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Solving Fuzzy Linear Programming Problems with Piecewise Linear Membership Function

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Abstract

In this paper, we concentrate on linear programming problems in which both the right-hand side and the technological coefficients are fuzzy numbers. We consider here only the case of fuzzy numbers with linear membership functions. The symmetric method of Bellman and Zadeh (1970) is used for a defuzzification of these problems. The crisp problems obtained after the defuzzification are non-linear and even non-convex in general. We propose here the "modified subgradient method" and "method of feasible directions" and uses for solving these problems see Bazaraa (1993). We also compare the new proposed methods with well known "fuzzy decisive set method". Finally, we give illustrative examples and their numerical solutions.

Keywords: Fuzzy linear programming; fuzzy number; augmented Lagrangian penalty function method; feasible directions of Topkis and Veinott; fuzzy decisive set method

MSC (2000) No.: 90C05, 90C70

1. Introduction

In fuzzy decision making problems, the concept of maximizing decision was proposed by Bellman and Zadeh (1970). This concept was adapted to problems of mathematical programming by Tanaka et al. (1984). Zimmermann (1983) presented a fuzzy approach to multi-objective linear programming problems. He also studied the duality relations in fuzzy linear programming. Fuzzy linear programming problem with fuzzy coefficients was formulated by Negoita (1970) and called robust programming. Dubois and Prade (1982) investigated linear fuzzy constraints. Tanaka and Asai (1984) also proposed a formulation of fuzzy linear programming with fuzzy constraints and give a method for its solution which bases on inequality relations between fuzzy numbers. Shaocheng (1994) considered the fuzzy linear programming problem with fuzzy constraints and defuzzificated it by first determining an upper bound for the objective function. Further he solved the obtained crisp problem by the fuzzy decisive set method introduced by Sakawa and Yana (1985). Guu and Yan-K (1999) proposed a two-phase approach for solving the fuzzy linear programming problems. Also applications of fuzzy linear programming include life cycle assessment [Raymond (2005)], production planning in the textile industry [Elamvazuthi et al. (2009)], and in energy planning [Canz (1999)].

We consider linear programming problems in which both technological coefficients and right-hand-side numbers are fuzzy numbers. Each problem is first converted into an equivalent crisp problem. This is a problem of finding a point which satisfies the constraints and the goal with the maximum degree. The idea of this approach is due to Bellman and Zadeh (1970). The crisp problems, obtained by such a manner, can be non-linear (even non-convex), where the non-linearity arises in constraints. For solving these problems we use and compare two methods. One of them called the augmented lagrangian penalty method. The second method that we use is the method of feasible directions of Topkis and Veinott (1993).

The paper is outlined as follows. In section 2, we study the linear programming problem in which both technological coefficients and right-hand-side are fuzzy numbers. The general principles of the augmented Lagrangian penalty method and method of feasible directions of Topkis and Veinott are presented in section 3 and 4, respectively. The fuzzy decisive set method is studied in section 5. In section 6, we examine the application of these two methods and then compare with the fuzzy decisive set method by concrete examples.

2. Linear Programming Problems with Fuzzy Technological Coefficients and Fuzzy Right Hand-side Numbers

We consider a linear programming problem with fuzzy technological coefficients and fuzzy right-hand-side numbers:

$$\text{Maximize} \quad \sum_{j=1}^{n} c_j x_j$$

$$\text{Subject to} \quad \sum_{j=1}^{n} \tilde{a}_{ij} x_j \leq \tilde{b}_i, \quad 1 \leq i \leq m \tag{1}$$

$$x_j \geq 0, \quad 1 \leq j \leq n,$$

where at least one $x_j > 0$ and $\tilde{a}_{ij}$ and $\tilde{b}_i$ are fuzzy numbers with the following linear membership functions:

$$\mu_{\tilde{a}_{ij}}(x) = \begin{cases} 1, & x < a_{ij}, \\ \dfrac{a_{ij} + d_{ij} - x}{d_{ij}}, & a_{ij} \le x < a_{ij} + d_{ij}, \\ 0, & x \ge a_{ij} + d_{ij}, \end{cases}$$

where $x \in R$ and $d_{ij} > 0$ for all $i = 1, \ldots, m,\ \ j = 1, \ldots, n,$ and

$$\mu_{\tilde{b}_i}(x) = \begin{cases} 1, & x < b_i, \\ \dfrac{b_i + p_i - x}{p_i}, & b_i \le x < b_i + p_i, \\ 0, & x \ge b_i + p_i, \end{cases}$$

where $p_i > 0,$ for $i = 1, \ldots, m.$ For defuzzification of the problem (1), we first calculate the lower and upper bounds of the optimal values. The optimal values z_l and z_u can be defined by solving the following standard linear programming problems, for which we assume that all they the finite optimal value

$$\begin{aligned} z_l = \text{Maximize} \quad & \textstyle\sum_{j=1}^{n} c_j x_j \\ \text{Subject to} \quad & \textstyle\sum_{j=1}^{n} (a_{ij} + d_{ij}) x_j \le b_i, \quad 1 \le i \le m \\ & x_j \ge 0, \quad 1 \le j \le n \end{aligned} \tag{2}$$

and

$$\begin{aligned} z_u = \text{Maximize} \quad & \textstyle\sum_{j=1}^{n} c_j x_j \\ \text{Subject to} \quad & \textstyle\sum_{j=1}^{n} a_{ij} x_j \le b_i + p_i, \quad 1 \le i \le m \\ & x_j \ge 0, \qquad 1 \le j \le n. \end{aligned} \tag{3}$$

The objective function takes values between z_l and z_u while technological coefficients take values between a_{ij} and $a_{ij} + d_{ij}$ and the right-hand side numbers take values between b_i and $b_i + p_i$.

Then, the fuzzy set of optimal values, G, which is a subset of $\Re^n$, is defined by

$$\mu_G(x) = \begin{cases} 0, & \sum_{j=1}^{n} c_j x_j < z_l, \\ \dfrac{\sum_{j=1}^{n} c_j x_j - z_l}{z_u - z_l}, & z_l \le \sum_{j=1}^{n} c_j x_j < z_u, \\ 1, & \sum_{j=1}^{n} c_j x_j \ge z_u. \end{cases} \tag{4}$$

The fuzzy set of the i constraint, c_i, which is a subset of $\Re^n$ is defined by

$$\mu_{c_i}(x) = \begin{cases} 0, & b_i < \sum_{j=1}^{n} a_{ij} x_j, \\ \dfrac{b_i - \sum_{j=1}^{n} a_{ij} x_j}{\sum_{j=1}^{n} d_{ij} x_j + p_i}, & \sum_{j=1}^{n} a_{ij} x_j \le b_i < \sum_{j=1}^{n} (a_{ij} + d_{ij}) x_j + p_i, \\ 1, & b_i \ge \sum_{j=1}^{n} \left(a_{ij} + d_{ij}\right) x_j + p_i. \end{cases} \tag{5}$$

By using the definition of the fuzzy decision proposed by Bellman and Zadeh (1970) [see also Lai and Hwang (1984)], we have

$$\mu_D(x) = \min(\mu_G(x), \min_i(\mu_{C_i}(x))). \tag{6}$$

In this case, the optimal fuzzy decision is a solution of the problem

$$\max_{x \ge 0}(\mu_D(x)) = \max_{x \ge 0} \min(\mu_G(x), \min_i(\mu_{C_i}(x))). \tag{7}$$

Consequently, the problem (1) transform to the following optimization problem

$$\begin{aligned} &\text{Maximize} \quad \lambda \\ &\text{Subject to} \quad \mu_G(x) \ge \lambda \\ &\qquad \mu_{C_i}(x) \ge \lambda, \ 1 \le i \le m \\ &\qquad x \ge 0 \\ &\qquad 0 \le \lambda \le 1. \end{aligned} \tag{8}$$

By using (4) and (5), the problem (8) can be written as:

$$\begin{aligned} &\text{Maximize} \quad \lambda \\ &\text{Subject to} \quad \lambda(z_l - z_u) - \sum_{j=1}^{n} c_j x_j + z_l \le 0 \\ &\qquad \sum_{j=1}^{n} (a_{ij} + \lambda d_{ij}) x_j + \lambda p_i - b_i \le 0, \ 1 \le i \le m \\ &\qquad x \ge 0, \ \ 0 \le \lambda \le 1. \end{aligned} \tag{9}$$

Notice that, the constraints in problem (9) containing the cross product terms λx_j are not convex. Therefore the solution of this problem requires the special approach adopted for solving general non convex optimization problems.

3. The Augmented Lagrangian Penalty Function Method

The approach used is to convert the problem into an equivalent unconstrained problem. This method is called the penalty or the exterior penalty function method, in which a penalty term is added to the objective function for any violation of the constraints. This method generates a sequence of infeasible points, hence its name, whose limit is an optimal solution to the original problem. The constraints are placed into the objective function via a penalty parameter in a way that penalizes any violation of the constraints.

In this section, we present and prove an important result that justifies using exterior penalty functions as a means for solving constrained problems.

Consider the following primal and penalty problems:

Primal problem:

$$\begin{aligned} &\text{Minimize} \quad -\lambda \\ &\text{Subject to} \quad \lambda(z_l - z_u) - \sum_{j=1}^{n} c_j x_j + z_l \le 0 \\ &\qquad \sum_{j=1}^{n} (a_{ij} + \lambda d_{ij}) x_j + \lambda p_i \le 0, \quad i = 1, \ldots, m \\ &\qquad -x_j \le 0, \quad j = 1, \ldots, n \\ &\qquad -\lambda \le 0, \\ &\qquad \lambda - 1 \le 0, \end{aligned} \tag{10}$$

Penalty problem:

Let ρ be a continuous function of the form

$$\rho(x_1,\ldots, x_n, \alpha) = \sum_{i=1}^{m}\phi\left(\sum_{j=1}^{n}(a_{ij}+\alpha d_{ij})x_j + \lambda p_i - b_i\right) + \sum_{j=1}^{n}\phi(-x_j) + \phi(-\lambda)+\phi(\lambda-1)+\phi\left(\lambda(z_l - z_u) - \sum_{j=1}^{n} c_j x_j + z_l\right), \quad (11)$$

where ϕ is continuous function satisfying the following:

$$\phi(y) = 0, \quad \text{if} \quad y \le 0 \quad \text{and} \quad \phi(y) > 0, \quad \text{if} \quad y > 0. \quad (12)$$

The basic penalty function approach attempts to solve the following problem:

Minimize $\theta(\mu)$

Subject to $\mu \ge 0$,

where $\theta(\mu) = \inf\{-\lambda + \mu\rho(x,\lambda) : x \in R^n, \lambda \in R\}$.

From this result, it is clear that we can get arbitrarily close to the optimal objective value of the primal problem by computing $\theta(\mu)$ for a sufficiently large μ. This result is established in Theorem 3.1.

Theorem 3.1. Consider the problem (10). Suppose that for each μ, there exists a solution $(x,\lambda)_\mu \in R^{n+1}$ to the problem to minimize $-\lambda + \mu\rho(x,\lambda)$ subject to $x \in R^n$ and $\lambda \in R$, and that $\{(x,\lambda)_\mu\}$ is contained in a compact subset of R^{n+1}. Then,

$$\lim_{\mu\to\infty}\theta(\mu) = \sup\{-\lambda : x \in R^n, \lambda \in R, g(x,\lambda) \le 0\},$$

where $g = (g_0, g_1,\ldots, g_m, g_{m+1},\ldots, g_{m+n}, g_{m+n+1}, g_{m+n+2})$ and

$$\begin{aligned} & g_0(x,\lambda) = \lambda(z_l = z_u) - \sum_{j=1}^{n} c_j x_j + z_l \\ & g_i(x,\lambda) = \sum_{j=1}^{n}(a_{ij} + \lambda d_{ij})x_j + \lambda p -_i, i = 1,\ldots m \\ & g_{m+j}(x,\alpha) = -x_j, \; j = 1,\ldots, n \\ & g_{n+m+1}(x,\alpha) = -\alpha \\ & g_{n+m+2}(x,\alpha) = \alpha - 1 \end{aligned} \quad (13)$$

and

$$\theta(\mu) = \inf\{-\lambda + \mu\rho(x,\lambda) : x \in R^n, \lambda \in R\} = -\lambda + \mu\rho[(x,\lambda)_\mu].$$

Furthermore, the limit $(\bar{x},\bar{\lambda})$ of any convergent subsequence of $\{(x,\lambda)_\mu\}$ is an optimal solution to the original problem, and $\mu\rho[(x,\lambda)_\mu] \to 0$ as $\mu \to \infty$.

Proof:

For proof, see Bazaraa (1993).

3.1. Augmented Lagrangian Penalty Functions

An augmented lagrangian penalty function for the problem (10) is as:

$$F_{AL}(x,\lambda,u) = -\lambda + \sum_{i=0}^{m+n+2} \mu_i \max\left\{ g_i(x,\lambda) + \frac{u_i}{2\mu_i}, 0 \right\}^2 - \sum_{i=0}^{m+n+2} \frac{u_i^2}{2\mu_i} , \tag{14}$$

where u_i are lagrange multiplier. The following result provides the basis by virtue of which the AL penalty function can be classified as an exact penalty function.

Theorem 3.1.1. Consider problem P to (10), and let the KKT solution $(\bar{x},\bar{\lambda},\bar{u})$ satisfy the second-order sufficiency conditions for a local minimum. Then, there exists a $\bar{\mu}$ such that for $\mu_i \geq \bar{\mu}$, the AL penalty function $F_{AL}(..,\bar{u})$, defined with $u = \bar{u}$, also achieves a strict local minimum at $(\bar{x},\bar{\lambda})$.

Proof:

For proof, see Bazaraa (1993).

Algorithm

The method of multipliers is an approach for solving nonlinear programming problems by using the augmented lagrangian penalty function in a manner that combines the algorithmic aspects of both Lagrangian duality methods and penalty function methods.

Initialization Step: Select some initial Lagrangian multipliers $u = \bar{u}$ and positive Values μ_i for $i = 0,\ldots,m+n+2,$ for the penalty parameters. Let (x^0, λ_0) be a null vector, and denote $VIOL(x^0, \lambda_0) = \infty$, where for any $x \in R^n$ and $\lambda \in R,$

$$\text{VIOL } (x,\lambda) = \max\{g_i(x,\lambda), i \in I = \{i : g_i(x,\lambda) > 0\}\}$$

is a measure of constraint violations. Put k = 1 and proceed to the"inner loop" of the algorithm.

Inner Loop: (Penalty Function Minimization)

Solve the unconstrained problem to

$$\text{Minimize} \quad F_{AL}(x, \lambda, \bar{u}),$$

and let (x^k, λ_k) denote the optimal solution obtained. If

$$\text{VIOL}(x^k, \lambda_k) = 0,$$

then stop with (x^k, λ_k) as a KKT point, (Practically, one would terminate if $\text{VIOL}(x^k, \lambda_k)$ is lesser than some tolerance $\varepsilon > 0.$) Otherwise, if

$$\text{VIOL}(x^k, \lambda_k) \le \frac{1}{4} VIOL\ (x^{k-1}, \lambda_{k-1}),$$

proceed to the outer loop. On the other hand, for each constraint $i = 0,\ldots,m$ for which $g_i(x^k, \lambda_k) > \frac{1}{4} VIOL(x^{k-1}, \lambda_{k-1}),$ replace the corresponding penalty parameter μ_i by $10\mu_i$, repeat this inner loop step.

Outer Loop: (Lagrange Multiplier Update)

Replace $\bar{u}$ by $\bar{u}_{new},$ where

$$(\bar{u}_{new})_i = \bar{u}_i + \max\{2\mu g_i(x^k, \lambda_k), -\bar{u}\}, i = 0,\ldots,m+n+2.$$

Increment *k* by 1, and return to the inner loop.

4. The Modification of Topkis and Veinott Revised Feasible Directions Method

The first, we describe the method of revised feasible directions of Topkis and Veinott. So we propose a modification from this method. At each iteration, the method generates an improving feasible direction and then optimizes along that direction. We now consider the following problem, where the feasible region is defined by a system of inequality constraints that are not necessarily linear:

$$\begin{aligned} &\text{Minimize} \quad -\lambda \\ &\text{Subject to} \quad \lambda(z_l - z_u) - \sum_{j=1}^{n} c_j x_j + z_l \le 0 \\ &\qquad \sum_{j=1}^{n} (a_{ij} + \lambda d_{ij}) x_j + \lambda p_i + b_i \le 0, \quad i = 1,\dots,m \\ &\qquad -x_j \le 0, \quad j = 1,\dots,n \\ &\qquad -\lambda \le 0, \\ &\qquad \lambda - 1 \le 0. \end{aligned} \tag{15}$$

Theorem below gives a sufficient condition for a vector d to be an improving feasible direction.

Theorem 4.1. Consider the problem in (15). Let $(\hat{x}, \hat{\lambda})$ be a feasible solution, and let I be the set of binding constraints, that is $I = \{i : g_i(\hat{x}, \hat{\lambda}) = 0\}$, where g_i's are as (13). If $\nabla(-\lambda)(\hat{x}, \hat{\lambda})^t d < 0 (i.e - d_{n+1} < 0)$ and $\nabla g_i(\hat{x}, \hat{\lambda})^t d < 0$ for $i \in I$, then d is an improving feasible direction.

Proof:

For proof see Bazaraa (1993).

Theorem 4.2. Let $(\hat{x}, \hat{\lambda}) \in R^{n+1}$ be a feasible solution of (15). Let $(\bar{z}, \bar{d})$ be an optimal solution to the following direction finding problem:

$$\begin{aligned} &\text{Minimize} \quad z \\ &\text{Subject to} \quad \nabla(-\lambda)(\hat{x}, \hat{\lambda})^t d \le 0 \\ &\qquad \nabla g_i(\hat{x}, \hat{\lambda})^t d - z \le g_i(\hat{x}, \hat{\lambda}), i = 0,\dots,m+n+2 \\ &\qquad -1 \le d_j \le 1, \; j = 0,\dots,n+1, \end{aligned} \tag{16}$$

if $\bar{z} < 0$, then $\bar{d}$ is an improving feasible direction . Also, $(\hat{x}, \hat{\lambda})$ is a Fritz John point, if and only if $\bar{z} = 0$.

After simplify, we can rewrite the problem (16) as follows:

$$
\begin{aligned}
\text{Minimize} \quad & z \\
\text{Subject to} \quad & -d_{n+1} - z \le 0 \\
& -\sum_{j=1}^{n} c_j d_j + (z_u - z_l) d_{n+1} - z \le -g_0(x, \lambda) \\
& \sum_{j=1}^{n} (a_{ij} + \lambda d_{ij}) d_j + (p_i + \sum_{j=1}^{n} d_{ij} x_j) d_{n+1} - z \le -g_i(x, \lambda), i = 1, \ldots, m \\
& -d_j - z \le x_j,\ j = 1, \ldots, n \\
& d_{n+1} + z \le 1 - \lambda \\
& -1 \le d_j \le 1,\ j = 1, \ldots, n+1.
\end{aligned}
\qquad (17)
$$

This revised method was proposed by Topkis and Veinott (1967) and guarantees convergence to a Fritz John point.

Generating a Feasible Direction

The problem under consideration is

$$
\begin{aligned}
\text{Minimize} \quad & -\lambda \\
\text{Subject to} \quad & g_i(x, \lambda) \le 0, i = 0, \ldots, m + n + 2,
\end{aligned}
$$

where g_i's are as (13). Given a feasible point $(\hat{x}, \hat{\lambda})$, a direction is found by solving the direction-finding linear programming problem DF $(\hat{x}, \hat{\lambda})$ to (17). Here, both binding and non binding constraints play a role in determining the direction of movement.

4. 1. Algorithm of Topkis and Veinott Revised Feasible Directions Method

A summary of the method of feasible directions of Topkis and Veinott for solving the problem (15), is given below. As will be shown later, the method converges to a Fritz John point.

Initialization Step: Choose a point (x^0, λ_0) such that $g_i(x^0, \lambda_0) \le 0$ for $i = 0, \ldots, m + n + 2$, where g_i are as (13). Let k = 1 and go to the main step.

Main Step:

1. Let (z_k, d^k) be an optimal solution to linear programming problem (17).
   If $z_k = 0$, step; (x^k, λ_k) is a Fritz John point. Otherwise, $z_k < 0$ and we go to 2.

2. Let l_k be an optimal solution to the following line search problem:

   Minimize $\quad -\lambda_k - l d_{n+1}$

   Subject to $\quad 0 \le l \le l_{\max}$,

where

$$l_{\max} = \sup\{l : g_i(y^k + l d^k) \le 0, i = 0,\dots, m+n+2\},$$

and $y^k = (x^k, \lambda_k)$ and g_i, for all $i = 0,\dots, m+n+2$, are as (13).

Let $y^{k+1} = y^k + l_k d^k$. Replace k by k +1, and return to step 1.

Theorem 4.1.1. Consider the problem in (15). Suppose that the sequence $\{(x^k, \lambda_k)\}$ is generated by the algorithm of Topkis and Veinott. Then, any accumulation point of $\{(x^k, \lambda_k)\}$ is a Fritz John point.

4.2. The Modification of Algorithm

In above algorithm, we need to obtain the gradient of the objective function and also the gradient of the constraint functions.

In this modification we do not need a feasible point. Note that we can forgo from the line search problem of step 2 in the main step, since, obviously, optimal solution for this line search problem is $l_{\max}$. Hence, in step 2 of the main step, we have $l_k = l_{\max}$.

Initialization Step (The method of find a the initial feasible point)

1. Set λ_1 = 1 and k = 1 and go to 2.

2. Test whether a feasible set satisfying the constraints of the problem (15) exists or not, using phase one of the simplex method, i.e., solving the problem below:

Minimize $1x_a$

Subject to $g_i(x,\lambda_k)+s+x_a=0, i=0,...,m+n+2$,

where s is the vector of slack variables and x is the vector of artificial variables. Let (x^k, s^k, x_a^k) be an optimal solution of this the problem. If $x_a^k=0,$ than (x^k,λ_k) is an initial feasible point for the problem (15) and go to 4; otherwise, go to 3.

3. Set $\lambda_{k+1}=\frac{\lambda_k}{2}$ and $k=k+1$, return to 2.

4. Set $x^1=x^k$, $\lambda_1=\lambda_k$, k = 1 and go to the main step.

Main Step:

1. Let (z_k, d^k) be an optimal solution to linear programming problem (17).
   If $z_k=0,$ step (x^k,λ_k) is a Fritz John point. Otherwise, $z_k<0$ and we go to 2.

2. Set $l_k=l_{\max}$, where

$$l_{\max}=\sup\{l: g_i(y^k+ld^k)\le 0, i=0,...,m+n+2\},$$

$y^k=(x^k,\lambda_k)$, and g_i is as (13). Let $y^{k+1}=y^k+l_kd^k$, replace k by k + 1, and return to 1.

The algorithm for finding $l_{\max}=\sup\{l: g_i(y+ld)\le 0\}$, by employing the bisection method. This algorithm is as below:

Initialization Step:

1. Set $l_1=1$ and k = 1.
2. If for at least one i , obtain $g_i(y+l_kd)>0,$ then go to 3, otherwise, set $l_{k+1}=l_k$, k = k + 1 and repeat 2.
3. Set $a_1=l_k-1,$ $b_1=l_k$, and go to the main step.

Main Sept:

1. Set $l_k = \dfrac{a_k + b_k}{2}$, if $\|a_k - b_k\| < \varepsilon$ (where? is a small positive scaler); stop, $l_{\max} = l_k$. Otherwise, go to 2.

3. If for at least one i obtain $g_i(y + l_k d) > 0$, then set $b_{k+1} = l_k$. Otherwise, set $a_{k+1} = l_k$, $k = k + 1$ and repeat 2.

5. Numerical Examples

Example 5.1. Solve the optimization problem

$$\begin{array}{ll} \text{Maximize} & 2x_1 + 3x_2 \\ \text{Subject to} & \tilde{1}x_1 + \tilde{2}x_2 \le 4 \\ & \tilde{3}x_1 + \tilde{1}x_2 \le 6 \\ & x_1, x_2 \ge 0, \end{array} \tag{18}$$

which take fuzzy parameters as $\tilde{1} = L(1,1), \tilde{2} = L(2,3), \tilde{3} = L(3,2)$ and $\tilde{1} = L(1,3)$, as used by Shaocheng (1994).

That is,

$$(a_{ij}) = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, \quad (d_{ij}) = \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix}, \quad (a_{ij} + d_{ij}) = \begin{bmatrix} 2 & 5 \\ 5 & 4 \end{bmatrix}.$$

For example, L $(a_{11} = 1, d_{11} = 1)$ is as:

$$\mu_{a11}(x) = \begin{cases} 1, & x < a_{11}, \\ \frac{a_{11} + d_{11} - x}{d_{11}}, & a_{11} \le x < a_{11} + d_{11}, \\ 0, & x \ge 1 + 1, \end{cases}$$

or

$$\mu_{a11}(x) = \begin{cases} 1, & x < 1, \\ \frac{1+1-x}{1}, & 1 \le x < 1+1, \\ 0, & x \ge 1+1. \end{cases}$$

For solving this problem we must solve the following two subproblems:

z_1 = maximize $2x_1 + 3x_2$

Subject to
$$1x_1 + 2x_2 \leq 4$$
$$3x_1 + 1x_2 \leq 6$$
$$x_1, x_2 \geq 0$$

and

z_2 = maximize $2x_1 + 3x_2$

Subject to
$$2x_1 + 5x_2 \leq 4$$
$$5x_1 + 4x_2 \leq 6$$
$$x_1, x_2 \geq 0.$$

Optimal solutions of these sub problems are,

$$x_1 = 1.6 \quad x_2 = 1.2 \quad z_1 = 6.8$$

and

$$x_1 = 0.86 \quad x_2 = 10.47 \quad z_2 = 3.06,$$

respectively. By using these optimal values, problem (18) can be reduced to the following equivalent non-linear programming problem:

Maximize λ

Subject to
$$\frac{2_{x1}+3_{x2}-3.06}{6.8-3.06} \geq \lambda$$
$$\frac{4-x_1-2_{x2}}{x_1+3x_2} \geq \lambda$$
$$\frac{6-3x_1-x_2}{2x_1+3x_2} \geq \lambda$$
$$0 \leq \lambda \leq 1$$
$$x_1, x_2 \geq 0.$$

That is,

Maximize λ

Subject to
$$2x_1 + 3x_2 + 3.74\lambda \geq 3.06 \tag{19}$$
$$(1+\lambda)x_1 + (2+3\lambda)x_2 \leq 4$$
$$(3+2\lambda)x_1 + (1+3\lambda)x_2 \leq 6$$
$$0 \leq \lambda \leq 1$$
$$x_1, x_2 \geq 0.$$

Let us solve problem (19) by using the modification method of feasible directions of Topkis and Veinott.

Initialization Step:

The problem the phase 1 is as:

$$\begin{array}{ll} \text{Minimize} & x_{a1}+x_{a2}+x_{a3} \\ \text{Subject to} & 2x_{a1}+3x_{a3}-s_1+x_{a1}=3.06+3.74\lambda \\ & (1+\lambda)x_1+(2+3\lambda)x_2+s_2+x_{a2}=4 \\ & (3+2\lambda)x_1+(1+3\lambda)x_2+s_3+x_{a3}=6 \\ & x_1,x_2,s_1,s_2,s_3,x_{a1},x_{a2},x_{a3}\geq 0, \end{array} \tag{20}$$

where x_{a1}, x_{a2}, x_{a3} are artificial variables and s_1, s_2, s_3 are slack variables. Set $\lambda = 1$, then, in optimal solution of above problem we have:

$$x_{a1}=3.741176, \quad x_{a2}=x_{a3}=0,$$

and since $x_{a1}\neq 0$ so the feasible set is empty, the new value of $\lambda=\frac{1}{2}$ is tried. For this $\lambda=\frac{1}{2}$, then $x_{a1}=0.734878\neq 0$ so the feasible set is empty. The new value of $\lambda=0.25$, then the optimal solution of the problem (20) is as follows:

$$\begin{array}{l} x_1=0.71355258 \\ x_2=0.95198976 \\ s_1=0.28807386 \\ s_2=0.49008779 \\ s_3=1.83658493 \\ x_{a1}=x_{a2}=x_{a3}=0. \end{array}$$

Hence, we are start from the point $(x^0,\lambda_0)=(0.71355258),0.95198976)^t$. We first formulate the problem (19) in the form

$$\begin{array}{ll} \text{Minimize} & -\lambda \\ \text{Subject to} & g_1(x_1,x_2,\lambda)=-2x_1-3x_2+3.74\lambda+3.06\leq 0 \\ & g_2(x_1,x_2,\lambda)=(1+\lambda)x_1+(2+3\lambda)x_2-4\leq 0 \\ & g_3(x_1,x_2,\lambda)=(3+2\lambda)x_1+(1+3\lambda)x_2-6\leq 0 \\ & g_4(x_1,x_2,\lambda)=-x_1\leq 0 \\ & g_5(x_1,x_2,\lambda)=-x_2\leq 0 \\ & g_6(x_1,x_2,\lambda)=-\lambda\leq 0 \\ & g_7(x_1,x_2,\lambda)=\lambda-1\leq 0. \end{array} \tag{21}$$

Iteration 1:

Search Direction: The direction finding problem is as follows:

Minimize z

Subject to

$$
\begin{aligned}
-d_3 - z &\le 0\\
-2d_1 - 3d_2 + 3.74d_3 - z &\le 0.2880747\\
1.25d_1 + 2.75d_2 + 3.569516d_3 - z &\le 0.490088\\
-d_1 - z &\le 0.7135528\\
-d_2 - z &\le 0.85198976\\
-d_3 - z &\le 0.25\\
-d_3 - z &\le 0.75\\
-1 \le d_j \le 1,\ j &= 1,2,3.
\end{aligned}
$$

The optimal solution to the above problem is

$$(d^1, z_1) = (0.4628627, -0.2230787, 0.1148866, -0.1148866)^t.$$

Line Search: The maximum value l such that $(x^0, \lambda_0) + ld^0$ is feasible is given by $l_{\max} = 1.047935486$. Hence $l_{\max} = 1.047935486$ is optimal solution. We then have $(x^1, \lambda_1) = (x^0, \lambda_0) + l_1 d_0 = (1.19860214, 0.71821759, 0.37039364)^t$.

The process is then repeated. Then, we have:

$$
\begin{aligned}
(x^2, \lambda_2) &= (1.13933356, 0.75573426, 0.39606129)^t\\
(x^3, \lambda_3) &= (1.14780263, 0.75037316, 0.39725715)^t\\
(x^4, \lambda_4) &= (1.14723602, 0.75074045, 0.39749963)^t\\
(x^5, \lambda_5) &= (1.14731541, 0.75068995, 0.39751106)^t\\
(x^6, \lambda_6) &= (1.14731003, 0.75069345, 0.39751336)^t\\
(x^7, \lambda_7) &= (1.14731079, 0.75069296, 0.39751347)^t.
\end{aligned}
$$

The optimal solution for the main problem (18) is as $(x_1^*, x_2^*) = (1.14731079, 075069296)^t$, which has the best membership grad $\lambda^* = 0.39751347$.

The progress of the algorithm of the method of feasible directions of Topkis and Veinott of Example 1 is depicted in Figure 1.

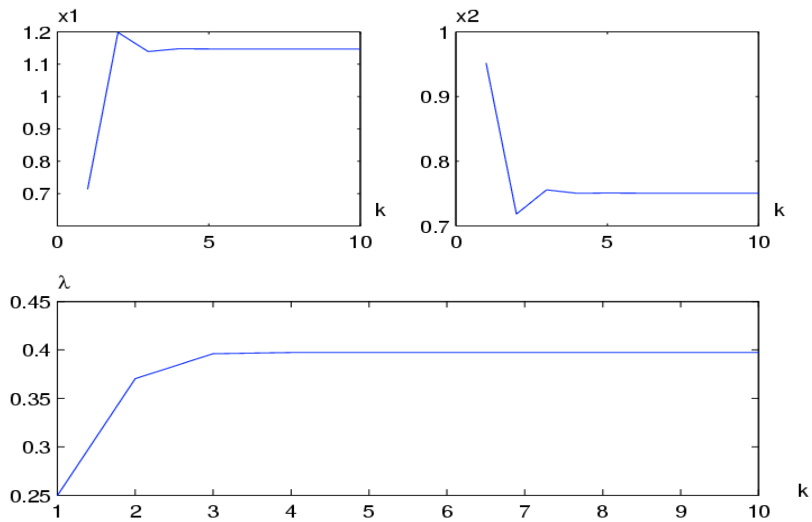

Figure 1. Approximate solution $x_1(.), x_2(.), \lambda(.)$.

Now, we solve this problem (18) with the augmented lagrangian penalty function method. We convert the problem (18) to (21). Select initial Lagrangian multipliers and positive values for the penalty parameters

$$\bar{u}_i = 0, \mu_i = 0.1, \quad i = `,...,7.$$

The starting point is taken as $(x^0, \lambda_0) = (1,1,1)^t$ and $\varepsilon = 0.00001$. Since $VIPOL(x^0, \lambda_0) = 3 > \varepsilon$, we choose the inner loop. The augmented Lagrangian penalty function is as

$$\begin{aligned} F_{AL}(x,\lambda,\bar{u}) = -\lambda + \tfrac{1}{10}[-2x_2 + \tfrac{187}{50}\lambda + \tfrac{153}{50}]^2 \\ + \tfrac{1}{10}[(1+\lambda)x_1 + (2+3\lambda)x_2 - 4]^2 \\ + \tfrac{1}{10}[(3+2\lambda)x_1 + (1+3\lambda)x_2 - 6]^2. \end{aligned}$$

Solving problem minimize $F_{AL}(x,\lambda,\bar{u})$, we obtain

$$(x^1, \lambda_1) = (0.98870612, 0.80516031, 0.54697368)^t,$$

$$VIOL(x^1, \lambda_1) = 0.71278842 > \varepsilon \text{ and } VIOL(x^1, \lambda_1) \le \tfrac{1}{4} VIOL(x^0, \lambda_0) = \tfrac{3}{4} = 0.75.$$

Hence, we go to outer loop step. The new Lagrangian multipliers are as

$$\bar{u}_{new} = (0.14255768\ ,0.09220549\ ,0.03481512\ ,0,0,0,0).$$

Set k = 1, and we go to the inner loop step. The process is then repeated. Then, we

$$(x^2, \lambda_2) = (1.13080371{,}0.77621059{,}0.39605607)^t$$
$$VIOL(x^2, \lambda_2) = 0.05335$$
$$(x^3, \lambda_3) = (1.14625379{,}0.75082359{,}0.39866258)^t$$
$$VIOL(x^3, \lambda_3) = 0.006$$
$$(x^4, \lambda_4) = (1.14719466{,}0.75091608{,}0.39747561)^t$$
$$VIOL(x^4, \lambda_4) = 0.00042$$
$$(x^5, \lambda_5) = (1.14724203{,}0.75073450{,}0.39753954)^t$$
$$VIOL(x^5, \lambda_5) = 0.00012$$
$$(x^6, \lambda_6) = (1.14727415{,}0.75071332{,}0.39751025)^t$$
$$VIOL(x^6, \lambda_6) = 0.0000027.$$

The optimal solution for the main problem (18) is the point

$$(x_1^*, x_2^*) = (1.14757415{,}0.75071332)^t,$$

which has the best membership grad λ^* = 0.39751025.

The progress of the algorithm of the method of the augmented Lagrangian penalty function of Example 1 is depicted in Figure 2.

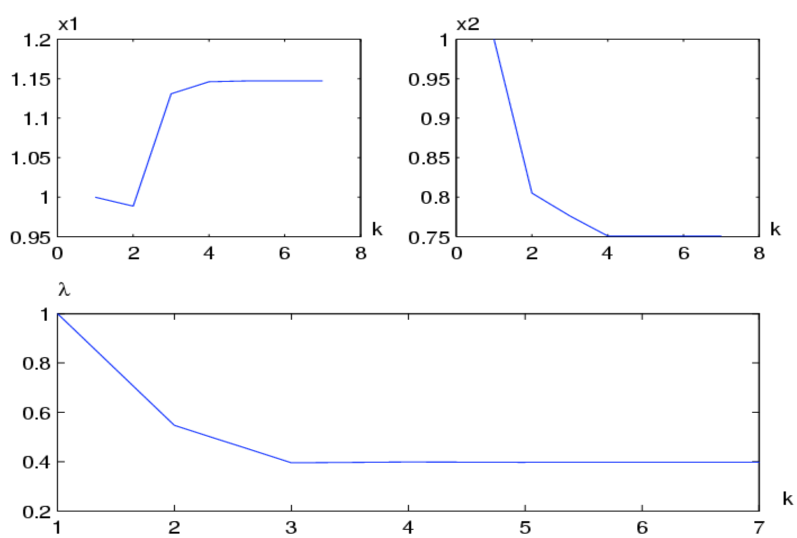

Figure 2. Approximate solution $x_1(.), x_2(.), \lambda(.)$.

Let us solve problem (19) by using the fuzzy decisive set method.

For λ = 1, the problem can be written as

$$
\begin{aligned}
&2x_1 + 3x_2 \geq 6.8 \\
&2x_1 + 5x_2 \leq 4 \\
&5x_1 + 4x_2 \leq 6 \\
&x_1 + x_2 \geq 0,
\end{aligned}
$$

and since the feasible set is empty, by taking $\lambda^L = 0$ and $\lambda^R = 1$, the new value of $\lambda = \frac{0+1}{2} = \frac{1}{2}$ is tried.

For $\lambda = \frac{1}{2} = 0.5$, the problem can be written as

$$
\begin{aligned}
&2x_1 + 3x_2 \geq 4.9294 \\
&\tfrac{3}{2}x_1 + \tfrac{7}{2}x_2 \leq 4 \\
&4x_1 + \tfrac{5}{2}x_2 \leq 6 \\
&x_1 + x_2 \geq 0,
\end{aligned}
$$

and since the feasible set is empty, by taking $\lambda^L = 0$ and $\lambda^R = \frac{1}{2}$, the new value of $\lambda = \frac{0+\frac{1}{2}}{2} = \frac{1}{4}$ is tried.

For $\lambda = \frac{1}{4} = 0.25$, the problem can be written as

$$
\begin{aligned}
&2x_1 + 3x_2 \geq 3.9941 \\
&\tfrac{5}{4}x_1 + \tfrac{11}{4}x_2 \leq 4 \\
&\tfrac{7}{2}x_1 + \tfrac{7}{4}x_2 \leq 6 \\
&x_1 + x_2 \geq 0,
\end{aligned}
$$

and since the feasible set is nonempty, by taking $\lambda^L = \frac{1}{4}$ and $\lambda^R = \frac{1}{2}$, the new value of $\lambda = \frac{1/4+1/2}{2} = \frac{3}{3}$ is tried.

For $\lambda = \frac{3}{8} = 0.375$, the problem can be written as

$$2x_1 + 3x_2 \geq 4.4618$$
$$\tfrac{11}{8}x_1 + \tfrac{25}{8}x_2 \leq 4$$
$$\tfrac{15}{4}x_1 + \tfrac{17}{8}x_2 \leq 6$$
$$x_1 + x_2 \geq 0,$$

and since the feasible set is nonempty, by taking $\lambda^L = \frac{1}{4}$ and $\lambda^R = \frac{1}{2}$, the new value of $\lambda = \frac{1/4+1/2}{2} = \frac{3}{3}$ is tried.

For $\lambda = \frac{3}{8} = 0.375$, the problem can be written as

$$2x_1 + 3x_2 \geq 4.4618$$
$$\tfrac{11}{8}x_1 + \tfrac{25}{8}x_2 \leq 4$$
$$\tfrac{15}{4}x_1 + \tfrac{17}{8}x_2 \leq 6$$
$$x_1 + x_2 \geq 0,$$

and since the feasible set is nonempty, by taking $\lambda^L = \frac{3}{8}$ and $\lambda^R = \frac{1}{2}$, the new value of $\lambda = \frac{3/8+1/2}{2} = \frac{7}{16}$ is tried.

For $\lambda = \frac{7}{16} = 0.4375$, the problem can be written as

$$2x_1 + 3x_2 \geq 4.6956$$
$$\tfrac{23}{16}x_1 + \tfrac{53}{16}x_2 \leq 4$$
$$\tfrac{31}{8}x_1 + \tfrac{37}{16}x_2 \leq 6$$
$$x_1 + x_2 \geq 0,$$

and since the feasible set is empty, by taking $\lambda^L = \frac{3}{8}$ and $\lambda^R = \frac{7}{16}$, the new value of $\lambda = \frac{3/8+7/16}{2} = \frac{13}{32}$ is tried.

The following values of λ are obtained in the next twenty six iterations:

$\lambda = 0.390625$
$\lambda = 0.3984375$
$\lambda = 0.39453125$
$\lambda = 0.396484375$
$\lambda = 0.3974609375$
$\lambda = 0.3979492187\,5$
$\lambda = 0.3977050781\,25$
$\lambda = 0.3975830078\,125$
$\vdots$
$\lambda = 0.3975558244\,8561$.

Consequently, we obtain the optimal value of λ at the thirty second iteration by using the fuzzy decisive set method.

Note that, the optimal value of λ found at the seven iteration of the method of feasible direction of Topkis and Veinott and at the sixth iteration of the augmented Lagrangian penalty function method is approximately equal to the optimal value of λ calculated at the twenty first iteration of the fuzzy decisive set method.

Example 6. 2. Solve the optimization problem

$$\begin{aligned} &\text{Maximize} \quad x_1 + x_2 \\ &\text{Subject to} \quad \tilde{1}x_1 + \tilde{2}x_2 \le \tilde{3} \\ &\qquad\qquad\quad \tilde{2}x_1 + \tilde{3}x_2 \le \tilde{4} \\ &\qquad\qquad\quad x_1, x_2 \ge 0\,, \end{aligned} \tag{22}$$

which take fuzzy parameters as:

$$\tilde{1} = L(1,1), \quad \tilde{2} = L(2,1), \quad \tilde{2} = L(2,2), \quad \tilde{3} = L(3,2),$$
$$b_1 = \tilde{3} = L(3,2), \quad b_{21} = \tilde{4} = L(4,3),$$

as used by Shaocheng (1994). That is,

$$(a_{ij}) = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \quad (d_{ij}) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \quad (a_{ij} + d_{ij}) = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$
$$(b_i) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad (p_i) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad (b_i + p_i) = \begin{bmatrix} 5 \\ 7 \end{bmatrix}.$$

To solve this problem, we must solve the following two subproblems

$$z_l = \text{ Maximize } \quad x_1 + x_2$$

$$\begin{aligned} \text{Subject to} \quad & 2x_1 + 3x_2 \le 3 \\ & 4x_1 + 5x_2 \le 4 \\ & x_1, x_2 \ge 0 \end{aligned}$$

and

$$z_u = \text{ Maximize } \quad x_1 + x_2$$

$$\begin{aligned} \text{Subject to} \quad & x_1 + 2x_2 \le 5 \\ & 2x_1 + 3x_2 \le 7 \\ & x_1, x_2 \ge 0\,. \end{aligned}$$

Optimal solutions of these subproblems are as follows:

$$\begin{aligned} x_1 &= 1 & & & x_1 &= 3.5 \\ x_2 &= 0 & &\text{and} & x_2 &= 0 \\ z_l &= 1 & & & z_u &= 3.5, \end{aligned}$$

respectively. By using these optimal values, problem (22) can be reduced to the following equivalent non-linear programming problem:

$$\text{Maximize} \quad \lambda$$

$$\begin{aligned} \text{Subject to} \quad & x_1 + x_2 - 2.5\lambda \ge 1 \\ & (1+\lambda)x_1 + (2+\lambda)x_2 + 2\lambda \le 3 \\ & (2+2\lambda)x_1 + (3+2\lambda)x_2 + 3\lambda \le 4 \\ & 0 \le \lambda \le 1 \\ & x_1, x_2 \ge 0. \end{aligned} \tag{23}$$

Let’s solve problem (23) by using the modification method of feasible directions of Topkis and Veinott.

Initialization Step:

The problem the phase 1 is as follows:

$$\text{Minimize} \quad x_{a1} + x_{a2} + x_{a3}$$

$$\text{Subject to} \qquad x_{a1} + x_2 - 2.5\lambda - s_1 + x_{a1} = 1$$
$$(1+\lambda)x_1 + (2+\lambda)x_2 + 2\lambda + s_2 + x_{a2} = 3 \tag{24}$$
$$(2+2\lambda)x_1 + (3+2\lambda)x_2 + 3\lambda + s_3 + x_{a3} = 4$$
$$x_1, x_2, s_1, s_2, s_3, x_{a2}, x_{a3} \geq 0,$$

where x_{a1}, x_{a2}, x_{a3} are artificial variables, s_1, s_2, s_3 are slack variables and λ is fixed scaler. Set $\lambda = 1$. Then, $x_{a1} = 3.25$ and since $x_{a1} \neq 0$ so the feasible set is empty, the new value of $\lambda = \frac{1}{2}$ is tried. Then we have

$$\lambda = 0.5 \Rightarrow x_{a1} = 1.1411667 \neq 0$$
$$\lambda = 0.25 \Rightarrow x_{a1} = 0.325 \neq 0$$
$$\lambda = 0.125 \Rightarrow x_{a1} = x_{a2} = x_{a3} = 0.$$

Hence, we are start from the point $(x^0, \lambda_0) = (1.41376683, 0.002421, 0.125)$.

We first formulate the problem (19) in the form

$$\text{Minimize} \qquad -\lambda$$
$$\text{Subject to} \qquad g_1(x_1, x_2, \lambda) = -x_1 - x_2 + 2.5\lambda + 1 \leq 0$$
$$g_2(x_1, x_2, \lambda) = (1+\lambda)x_1 + (2+\lambda)x_2 + 2\lambda - 3 \leq 0$$
$$g_3(x_1, x_2, \lambda) = (2+2\lambda)x_1 + (3+2\lambda)x_2 + 3\lambda - 4 \leq 0$$
$$g_4(x_1, x_2, \lambda) = -x_1 \leq 0 \tag{25}$$
$$g_5(x_1, x_2, \lambda) = -x_2 \leq 0$$
$$g_6(x_1, x_2, \lambda) = -\lambda \leq 0$$
$$g_7(x_1, x_2, \lambda) = \lambda - 1 \leq 0.$$

The direction finding problem for each the arbitrary constant point (x_1, x_2, λ) is as follows:

$$\text{Minimize} \qquad z$$
$$\text{Subject to} \qquad -d_3 - z \leq 0$$
$$-d_1 - d_2 + 2.5d_3 - z \leq -g_1(x_1, x_2, \lambda)$$
$$(1+\lambda)d_1 + (2+\lambda)d_2 + (x_1 + x_2)d_3 - z \leq -g_2(x_1, x_2, \lambda)$$
$$(2+2\lambda)d_1 + (3+2\lambda)d_2 + (2x_1 + 2x_2)d_3 - z \leq -g_3(x_1, x_2, \lambda)$$
$$-d_1 - z \leq -x_1$$
$$-d_2 - z \leq -x_2$$
$$d_3 - z \leq 1 - \lambda$$
$$-1 \leq d_1, d_2, d_3 \leq 1.$$

Iteration 1

Search Direction: For the initial point $(x^0, \lambda_0) = (1.41376683\ ,0.002421\ ,0.125)^t$ the direction finding problem is as follows:

Minimize z

Subject to
$$-d_3 - z \le 0$$
$$-d_1 - d_2 + 2.5d_3 - z \le 0.10368784$$
$$1.125d_1 + 2.125d_2 + 3.4162d_3 - z \le 1.15436768$$
$$2.25d_1 + 3.25d_2 + 5.8324d_3 - z \le 0.436156$$
$$-d_1 - z \le 1.41377$$
$$-d_2 - z \le 0.0024$$
$$-d_3 - z \le 0.125$$
$$d_3 - z \le 0.875$$
$$-1 \le d_j \le 1, \quad j = 1,2,3.$$

The optimal solution to the above problem is

$$(d^1, z_1) = (0.00455939\ ,0.040353486\ ,0.042774491\ ,-0.042774491\ )^t.$$

Line Search: The maximum value of l such that $x^0 + ld^0$ is feasible is given by $l_{\max} = 1.09670256.$ Hence $l_1 = 1.09670256.$ We then have

$$(x^1, \lambda_1) = (x^0, \lambda_0) + l_1 d^0 = (1.41998447,0.04667676,0.17191087)^t.$$

The process is then repeated. Then, we have:

$$(x^2, \lambda_2) = (1.45693175\ ,0.00000018\ ,0.18193177\ )^t$$
$$(x^3, \lambda_3) = (1.45719935\ ,0.00103274\ ,0.18296452\ )^t$$
$$(x^4, \lambda_4) = (1.45801560\ ,-0.00000000\ ,0.18318790\ )^t$$
$$(x^5, \lambda_5) = (1.45802153\ ,0.00002256\ ,0.18321046\ )^t$$
$$(x^6, \lambda_6) = (1.45803936\ ,0.00000000\ ,0.18321584\ )^t$$
$$(x^7, \lambda_7) = (1.45803949\ ,0.00000049\ ,0.18321584\ )^t$$
$$(x^8, \lambda_8) = (1.45803988\ ,-0.00000000\ ,0.18321594\ )^t.$$

The optimal solution for the main problem (18) is

$$(x_1^*, x_2^*) = (1.45803988,-0.00000000)^t,$$

which has the best membership grad $\lambda^* = 0.18321594$.

The progress of the algorithm of the method of feasible directions of Topkis and Veinott of Example 2 is depicted in Figure 3.

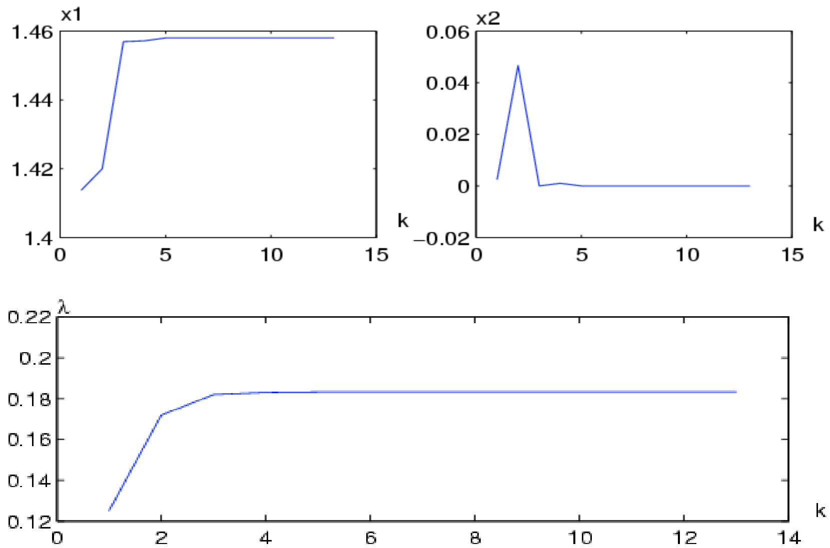

Figure3. Approximate solution $x_1(.), x_2(.), \lambda(.)$.

Now, we solve the problem (22) with the augmented Lagrangian penalty function method. We convert the problem (22) to (25). Select initial Lagrangian multipliers and positive values for the penalty parameters

$$\bar{u}_i = 0 \;, \quad \mu_i = 0.1 \quad , \; i = 1,..., \; 7 \,.$$

The starting point is taken as $(x^0, \lambda_0) = (1,1,1)^t$ and $\varepsilon = 0.00001$. Since $VIOL(x^0, \lambda_0) = 3 > \varepsilon$ we going to inner loop. The augmented Lagrangian penalty function is as:

$$\begin{aligned} F_{AL}(x, \lambda, \bar{u}) = -\lambda + \tfrac{1}{10}[-x_1 - x_2 + 2.5\lambda + 1]^2 \\ + [(1+\lambda)x_1 + (2+\lambda)x_2 + 2\lambda - 3]^2 \\ + \tfrac{1}{10}[(2+2\lambda)x_1 + (3+2\lambda)x_2 + 3\lambda - 4]^2, \end{aligned}$$

with solving problem minimize $F_{AL}(x, \lambda, \bar{u})$ we obtain

$$(x^1, \lambda_1) = (0.88778718 \;, 0.10414525 \;, 0.50264920 \;)^t,$$

and $VIOL(x^1, \lambda_1) = 1.26469058 > \varepsilon$ and also $VIOL(x^1, \lambda_1) \le \tfrac{1}{4} VIOL(x^0, \lambda_0) = \tfrac{8}{4} = 2$.

Hence, we go to outer loop step. The new Lagrangian multipliers are as

$$\bar{u}_{new} = (0.25293811\ ,0,0.11862916\ ,0,0,0,0).$$

Set k = 1, and we go to the inner loop step. The process is then repeated. Then, we have:

$$(x^3, \lambda_3) = (1.59043285\ ,-0.00435813\ ,0.10791938\ )^t$$
$$VIOL\ (x^3, \lambda_3) = 0.00436$$
$$(x^4, \lambda_4) = (1.42634348\ ,0.00012926\ ,0.19523932\ )^t$$
$$VIOL\ (x^4, \lambda_4) = 0.06162557$$
$$(x^5, \lambda_5) = (1.45584285\ ,0.00027062\ ,0.18271767\ )^t$$
$$VIOL\ (x^5, \lambda_5) = 0.00068072$$
$$(x^6, \lambda_6) = (1.4887092\ ,-0.00014049\ ,0.18342549\ )^t$$
$$VIOL\ (x^6, \lambda_6) = 0.00273355$$
$$(x^7, \lambda_7) = (1.45814601\ ,-0.00000151\ ,0.18318092\ )^t$$
$$VIOL\ (x^7, \lambda_7) = 0.0000388$$
$$(x^8, \lambda_8) = (1.45802558\ ,0.00000024\ ,0.18322008\ )^t$$
$$VIOL\ (x^8, \lambda_8) = 0.0000244$$
$$(x^9 . \lambda_9) = (1.45803637\ ,0.00000034\ ,0.18321458\ )^t$$
$$VIOL\ (x^9, \lambda_9) = 0.$$

The optimal solution for the main problem (22) is

$$(x_1^*, x_2^*) = (1.45803637, 0.00000034)^t,$$

which has the best membership grad $\lambda^* = 0.18321458$.

The progress of the algorithm of the method of the augmented Lagrangian penalty function of Example 2 is depicted in Figure 4.

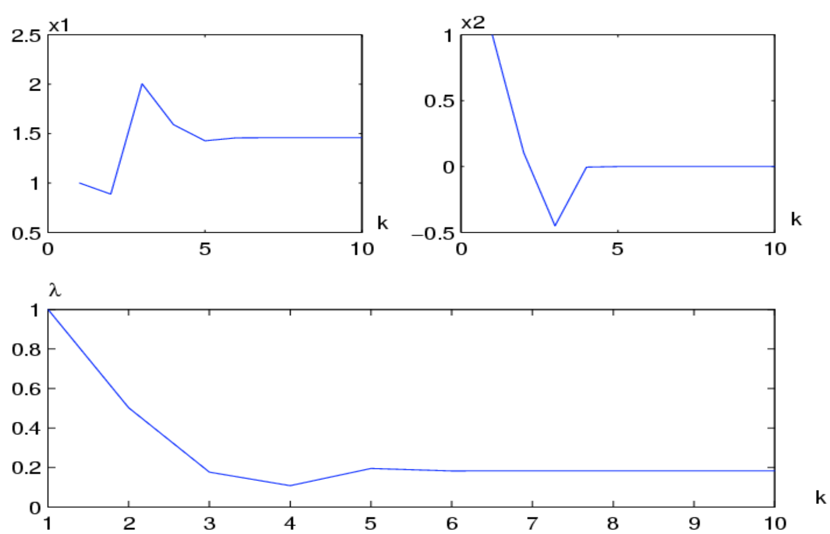

Figure 4. Approximate solution $x_1(.), x_2(.), \lambda(.)$.

Let us solve the problem (23) by using the fuzzy decisive set method.

For $\lambda = 1$, the problem can be written as

$$x_1 + x_2 \geq 3.5$$
$$2x_1 + 3x_2 \leq 1$$
$$4x_1 + 5x_2 \leq 1$$
$$x_1 + x_2 \geq 0,$$

and since the feasible set is empty, by taking $\lambda^L = 0$ and $\lambda^R = 1$, the new value of $\lambda = \frac{0+1}{2} = \frac{1}{2}$ is tried.

For $\lambda = \frac{1}{2} = 0.5$, the problem can be written as

$$x_1 + x_2 \geq 1.25$$
$$\tfrac{6}{2}x_1 + \tfrac{5}{2}x_2 \leq 2$$
$$3x_1 + 4x_2 \leq \tfrac{5}{2}$$
$$x_1 + x_2 \geq 0,$$

and since the feasible set is empty, by taking $\lambda^L = 0$ and $\lambda^R = \frac{1}{2}$, the new value of $\lambda = \frac{0+1/2}{2} = \frac{1}{4}$ is tried. For $\lambda = \frac{1}{4} = 0.25$, the problem can be written as

$$x_1 + x_2 \geq 1.625$$
$$\tfrac{5}{4}x_1 + \tfrac{9}{4}x_2 \leq \tfrac{5}{2}$$
$$\tfrac{5}{2}x_1 + \tfrac{7}{2}x_2 \leq \tfrac{13}{4}$$
$$x_1 + x_2 \geq 0,$$

and since the feasible set is empty, by taking $\lambda^L = 0$ and $\lambda^R = \frac{1}{4}$, the new value of $\lambda = \frac{0+1/4}{2} = \frac{1}{8}$ is tried. For $\lambda = \frac{1}{8} = 0.125$, the problem can be written as

$$x_1 + x_2 \geq 1.3125$$
$$\tfrac{9}{8}x_1 + \tfrac{17}{8}x_2 \leq \tfrac{22}{8}$$
$$\tfrac{9}{5}x_1 + \tfrac{13}{4}x_2 \leq \tfrac{29}{8}$$
$$x_1 + x_2 \geq 0,$$

and since the feasible set is nonempty, by taking $\lambda^L = \frac{1}{8}$ and $\lambda^R = \frac{1}{4}$, the new value of $\lambda = \frac{1/8+1/4}{2} = \frac{3}{16}$ is tried.

The following values of λ are obtained in the next twenty one iterations:

$$\lambda = 0.1875$$
$$\lambda = 0.15625$$
$$\lambda = 0.171875$$
$$\lambda = 0.1796875$$
$$\lambda = 0.18359375$$
$$\lambda = 0.181640625$$
$$\lambda = 0.182617187$$
$$\lambda = 0.183105468$$
$$\vdots$$
$$\lambda = 0.183215915\,.$$

Consequently, we obtain the optimal value of λ at the twenty fifth iteration of the fuzzy decisive set method. Note that, the optimal value of λ found at the second iteration of the method of feasible direction of Topkis and Veinott and at the sixth iteration of the augmented Lagragian penalty function method is approximately equal to the optimal value of λ calculated at the twenty fifth iteration of the fuzzy decisive set method.

7. Conclusions

This paper presents a method for solving fuzzy linear programming problems in which both the right-hand side and the technological coefficients are fuzzy numbers. After the defuzzification using method of Bellman and Zadeh, the crisp problems are non-linear and even non-convex in general. We use here the "modified subgradient method" and "method of feasible directions" for solving these problems. We also compare the new proposed methods with well known "fuzzy decisive set method". Numerical results show the applicability and accuracy of this method. This method can be applied for solving any fuzzy linear programming problems with fuzzy coefficients in constraints and fuzzy right hand side values.

REFERENCES

Bazaraa, M. S., Sherali, H. D. and Shetty, C. M. (1993). Non-linear programming theory and algo-rithms, John Wiley and Sons, New York, 360-430.

Bellman, R. E. and Zadeh, L. A. (1970). Decision-making in a fuzzy environment, Management Science 17, B141-B164.

Canz, T. (1999). Fuzzy linear programming for DSS in energy planning, Int. J. of global energy issues 12, 138-151.

Dubois, D and Prade, H. (1982). System of linear fuzzy constraints, Fuzzy set and systems 13, 1-10.
Elamvazuthi, I., Ganesan, T. Vasant, P. and Webb, F. J. (2009). Application of a fuzzy programming technique to production planning in the textile industry, Int. J. of computer science and information security 6, 238-243.
Guu, Sy-Ming and Yan-K, Wu (1999). Two-phase approach for solving the fuzzy the fuzzy linear programming problems, Fuzzy set and systems 107, 191-195.
Negoita, C. V. (1970). Fuzziness in management, OPSA/TIMS, Miami.
Sakawa, M and Yana, H. (1985). Interactive decision making for multi-objective linear fractional programming problem with fuzzy parameters, Cybernetics Systems 16 , 377-397.
Shaocheng, T. (1994). Interval number and fuzzy number linear programming, Fuzzy sets and systems 66, 301-306.
Tanaka, H and Asai, K. (1984). Fuzzy linear programming problems with fuzzy numbers, Fuzzy sets and systems 13, 110.
Tanaka, H, Okuda, T. and Asai, K. (1984). On fuzzy mathematical programming, J. Cybernetics 3, 291-298.
Tan, Raymond R. (2005). Application of symmetric fuzzy linear programming in life cycle assessment, Environmental modeling & software 20, 1343-1346.
Zimmermann, H. J. (1983). Fuzzy mathematical programming, Comput. Ops. Res. Vol. 10 No. 4, 291-298.

APPENDIX

The Algorithm of the Fuzzy Decisive Set Method

This method is based on the idea that, for a fixed value of λ , the problem (9) is linear programming problems. Obtaining the optimal solution λ^* to the problem (9) is equivalent to determining the maximum value of λ so that the feasible set is nonempty. The algorithm of this method for the problem (9) is presented below.

Algorithm

Step 1. Set $\lambda = 1$ and test whether a feasible set satisfying the constraints of the problem (9) exists or not, using phase one of the Simplex method. If a feasible set exists, set $\lambda = 1$, otherwise, set $\lambda^L = 0$ and $\lambda^R = 1$ and o to the next step.

Step 2. For the value of $\lambda = \frac{\lambda^L + \lambda^R}{2}$, update the value of λ^L and λ^R using the bisection method as follows:

$\lambda^L = \lambda$, if feasible set is nonempty for λ,
$\lambda^R = \lambda$, if feasible set is empty for λ.

Consequently, for each λ, test whether a feasible set of the problem (9) exists or not using phase one of the Simplex method and determine the maximum value λ^* satisfying the constraints of the problem (9).