t)$. Expanding $\phi(x,t;p)$ in Taylor series with respect to p, one has

$$\phi(x,t;p) = u_0(x,t) + \sum_{k=1}^{\infty} p^k u_k(x,t) \tag{3}$$

where

$$u_k(x,t) = \frac{1}{k!}\frac{\partial^k \phi}{\partial p^k}\bigg|_{p=0}. \tag{4}$$

The convergence of the series (3) depends upon the auxiliary parameter $\hbar$. If it is convergent at $p=1$, one has

$$\phi(x,t;p) = u_0(x,t) + \sum_{k=1}^{\infty} u_k(x,t)$$

which must be one of the solutions of the original nonlinear equation, as proven by Liao (1992). Now we define the vector

$$\vec{u}_n(x,t) = \{u_0(x,t), u_1(x,t), u_2(x,t), \ldots\ldots, u_n(x,t)\}, \qquad (5)$$

So the m^{th}-order deformation equations are

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar R_m(\vec{u}_{m-1}(x,t)), \qquad (6)$$

with the initial conditions

$$u_i(x,0) = 0, \qquad (7)$$

where

$$R_m(\vec{u}_{m-1}(x,t)) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(x,t;p)]}{\partial p^{m-1}}\bigg|_{p=0},$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases}$$

Now, the solution of the m^{th}-order deformation equation (6) for $m \ge 1$ becomes

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar J_t^{\alpha_i}[R_m(\vec{u}_{m-1}(x,t))] + c,$$

where c is the integration constants determined by the initial condition (7). In this way, it is easy to obtain $u_m(x,t)$ for $m \ge 1$, at m^{th}-order, we have

$$u(x,t) = \lim_{N\to\infty} \Phi_N(x,t), \qquad (8)$$

where

$$\Phi_N(x,t) = \sum_{m=0}^{N-1} u_m(x,t) \ .$$

The important thing of HAM is the introduction of auxiliary parameter $\hbar$, which helps to construct the so called zero-order deformation equation, which gives more general homotopy than the traditional one [Liao (2003)]. The basic difference of this method with other analytical methods is that it gives a family of solution in terms of $\hbar$. There is an important role of the homotopy parameter $p\,(0 \le p \le 1)$ also. If $\hbar = -1$ and $p = 1$, then the solution is same as the solution obtained by another mathematical tool HPM. But the beauty of the HAM is that the region of convergence can be increased by controlling parameters $\hbar$ and p. It is also shown by many researchers during the solution of linear and nonlinear differential equations that the convergence region and the rate of series solutions can be controlled through the plotting of $\hbar$-curves and then choosing a proper value of $\hbar$ from convergence region to get the better approximation of the solutions.

The method is valid for the solution in the small region of time $0 \le t \le 1$. However, introducing more terms in the series solution (8), the convergence region of the time can be increased. Again with the proper choices of base functions the convergence region of t can also be increased. Using the homotopy-pade technique greatly accelerates the convergence of the series solution and also the convergence region of time. Again for the solution of those nonlinear problems HAM is good mathematical tool if we have knowledge about a given problems a prior. Thus for same physical nonlinear problems it is difficult to approximate the solution when there will be lack of knowledge about proper set of base functions. Even researchers are facing some problems when they are using the method the nonlinear problems with discontinue or chaotic solutions.

3. Applications

We will apply the HAM to solve the following systems of linear homogeneous/nonhomogeneous Fractional PDEs.

Example 1. Consider the following system of linear fractional PDEs with $0 < \alpha, \beta \le 1$

$$D_t^{\alpha} u - D_x^{\beta} \mathrm{v} + u + \mathrm{v} = 0 \, , \tag{9a}$$

$$D_t^{\alpha} \mathrm{v} - D_x^{\beta} u + u + \mathrm{v} = 0 \, , \tag{9b}$$

with initial conditions as

$$u(x,0) = \sinh\ x \, , \ \mathrm{v}(x,0) = \cosh\ x \ . \tag{10}$$

To solve system of equations (9a) – (10) by means of HAM, we choose the initial approximations

$$u_0(x,t) = \sinh x, \; v_0(x,t) = \cosh x \tag{11}$$

and the linear operator

$$L[\phi_i(x,t;p)] = \frac{\partial^\alpha \phi_i(x,t;p)}{\partial t^\alpha}, \quad i = 1,2, \tag{12}$$

with the property

$$L[c_i] = 0, \tag{13}$$

where $c_i (i = 1,2)$ are integral constants. Furthermore, for equations (9a) and equation (9b), we define a system of nonlinear operators as

$$N_1[\phi_i(x,t;p)] = \frac{\partial^\alpha \phi_1(x,t;p)}{\partial t^\alpha} - \frac{\partial^\beta \phi_2(x,t;p)}{\partial x^\beta} + \phi_1(x,t;p) + \phi_2(x,t;p)$$

$$N_2[\phi_i(x,t;p)] = \frac{\partial^\alpha \phi_2(x,t;p)}{\partial t^\alpha} - \frac{\partial^\beta \phi_1(x,t;p)}{\partial x^\beta} + \phi_1(x,t;p) + \phi_2(x,t;p).$$

Now, we construct the zeroth-order deformation equations

$$(1-p)L[\varphi_i(x,t;p) - Z_{i,0}(x,t)] = p\,\hbar_i\, N_i[\varphi_i(x,t;p)], \; i = 1,2. \tag{14}$$

Obviously, when $p = 0$ and $p = 1$,

$$\phi_1(x,t;0) = Z_{1,0}(x,t) = u_0(x,t) \text{ and } \phi_1(x,t;1) = Z_1(x,t) = u(x,t),$$

$$\phi_2(x,t;0) = Z_{2,0}(x,t) = v_0(x,t) \text{ and } \phi_2(x,t;1) = Z_2(x,t) = v(x,t).$$

Therefore, as the embedding parameter p increases from zero to unity, $\phi_i(x,t;p)$ varies from the initial guess $Z_{i,0}(x,t)$ to the solution $Z_i(x,t)$ for $i = 1,2$. Expanding $\phi_i(x,t;p)$ in Taylor series with respect to p, one can find

$$\phi_i(x,t;p) = Z_{i,0}(x,t) + \sum_{m=1}^{\infty} Z_{i,m}(x,t).\,p^m,$$

where

$$Z_{i,m}(x,t) = \frac{1}{m!}\frac{\partial^m \phi_i(x,t;p)}{\partial p^m}\bigg|_{p=0}.$$

If the auxiliary linear operator, the initial guesses and the auxiliary parameters $\hbar_i$ are properly chosen, the above series is convergent at $p=1$, then one has

$$u(x,t) = Z_{1,0}(x,t) + \sum_{m=1}^{\infty} Z_{1,m}(x,t),$$

$$\mathrm{v}(x,t) = Z_{2,0}(x,t) + \sum_{m=1}^{\infty} Z_{2,m}(x,t),$$

which must be one of the solutions of the original equations, as proved by Liao (1999),

Now, we define the vector

$$\vec{Z}_{i,n}(x,\ t) = \{Z_{i,0}(x,\ t), Z_{i,1}(x,\ t), \ldots., Z_{i,n}(x,\ t)\}.$$

Then, the m^{th} order deformation equations are

$$L[Z_{i,m}(x,\ t) - \chi_m Z_{i,m-1}(x,\ t)] = \hbar_i R_{i,m}(\vec{Z}_{i,m-1}(x,\ t)), \tag{15}$$

with the initial conditions

$$Z_{i,m}(x,\ 0) = 0, \tag{16}$$

here

$$R_{1,m}(\vec{Z}_{1,m-1}(x,\ \tau)) = D_t^{\alpha}(Z_{1,m-1}) - D_x^{\beta}(Z_{2,m-1}) + Z_{1,m-1} + Z_{2,m-1},$$

$$R_{2,m}(\vec{Z}_{2,m-1}(x,\ \tau)) = D_t^{\alpha}(Z_{2,m-1}) - D_x^{\beta}(Z_{1,m-1}) + Z_{1,m-1} + Z_{2,m-1},$$

Now, the solution of the m^{th} order deformation equations (15) for $m \geq 1$ becomes

$$Z_{i,m}(x,\ t) = \chi_m Z_{i,m-1}(x,\ t) + \hbar_i J_t^{\alpha}[R_{i,m}(\vec{Z}_{i,m-1}(x,\ t))\,dt] + C_i,\ i = 1, 2, \tag{17}$$

where the integration constants C_1 and C_2 are determined by the initial condition (16). We now successively obtain

$$Z_{1,0}(x,t) = \sinh x$$

$$Z_{1,1}(x,t) = \hbar_1\left(-f_1(x) + \sinh x + \cosh x\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)},$$

$$Z_{1,2}(x,t) = \hbar_1(1+\hbar_1)\left(-f_1(x)+\sinh x+\cosh x\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \{\hbar_1(\hbar_1+\hbar_2)\cosh x + \hbar_1(\hbar_1+\hbar_2)\sinh x$$
$$-\hbar_1(\hbar_1+\hbar_2)f_1(x) - 2\hbar_1\hbar_2 g_1(x) + \hbar_1\hbar_2 g_2(x)\}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$Z_{1,3}(x,t) = \hbar_1(1+\hbar_1)^2\left(-f_1(x)+\sinh x+\cosh x\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \Big[\left(\hbar_1^{\,2}(1+\hbar_1)+\hbar_1\hbar_2(1+\hbar_2)\right)\left(-f_1(x)+\sinh x+\cosh x\right)$$
$$+\hbar_1\hbar_2(1+\hbar_2)\left(-2g_1(x)+g_2(x)\right)+(1+\hbar_1)\{\hbar_1(\hbar_1+\hbar_2)\left(-f_1(x)+\sinh x+\cosh x\right)$$
$$-2\hbar_1\hbar_2\left(-2g_1(x)+g_2(x)\right)\}\Big]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \Big[\left(\hbar_1^{\,2}(\hbar_1+\hbar_2)+\hbar_1\hbar_2(\hbar_1+\hbar_2)\right)(\sinh x+\cosh x)$$
$$-\left(\hbar_1^{\,2}(\hbar_1+\hbar_2)+\hbar_1\hbar_2(\hbar_1+\hbar_2)+2\hbar_1\hbar_2\right)f_1(x) + 3\hbar_1^{\,2}\hbar_2 f_2(x) - \hbar_1^{\,2}\hbar_2 f_3(x)$$
$$+\hbar_1\hbar_2(2\hbar_1+\hbar_2)\left(-2g_1(x)+g_2(x)\right)\Big]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},$$

$$Z_{2,0}(x,t) = \cosh x$$

$$Z_{2,1}(x,t) = \hbar_2\left(-g_1(x)+\sinh x+\cosh x\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)},$$

$$Z_{2,2}(x,t) = \hbar_2(1+\hbar_2)\left(-g_1(x)+\sinh x+\cosh x\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \Big[\hbar_2(\hbar_1+\hbar_2)\left(-g_1(x)+\sinh x+\cosh x\right)$$
$$+\hbar_1\hbar_2\left(-2f_1(x)+f_2(x)\right)\Big]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$Z_{2,3}(x,t) = \hbar_2(1+\hbar_2)^2\left(-g_1(x)+\sinh x+\cosh x\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \Big[\hbar_2(\hbar_1(\hbar_1+1)+\hbar_2(\hbar_2+1))$$
$$\left(-g_1(x)+\sinh x+\cosh x\right)+\hbar_1\hbar_2(1+\hbar_1)\left(-2f_1(x)+f_2(x)\right)+\hbar_2(\hbar_1+\hbar_2)(1+\hbar_2)$$
$$\left(-g_1(x)+\sinh x+\cosh x\right)+\hbar_1\hbar_2(1+\hbar_2)\left(-2f_1(x)+f_2(x)\right)\Big]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$
$$+\Big[\hbar_2(\hbar_1+\hbar_2)^2(\sinh x+\cosh x)+\hbar_1\hbar_2(\hbar_1+2\hbar_2)\left(-2f_1(x)+f_2(x)\right)$$
$$-\hbar_1(\hbar_1^{\,2}+3\hbar_2^{\,2}+2\hbar_1\hbar_2)g_1(x)+3\hbar_1^{\,2}\hbar_2 g_2(x)-\hbar_1\hbar_2^{\,2}g_3(x)\Big]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},$$

where

$$f_1(x) = \frac{x^{-\beta}}{\Gamma(1-\beta)} + \frac{x^{2-\beta}}{\Gamma(3-\beta)} + \frac{x^{4-\beta}}{\Gamma(5-\beta)} + \frac{x^{6-\beta}}{\Gamma(7-\beta)},$$
$$f_2(x) = \frac{x^{-2\beta}}{\Gamma(1-2\beta)} + \frac{x^{2-2\beta}}{\Gamma(3-2\beta)} + \frac{x^{4-2\beta}}{\Gamma(5-2\beta)} + \frac{x^{6-2\beta}}{\Gamma(7-2\beta)},$$
$$f_3(x) = \frac{x^{-3\beta}}{\Gamma(1-3\beta)} + \frac{x^{2-3\beta}}{\Gamma(3-3\beta)} + \frac{x^{4-3\beta}}{\Gamma(5-3\beta)} + \frac{x^{6-3\beta}}{\Gamma(7-3\beta)},$$

$$g_1(x)=\frac{x^{1-\beta}}{\Gamma(2-\beta)}+\frac{x^{3-\beta}}{\Gamma(4-\beta)}+\frac{x^{5-\beta}}{\Gamma(6-\beta)}+\frac{x^{7-\beta}}{\Gamma(8-\beta)},$$
$$g_2(x)=\frac{x^{1-2\beta}}{\Gamma(2-2\beta)}+\frac{x^{3-2\beta}}{\Gamma(4-2\beta)}+\frac{x^{5-2\beta}}{\Gamma(6-2\beta)}+\frac{x^{7-2\beta}}{\Gamma(8-2\beta)},$$
$$g_3(x)=\frac{x^{1-3\beta}}{\Gamma(2-3\beta)}+\frac{x^{3-3\beta}}{\Gamma(4-3\beta)}+\frac{x^{5-3\beta}}{\Gamma(6-3\beta)}+\frac{x^{7-3\beta}}{\Gamma(8-3\beta)}.$$

Proceeding in this manner, the components u_n and v_n, $n \geq 0$, of the HAM can be completely obtained and the series solutions are thus entirely determined.

Finally, we approximate the analytical solutions $u(x,t)$ and $\mathrm{v}(x,t)$ by the truncated series

$$u(x,t) = \lim_{N\to\infty} \Phi_N(x,t) \text{ and } \mathrm{v}(x,t) = \lim_{N\to\infty} \Psi_N(x,t), \tag{18}$$

where

$$\Phi_N(x,t) = \sum_{n=0}^{N-1} u_n(x,t) \text{ and } \Psi_N(r,t) = \sum_{n=0}^{N-1} \mathrm{v}_n(x,t) .$$

The above series solutions generally converge very rapidly. A classical approach of convergence of this type of series is already presented by Abbaoui and Cherruault (1995).

Example 2. Consider the following non-homogeneous system of linear Fractional PDEs with $0 < \alpha, \beta \leq 1$ as

$$D_t^{\alpha} u - D_x^{\beta} \mathrm{v} - u + \mathrm{v} = -2, \tag{19a}$$

$$D_t^{\alpha} \mathrm{v} + D_x^{\beta} u - u + \mathrm{v} = -2, \tag{19b}$$

with initial conditions are

$$u(x,0) = 1 + e^{x}, \quad \mathrm{v}(x,0) = -1 + e^{x}. \tag{20}$$

To solve system of equations (19a) – (20) by means of HAM, we choose the initial approximations

$$u_0(x,t) = 1 + e^{x}, \ \mathrm{v}_0(x,t) = -1 + e^{x}, \tag{21}$$

and the linear operator as equation (12) with the property given in (13). The system of nonlinear operators to describe (19a) and (19b) are

$$N_1[\phi_i(x,t;p)] = \frac{\partial^\alpha \phi_1(x,t;p)}{\partial t^\alpha} - \frac{\partial^\beta \phi_2(x,t;p)}{\partial x^\beta} - \phi_1(x,t;p) + \phi_2(x,t;p) + 2$$

$$N_2[\phi_i(x,t;p)] = \frac{\partial^\alpha \phi_2(x,t;p)}{\partial t^\alpha} + \frac{\partial^\beta \phi_1(x,t;p)}{\partial x^\beta} - \phi_1(x,t;p) + \phi_2(x,t;p) + 2 \cdot$$

Proceeding as the previous example we finally obtain

$$Z_{1,0}(x,t) = 1 + e^x,$$

$$Z_{1,1}(x,t) = -\hbar_1 f_1(x)\frac{t^\alpha}{\Gamma(1+\alpha)},$$

$$Z_{1,2}(x,t) = -\hbar_1(1+\hbar_1)f_1(x)\frac{t^\alpha}{\Gamma(1+\alpha)} + \left[\hbar_1\hbar_2\left(2g_1(x) - g_2(x) - 2\frac{x^{-\beta}}{\Gamma(1-\beta)}\right) + \hbar_1(\hbar_1-\hbar_2)f_1(x)\right]\frac{t^{2\alpha}}{\Gamma(1+2\alpha)},$$

$$\begin{aligned}Z_{1,3}(x,t) = &-\hbar_1(1+\hbar_1)^2 f_1(x)\frac{t^\alpha}{\Gamma(1+\alpha)} + \left[(1+\hbar_1)\left\{\hbar_1\hbar_2\left(2g_1(x) - g_2(x) - 2\frac{x^{-\beta}}{\Gamma(1-\beta)}\right) + \hbar_1(\hbar_1-\hbar_2)f_1(x)\right\}\right.\\ &\left.+\left\{\hbar_1\hbar_2(1+\hbar_2)\left(2g_1(x)-g_2(x)-2\frac{x^{-\beta}}{\Gamma(1-\beta)}\right) + \left(\hbar_1^2(1+\hbar_1) - \hbar_1\hbar_2(1+\hbar_2)\right)(f_1(x)+2) - 2\hbar_1(\hbar_1-\hbar_2)\right\}\right]\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\\ &+\left[\hbar_1^2\hbar_2\left(f_3(x) - 3f_2(x) + 2f_1(x) + g_2(x) - 2g_1(x) + 4\frac{x^{-\beta}}{\Gamma(1-\beta)} - 2\frac{x^{-2\beta}}{\Gamma(1-2\beta)} - 4\right)\right.\\ &\left.-\hbar_1(\hbar_1-\hbar_2)^2(f_1(x)+2) + \hbar_1\hbar_2(\hbar_1-\hbar_2)(g_2(x)-2g_1(x)+2) + 2\hbar_1(\hbar_1^2+\hbar_2^2)\right]\frac{t^{3\alpha}}{\Gamma(1+3\alpha)},\end{aligned}$$

$$Z_{2,0}(x,t) = -1 + e^x,$$

$$Z_{2,1}(x,t) = \hbar_2 g_1(x)\frac{t^\alpha}{\Gamma(1+\alpha)},$$

$$Z_{2,2}(x,t) = \hbar_2 g_1(x)\frac{t^\alpha}{\Gamma(1+\alpha)} + \left[\hbar_1\hbar_2\left(2f_1(x) - f_2(x) + 2\frac{x^{-\beta}}{\Gamma(1-\beta)}\right) - \hbar_2(\hbar_1-\hbar_2)g_1(x)\right]\frac{t^{2\alpha}}{\Gamma(1+2\alpha)},$$

$$\begin{aligned}Z_{2,3}(x,t) = &\hbar_2(1+\hbar_2)^2 g_1(x)\frac{t^\alpha}{\Gamma(1+\alpha)} + \left[(1+\hbar_2)\left\{\hbar_1\hbar_2\left(2f_1(x) - f_2(x) + 2\frac{x^{-\beta}}{\Gamma(1-\beta)}\right) - \hbar_2(\hbar_1-\hbar_2)g_1(x)\right\}\right.\\ &\left.+\left\{\hbar_1\hbar_2(1+\hbar_1)\left(2f_1(x) - f_2(x) - g_1(x) + 2 + 2\frac{x^{-\beta}}{\Gamma(1-\beta)}\right) + \hbar_2^2(1+\hbar_2)(g_2(x)-2) - 2\hbar_2(\hbar_1-\hbar_2)\right\}\right]\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\\ &+\left[\hbar_2^2\hbar_1\left(-g_3(x) + 3g_2(x) + 2f_1(x) - f_2(x) - 2g_1(x) + 4\frac{x^{-\beta}}{\Gamma(1-\beta)} + 2\frac{x^{-2\beta}}{\Gamma(1-2\beta)} - 4\right)\right.\\ &\left.+\hbar_2(\hbar_1-\hbar_2)^2(g_1(x)-2) + \hbar_1\hbar_2(\hbar_1-\hbar_2)(f_2(x)-2f_1(x)+2) + 2\hbar_2(\hbar_1^2+\hbar_2^2)\right]\frac{t^{3\alpha}}{\Gamma(1+3\alpha)},\end{aligned}$$

where

$$f_1(x) = \frac{x^{1-\beta}}{\Gamma(2-\beta)} + \frac{x^{2-\beta}}{\Gamma(3-\beta)} + \frac{x^{3-\beta}}{\Gamma(4-\beta)} + \frac{x^{4-\beta}}{\Gamma(5-\beta)},$$

$$f_2(x) = \frac{x^{1-2\beta}}{\Gamma(2-2\beta)} + \frac{x^{2-2\beta}}{\Gamma(3-2\beta)} + \frac{x^{3-2\beta}}{\Gamma(4-2\beta)} + \frac{x^{4-2\beta}}{\Gamma(5-2\beta)},$$

$$f_3(x) = \frac{x^{1-3\beta}}{\Gamma(2-3\beta)} + \frac{x^{2-3\beta}}{\Gamma(3-3\beta)} + \frac{x^{3-3\beta}}{\Gamma(4-3\beta)} + \frac{x^{4-3\beta}}{\Gamma(5-3\beta)},$$

$$g_1(x)=2\frac{x^{-\beta}}{\Gamma(1-\beta)}+\frac{x^{1-\beta}}{\Gamma(2-\beta)}+\frac{x^{2-\beta}}{\Gamma(3-\beta)}+\frac{x^{3-\beta}}{\Gamma(4-\beta)}+\frac{x^{4-\beta}}{\Gamma(5-\beta)},$$
$$g_2(x)=2\frac{x^{-2\beta}}{\Gamma(1-2\beta)}+\frac{x^{1-2\beta}}{\Gamma(2-2\beta)}+\frac{x^{2-2\beta}}{\Gamma(3-2\beta)}+\frac{x^{3-2\beta}}{\Gamma(4-2\beta)}+\frac{x^{4-2\beta}}{\Gamma(5-2\beta)},$$
$$g_3(x)=2\frac{x^{-3\beta}}{\Gamma(1-3\beta)}+\frac{x^{1-3\beta}}{\Gamma(2-3\beta)}+\frac{x^{2-3\beta}}{\Gamma(3-3\beta)}+\frac{x^{3-3\beta}}{\Gamma(4-3\beta)}+\frac{x^{4-3\beta}}{\Gamma(5-3\beta)}.$$

Finally, the approximate analytical solutions for $u(x,t)$ and $\mathrm{v}(x,t)$ can be obtained using equations in (18).

4. Numerical Results and Discussion

In this section, numerical results of the probability density functions $u(x,t)$ and $\mathrm{v}(x,t)$ for different fractional Brownian motions $\alpha,\beta=\frac{1}{3},\frac{1}{2},\frac{2}{3}$ and also for standard motion $\alpha,\beta=1$ are calculated for various values of t for both the Examples keeping $x=1$, which are depicted through Figs. 1-4. During the calculation of the series solution only four order terms are considered.

The values of $u(x,t)$ and $\mathrm{v}(x,t)$ for different particular cases with the proper choices of $\hbar,\ p$ and for various values of α and β are shown through Tables 1 – 4 for Examples 1 and 2 to describe the convergence of the solutions.

It is seen from Figure 1 that $u(x,t)$ decreases with the increase in t and with the decrease in the fractional values of α for all space fractional derivatives β. Where as for standard motion, i.e., for space derivative $\beta=1$, initially it decreases with the increase in time and also with α but afterwards it becomes opposite in nature.

It is also seen from Fig. 2 that $\mathrm{v}(x,t)$ has the same nature as $u(x,t)$. For standard space derivative $\beta=1$, it takes lesser time for changing the behavior with α.

Figures 3 and 4 which graphically describe Example 2, reveal the opposite nature of the solutions of $u(x,t)$ and $\mathrm{v}(x,t)$. Fig. 3 shows that $u(x,t)$ increases with the increase in t and decreases with increase in α but Figure 4 depicts that $\mathrm{v}(x,t)$ decreases with t and increases with the increase in α.

5. Conclusion

This paper has focused on the successful employment of the powerful mathematical tool (HAM) to investigate the solution of a system of linear homogeneous/non-homogeneous equations with fractional time- and space- derivatives. The method provides us a simple way to adjust and control the convergence of the series solution by choosing proper values of auxiliary and homotopy parameters. Thus it may be concluded that HAM in spite of its limitations is simple and represents a very powerful analytical approach for handling fractional related homogeneous/non-homogeneous system of PDE.

The different nature in the behavior of the probability density functions for different fractional Brownian motions, faster computation procedure of the present method and the convergence criterion of the series solution with the proper choices of auxiliary and homotopy parameters render the article a different dimension and the authors strongly believe that the present article will be highly acceptable by the researchers working in the field of fractional calculus.

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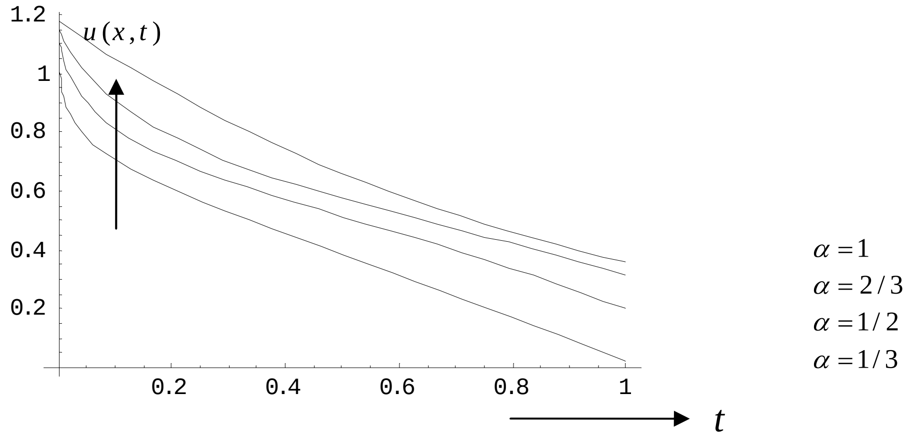
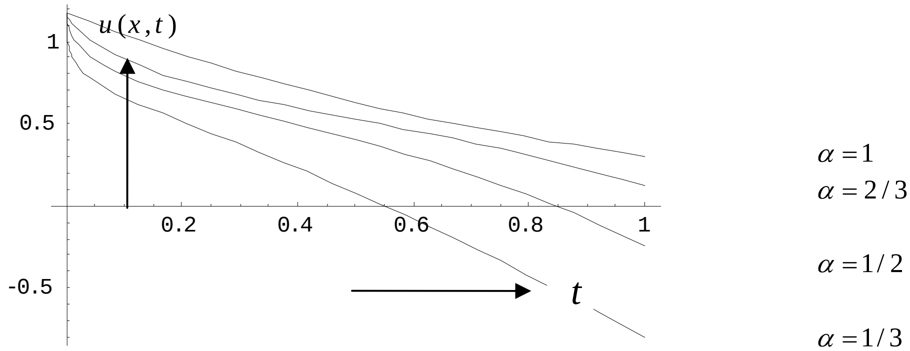


1. (a)



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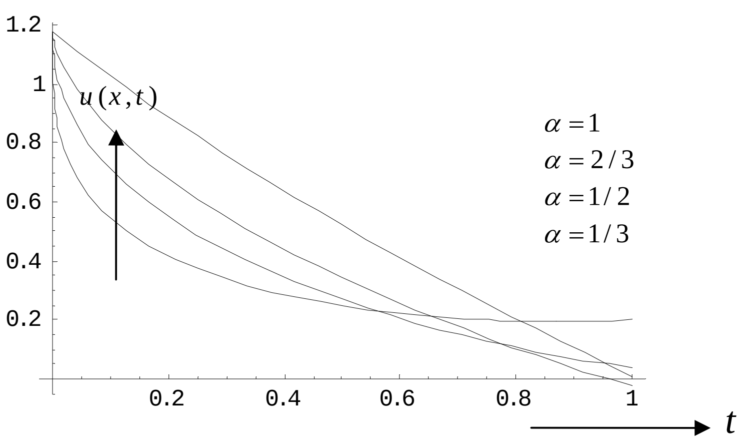
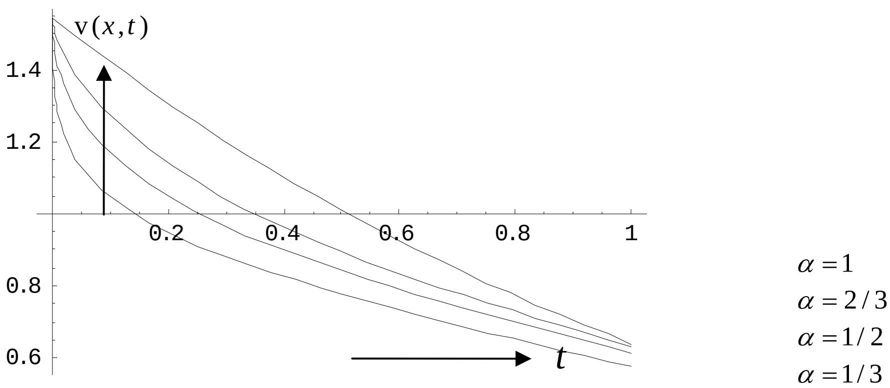
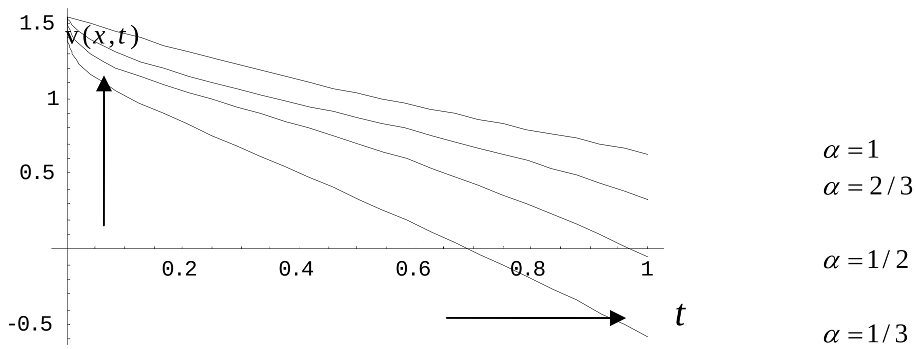


1. (c)

Figure 1. Plots of $u(x,t)$ w.r.t. t at $x = 1$ for Example 1 for (a) $\beta = \frac{1}{2}$, (b) $\beta = \frac{2}{3}$, (c) $\beta = 1$



2. (a)



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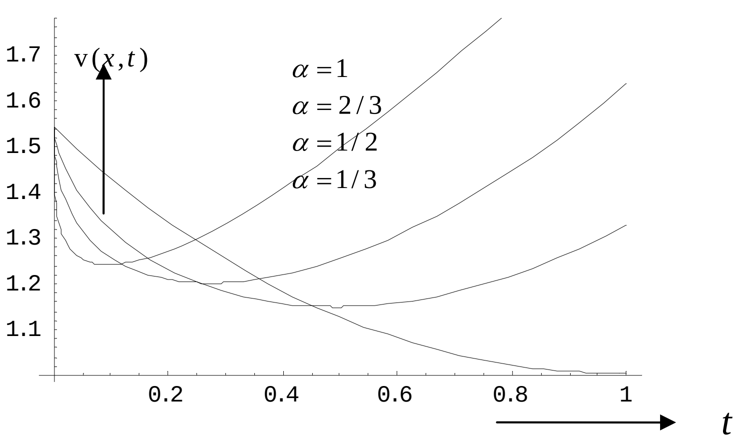
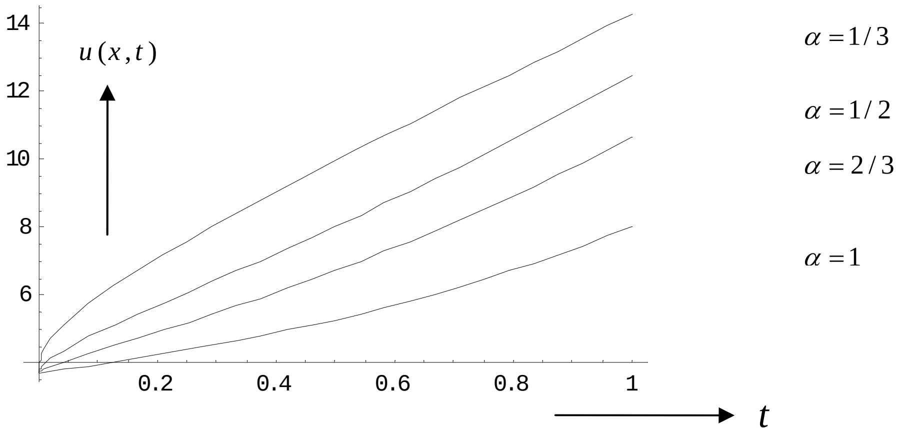
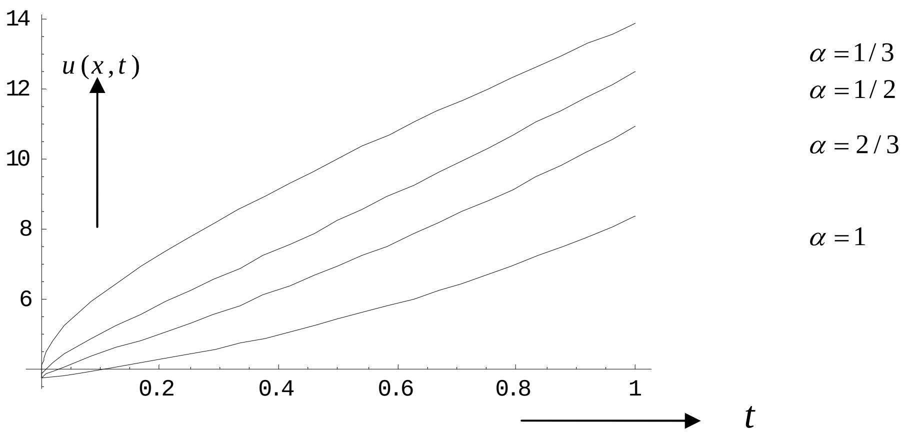


2. (c)

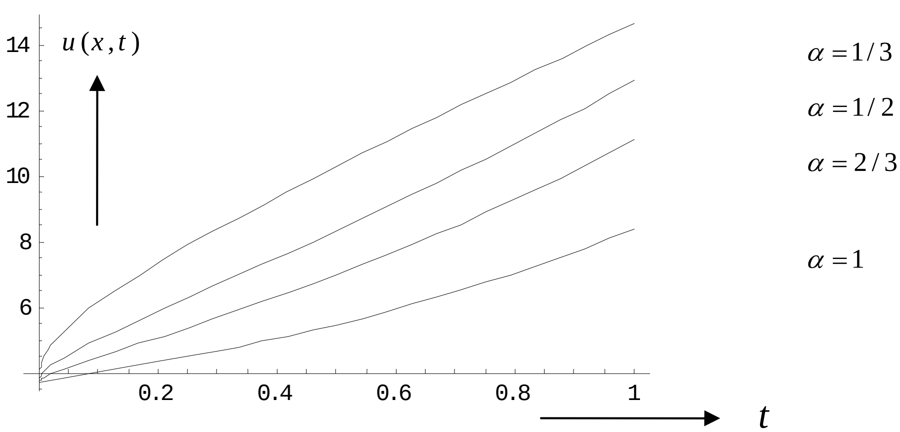
Figure 2. Plots of $\mathrm{v}(x,t)$ w.r.t. t at $x=1$ for Example 1 for (a) $\beta=\frac{1}{2}$, (b) $\beta=\frac{2}{3}$, (c) $\beta=1$



3. (a)



3. (b)



3. (c)

Figure 3. Plots of $u(x,t)$ w.r.t. t at $x=1$ for Example 2 for (a) $\beta=\frac{1}{2}$, (b) $\beta=\frac{2}{3}$, (c) $\beta=1$

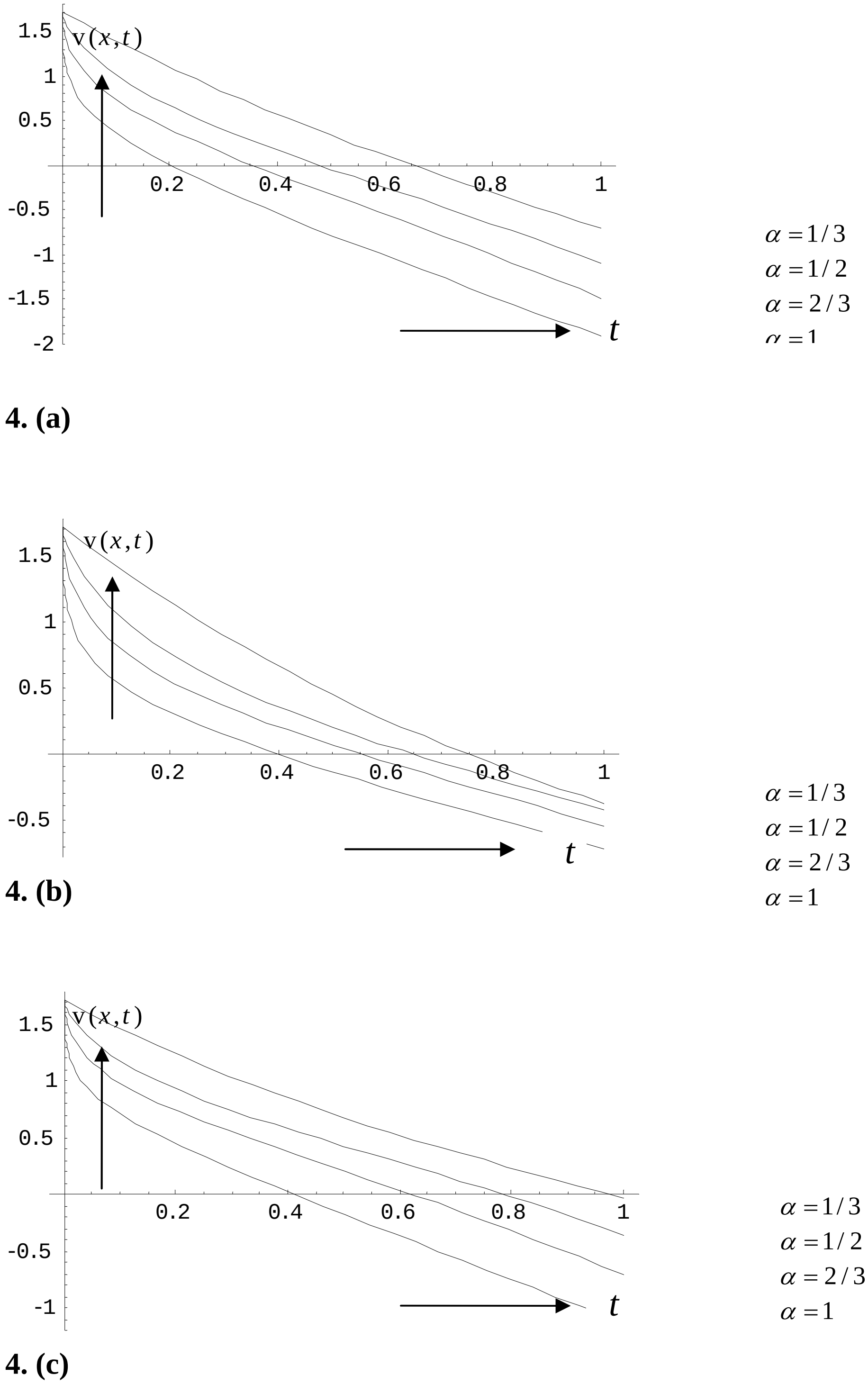


Figure 4. Plots of $v(x,t)$ w.r.t. t at $x=1$ for Example 2 for (a) $\beta=\frac{1}{2}$, (b) $\beta=\frac{2}{3}$, (c) $\beta=1$

Table 1. Comparison of HPM and HAM results of $u(x,t)$ for different values of α and β at $t=1$ and $x=1$ for Example 1:

β	α	u_{HPM}	u_{HAM}		
			$\hbar=-0.98$, $p=1.02083$	$\hbar=-0.96$, $p=1.04246$	$\hbar=-0.94$, $p=1.06483$
$\frac{1}{2}$	$1/3$	0.02096	0.02096	0.02096	0.02096
	$1/2$	0.19970	0.20073	0.20160	0.20224
	$2/3$	0.31539	0.31552	0.31526	0.31447
	1	0.35796	0.35219	0.34553	0.33777
$\frac{2}{3}$	$1/3$	- 0.80296	- 0.77114	- 0.73768	- 0.70225
	$1/2$	- 0.24257	- 0.21791	- 0.19228	- 0.16558
	$2/3$	0.11756	0.13284	0.14838	0.16410
	1	0.30074	0.29885	0.29616	0.29246
1	$1/3$	0.20097	0.15536	0.10717	0.05607
	$1/2$	0.04001	0.00430	- 0.03359	- 0.07393
	$2/3$	- 0.02439	- 0.05136	- 0.08021	- 0.11118
	1	0.01123	- 0.00764	- 0.02822	- 0.05069

Table 2. Comparison of HPM and HAM results of $v(x,t)$ for different values of α and β at $t=1$ and $x=1$ for Example 1

β	α	v_{HPM}	v_{HAM}		
			$\hbar=-0.98$, $p=1.0039$	$\hbar=-0.96$, $p=1.00807$	$\hbar=-0.94$, $p=1.06483$
$\frac{1}{2}$	$1/3$	0.57687	0.57687	0.57687	0.57687
	$1/2$	0.61165	0.61077	0.60964	0.60824
	$2/3$	0.63300	0.63142	0.62936	0.62681
	1	0.64005	0.63760	0.63448	0.63073
$\frac{2}{3}$	$1/3$	-0.58322	-0.50700	- 0.43481	- 0.36576
	$1/2$	-0.05610	-0.01153	0.03040	0.07022
	$2/3$	0.32820	0.34938	0.36891	0.38705
	1	0.63299	0.63451	0.63519	0.63510
1	$1/3$	2.04215	1.95702	1.87813	1.80442
	$1/2$	1.63863	1.58778	1.54059	1.49634
	$2/3$	1.32973	1.30420	1.28025	1.25748
	1	1.00689	1.00185	0.99651	0.99077

Table 3. Comparison of HPM and HAM results of $u(x,t)$ for different values of α and β at $t=1$ and $x=1$ for Example 2

β	α	u_{HPM}	u_{HAM} $\hbar=-0.98$, $p=1.0028$	$\hbar=-0.96$, $p=1.00586$	$\hbar=-0.94$, $p=1.00919$
$\frac{1}{2}$	1/3	14.2702	14.2701	14.2701	14.2701
	1/2	12.4736	12.5154	12.5561	12.5957
	2/3	10.6656	10.7284	10.7899	10.8501
	1	8.0198	8.0753	8.1304	8.1851
$\frac{2}{3}$	1/3	13.8877	13.9920	14.0896	14.1808
	1/2	12.5026	12.6100	12.7121	12.8091
	2/3	10.9367	11.0356	11.1309	11.2229
	1	8.3529	8.4180	8.4822	8.5457
1	1/3	14.6629	14.7023	14.7389	14.7727
	1/2	12.9007	12.9720	13.0402	13.1054
	2/3	11.0919	11.1754	11.2564	11.3348
	1	8.3918	8.4573	8.5219	8.5859

Table 4. Comparison of HPM and HAM results of $v(x,t)$ for different values of α and β at $t=1$ and $x=1$ for Example 2

β	α	v_{HPM}	v_{HAM} $\hbar=-0.98$, $p=1.029351$	$\hbar=-0.96$, $p=1.060337$	$\hbar=-0.94$, $p=1.093067$
$\frac{1}{2}$	1/3	-1.91682	-1.91682	-1.91682	-1.91682
	1/2	-1.48779	-1.48716	-1.48693	-1.48720
	2/3	-1.10245	-1.10718	-1.11318	-1.12070
	1	- 0.70454	- 0.73105	-0.76110	-0.79524
$\frac{2}{3}$	1/3	- 0.72396	- 0.69713	-0.66853	-0.63806
	1/2	- 0.55290	- 0.53172	-0.50955	-0.48640
	2/3	- 0.42890	- 0.41922	-0.40977	-0.40075
	1	- 0.37430	- 0.39385	-0.41634	-0.44228
1	1/3	-1.12940	-1.11168	-1.09216	-1.07060
	1/2	- 0.71415	- 0.69556	-0.67552	-0.65391
	2/3	- 0.35599	- 0.34267	-0.32876	-0.31426
	1	- 0.03171	- 0.04057	-0.05114	-0.06374