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New Exact Solutions of some Nonlinear Partial Differential Equations by the First Integral Method

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Abstract

The first integral method is an efficient method for obtaining exact solutions of nonlinear partial differential equations. The efficiency of the method is demonstrated by applying it for two selected equations. This method can be applied to nonintegrable equations as well as to integrable ones.

Keywords: First integral method; Gardner equation; (2+1)-dimensional nonlinear Schrödinger equation

MSC 2000 No.: 35D

1. Introduction

Nonlinear evolution equations have a major role in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, solid state physics, chemical inematics, chemical physics and geochemistry. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations. In recent years, quite a few methods for obtaining explicit traveling and solitary wave solutions of nonlinear evolution equations have been proposed. A variety of powerful methods, such as, tanh-sech method [Malfliet (1992), Khater et al. (2002), Wazwaz (2006)], extended tanh method [El-Wakil et al. (2007), Fan (2000), Wazwaz (2005)], hyperbolic function method (Xia and Zhang (2001)), sine-cosine method [Wazwaz (2004), Yusufoglu and Bekir (2006), Jacobi elliptic function expansion method [Inc and Ergut (2005)], F-expansion method [Zhang (2006)], and the First Integral method [Feng (2002), Ding and Li (1996)] .

The first integral method was first proposed by Feng (2002), in solving Burgers-KdV equation, which is based on the ring theory of commutative algebra. Recently, this useful method is widely used by many [See, Feng and Wang (2002), Raslan (2008), Abbasbandy and Shirzadi (2010) and reference therein].

The aim of this paper is to find new exact solutions of the Gardner equation and the (2+1) -dimensional nonlinear Schrödinger equation by the first integral method.

2. First integral Method

Consider the nonlinear partial differential equation in the form

$$F(u, u_x, u_y, u_t, u_{xx}, u_{xy}, ...) = 0, \tag{1}$$

where $u = u(x, y, t)$ is the solution of nonlinear partial differential equation (1). We use the transformations

$$u(x, y, t) = u(\xi), \qquad \xi = k(x + ly - \lambda t). \tag{2}$$

This enables us to use the following changes:

$$\frac{\partial}{\partial t}(.) = -k\lambda \frac{\partial}{\partial \xi}(.), \quad \frac{\partial}{\partial x}(.) = k\frac{\partial}{\partial \xi}(.), \quad \frac{\partial}{\partial y}(.) = kl\frac{\partial}{\partial \xi}(.), \quad \frac{\partial^2}{\partial x^2}(.) = k^2\frac{\partial^2}{\partial \xi^2}(.), \cdots. \tag{3}$$

We use (3) to change the nonlinear partial differential equation (1) to nonlinear ordinary differential equation

$$G(f(\xi), \frac{\partial f(\xi)}{\partial \xi}, \frac{\partial^2 f(\xi)}{\partial \xi^2}, ...) = 0. \tag{4}$$

Next, we introduce a new independent variable

$$X(\xi) = u(\xi), \qquad Y = \frac{\partial u(\xi)}{\partial \xi}, \tag{5}$$

which leads to the system of nonlinear ordinary differential equations

$$\frac{\partial X(\xi)}{\partial \xi} = Y(\xi),$$
$$\frac{\partial Y(\xi)}{\partial \xi} = F_1(X(\xi), Y(\xi)). \tag{6}$$

By the qualitative theory of ordinary differential equations [Ding and Li (1996)], if we can find the integrals to Equation (6) under the same conditions, then the general solutions to Equation (6) can be solved directly. However, in general, it is really difficult for us to realize this even for one first integral, because for a given plane autonomous system, there is no systematic theory that can tell us how to find its first integrals, nor is there a logical way for telling us what these first integrals are. We will apply the Division Theorem to obtain one first integral to Equation (6) which reduces Equation (4) to a first order integrable ordinary differential equation. An exact solution to Equation (1) is then obtained by solving this equation.

Now, let us recall the Division Theorem:

Division Theorem. Suppose that $P(w,z)$ and $Q(w,z)$ are polynomials in $C(w,z)$, and $P(w,z)$ is irreducible in $C(w,z)$. If $Q(w,z)$ vanishes at all zero points of $P(w,z)$, then there exists a polynomial $F_2(w,z)$ in $C(w,z)$ such that $Q(w,z) = P(w,z)F_2(w,z)$.

3. Application

Example 1. Let us first consider the Gardner equation [Wazwaz (2007), Biswas (2008)]

$$u_t = 6(u + \varepsilon^2 u^2)u_x + u_{xxx}. \tag{7}$$

By making the transformation $u(x,t) = u(\xi)$, $\xi = k(x - \lambda t)$, the Equation (7) becomes

$$-k\lambda \frac{\partial u(\xi)}{\partial \xi} = 6k(u(\xi) + \varepsilon^2 (u(\xi))^2)\frac{\partial u(\xi)}{\partial \xi} + k^3 \frac{\partial^3 u(\xi)}{\partial \xi^3}. \tag{8}$$

Hence,

$$\lambda \frac{\partial u}{\partial \xi} + 6u \frac{\partial u}{\partial \xi} + 6\varepsilon^2 u^2 \frac{\partial u}{\partial \xi} + k^2 \frac{\partial^3 u}{\partial \xi^3} = 0. \tag{9}$$

Integrating Equation (9) once with respect to ξ, then we have

$$\lambda u + 3u^2 + 2\varepsilon^2 u^3 + k^2 \frac{\partial^2 u}{\partial \xi^2} = R, \tag{10}$$

where R is the integration constant.

Rewrite this second-order ordinary differential equation as follows

$$\frac{\partial^2 u}{\partial \xi^2} + k_1 u^3 + k_2 u^2 + k_3 u - k_4 = 0, \tag{11}$$

where

$$k_1 = \frac{2\varepsilon^2}{k^2}, \qquad k_2 = \frac{3}{k^2}, \qquad k_3 = \frac{\lambda}{k^2}, \qquad k_4 = \frac{R}{k^2}.$$

Using (5) we get

$$\dot{X}(\xi) = Y(\xi), \tag{12}$$

$$\dot{Y}(\xi) = -k_1 (X(\xi))^3 - k_2 (X(\xi))^2 - k_3 X(\xi) + k_4. \tag{13}$$

According to the first integral method, we suppose the $X(\xi)$ and $Y(\xi)$, are the nontrivial solutions of (12), (13), and

$$Q(X,Y) = \sum_{i=0}^{N} a_i(X) Y^i = 0,$$

is an irreducible polynomial in the complex domain $C(X,Y)$, such that

$$Q(X(\xi), Y(\xi)) = \sum_{i=0}^{N} a_i(X(\xi)) Y^i(\xi) = 0, \tag{14}$$

where $a_i(X)(i = 0,1,...,N)$, are polynomials of X and $a_N(X) \neq 0$. Equation (14) is called the first integral to (12), (13). Due to the Division Theorem, there exists a polynomial $g(X) + h(X)Y$, in the complex domain $C(X,Y)$, such that

$$\frac{dQ}{d\xi} = \frac{dQ}{dX} . \frac{dX}{d\xi} + \frac{dQ}{dY} . \frac{dY}{d\xi} = (g(X) + h(X)Y) \sum_{i=0}^{N} a_i(X) Y^i . \tag{15}$$

In this example, we take two different cases, assuming that $N = 1$, and $N = 2$, in (14).

Case A

Suppose that $N = 1$. By comparing with the coefficients of $Y^i\ (i = 2,1,0)$ from both sides of (15), we have

$$\dot{a}_1(X) = h(X)a_1(X), \tag{16}$$

$$\dot{a}_0(X) = g(X)a_1(X) + h(X)a_0(X), \tag{17}$$

$$a_1(X)[-k_1(X(\xi))^3 - k_2(X(\xi))^2 - k_3X(\xi) + k_4] = g(X)a_0(X). \tag{18}$$

We obtain that $a_1(X)$, is constant and $h(X) = 0$, take $a_1(X) = 1$, and balancing the degrees of $g(X)$, $a_1(X)$ and $a_0(X)$, we conclude that $\deg(g(X)) = 1$, only.

Suppose that $g(X) = A_1X + A_0$, then we find $a_0(X)$.

$$a_0(X) = c_1 + A_0X + \frac{1}{2}A_1X^2, \tag{19}$$

where c_1 is arbitrary integration constant. Substituting $a_0(X), a_1(X)$ and $g(X)$ in the last equation in (18) and setting all the coefficients of X to be zero, then we obtain a system of nonlinear equations and by solving it, we obtain

$$A_0 = -\frac{i}{\varepsilon k}, \quad A_1 = -\frac{2i\varepsilon}{k}, \quad c_1 = -\frac{i(\varepsilon^2\lambda - 1)}{2\varepsilon^3 k}, \quad R = \frac{1-\varepsilon^2\lambda}{2\varepsilon^4}, \tag{20}$$

$$A_0 = \frac{i}{\varepsilon k}, \quad A_1 = \frac{2i\varepsilon}{k}, \quad c_1 = \frac{i(\varepsilon^2\lambda - 1)}{2\varepsilon^3 k}, \quad R = \frac{1-\varepsilon^2\lambda}{2\varepsilon^4}, \tag{21}$$

where k, ε and λ are arbitrary constants.

Using the conditions (20), into Equation (14), we obtain

$$Y(\xi) = \frac{i(\varepsilon^2\lambda - 1)}{2\varepsilon^3 k} + \frac{i}{\varepsilon k}X(\xi) + \frac{i\varepsilon}{k}(X(\xi))^2. \tag{22}$$

Combining (22) with (12), we obtain the exact solution to equation (11) and then the exact solution to Gardner equation can be written as

$$u(x,t) = -\frac{1}{2\varepsilon^2} - \frac{\sqrt{2\varepsilon^2\lambda - 3}}{2\varepsilon^2}\tan\left[\frac{\sqrt{2\varepsilon^2\lambda - 3}}{2i\varepsilon k}(k(x-\lambda t)+\xi_0)\right], \tag{23}$$

where ξ_0 is an arbitrary constant.

Similarly, in the case of (21), into Equation (14), we obtain

$$Y(\xi) = -\frac{i(\varepsilon^2\lambda - 1)}{2\varepsilon^3 k} - \frac{i}{\varepsilon k}X(\xi) - \frac{i\varepsilon}{k}(X(\xi))^2, \tag{24}$$

and then the exact solution of the Gardner equation can be written as

$$u(x,t) = -\frac{1}{2\varepsilon^2} + \frac{\sqrt{2\varepsilon^2\lambda - 3}}{2\varepsilon^2}\tan\left[\frac{\sqrt{2\varepsilon^2\lambda - 3}}{2i\varepsilon k}(k(x-\lambda t)+\xi_0)\right], \tag{25}$$

where ξ_0 is an arbitrary constant.

Case B

Suppose that $N = 2$, by equating with the coefficients of $Y^i\,(i = 3,2,1,0)$ from both sides of (15), we have

$$\dot{a}_2(X) = h(X)a_2(X), \tag{26}$$

$$\dot{a}_1(X) = g(X)a_2(X) + h(X)a_1(X), \tag{27}$$

$$\begin{aligned}\dot{a}_0(X) &= -2a_2(X)[-k_1(X(\xi))^3 - k_2(X(\xi))^2 - k_3X(\xi) + k_4] \\ &+ g(X)a_1(X) + h(X)a_0(X),\end{aligned} \tag{28}$$

$$a_1(X)[-k_1(X(\xi))^3 - k_2(X(\xi))^2 - k_3X(\xi) + k_4] = g(X)a_0(X). \tag{29}$$

We obtain that $a_2(X)$, is constant and $h(X) = 0$, take $a_2(X) = 1$, and balancing the degrees $g(X), a_1(X)$ and $a_0(X)$, we conclude that $\deg(g(X)) = 1$, only.

Suppose that $g(X) = A_1X + A_0$, then we find $a_0(X)$, and $a_1(X)$ as

$$a_1(X) = c_1 + A_0X + \frac{1}{2}A_1X^2, \tag{30}$$

$$a_0(X)=\frac{1}{4}(\frac{4\varepsilon^2}{k^2}+\frac{1}{2}A_1^2)X^4+\frac{1}{3}(\frac{6}{k^2}+\frac{3}{2}A_0A_1)X^3+\frac{1}{2}(c_1A_1+A_0^2+\frac{2\lambda}{k^2})X^2 \\ +(c_1A_0-\frac{2R}{k^2})X+c_2. \quad (31)$$

where c_2 is arbitrary integration constant. Substituting $a_0(X)$, $a_1(X)$ and $g(X)$, in the last equation in (29) and setting all the coefficients of X to be zero, then we obtain a system of nonlinear equations and by solving it with aid Maple, we obtain

$$c_2=\frac{c_1^2}{4},\qquad k=\pm\frac{4i\varepsilon}{A_1},\qquad \lambda=\frac{A_1+4c_1\varepsilon^4}{\varepsilon^2A_1},\quad A_0=\frac{A_1}{2\varepsilon^2},\quad R=-\frac{2c_1}{A_1}, \quad (32)$$

where A_1,ε and c_1 are arbitrary constants. Using the conditions (32), into Equation (14), we can get

$$Y(\xi)=-\frac{A_1\varepsilon^2(X(\xi))^2+A_1X(\xi)+2\varepsilon^2c_1}{4\varepsilon^2}. \quad (33)$$

Combining (33) with (12), we obtain the exact solution to equation (11) and the exact solution to Gardner equation can be written as

$$u(x,t)=-\frac{1}{2\varepsilon^2}-\frac{\sqrt{8c_1\varepsilon^4A_1-A_1^2}}{2A_1\varepsilon^2}\tan[\frac{\sqrt{8c_1\varepsilon^4A_1-A_1^2}}{8\varepsilon^2}(k(x-\frac{A_1+4c_1\varepsilon^4}{\varepsilon^2A_1}t)+\xi_0)], \quad (34)$$

where ξ_0 is an arbitrary constant.

Example 2. Considering the (2+1)-dimensional nonlinear Schrödinger equation [Zhou et al. (2004)] that reads

$$iu_t+au_{xx}-bu_{yy}+c|u|^2u=0, \quad (35)$$

where a,b and c are nonzero constants. Firstly, we introduce the transformations

$$u(x,y,t)=e^{i\theta}u(\xi),\quad \theta=\alpha x+\beta y+\delta t,\qquad \xi=k(x+ly-\lambda t), \quad (36)$$

where α,β,δ,k,l and λ are real constants. Substituting (36) into Equation (35) we obtain the $\lambda=2(a\alpha-b\beta l)$ and $u(\xi)$ satisfy into the ODE:

$$-(\delta + a\alpha^2 - b\beta^2)u(\xi) + (a - bl^2)k^2 \frac{\partial^2 u(\xi)}{\partial \xi^2} + c(u(\xi))^3 = 0. \tag{37}$$

Rewrite this second-order ordinary differential equation as follows

$$\frac{\partial^2 u(\xi)}{\partial \xi^2} + k_1 u^3 - k_2 u = 0, \tag{38}$$

where

$$k_1 = \frac{c}{(a - bl^2)k^2}, \qquad k_2 = \frac{\delta + a\alpha^2 - b\beta^2}{(a - bl^2)k^2}.$$

Using (5) we get

$$\dot{X}(\xi) = Y(\xi), \tag{39}$$

$$\dot{Y}(\xi) = -k_1 (X(\xi))^3 + k_2 X(\xi). \tag{40}$$

According to the first integral method, we suppose the $X(\xi)$ and $Y(\xi)$, are the nontrivial solutions of (39) and (40) also

$$Q(X,Y) = \sum_{i=0}^{N} a_i(X) Y^i = 0$$

is an irreducible polynomial in the complex domain $C(X,Y)$, such that

$$Q(X(\xi), Y(\xi)) = \sum_{i=0}^{N} a_i(X(\xi)) Y^i(\xi) = 0, \tag{41}$$

where $a_i(X)(i = 0,1,...,N)$, are polynomials of X and $a_N(X) \neq 0$. Equation (41) is called the first integral to (39), (40). Due to the Division Theorem, there exists a polynomial $g(X) + h(X)Y$, in the complex domain $C(X,Y)$, such that

$$\frac{dQ}{d\xi} = \frac{dQ}{dX}.\frac{dX}{d\xi} + \frac{dQ}{dY}.\frac{dY}{d\xi} = (g(X) + h(X)Y) \sum_{i=0}^{N} a_i(X) Y^i. \tag{42}$$

Suppose that $N = 1$, by comparing with the coefficients of $Y^i (i = 2,1,0)$ from both sides of (42), we have

$$\dot{a}_1(X) = h(X)a_1(X), \tag{43}$$

$$\dot{a}_0(X) = g(X)a_1(X) + h(X)a_0(X), \tag{44}$$

$$a_1(X)[-k_1(X(\xi))^3 + k_2X(\xi)] = g(X)a_0(X). \tag{45}$$

We obtain that $a_1(X)$, is constant and $h(X)=0$, take $a_1(X)=1$, and balancing the degrees of $g(X)$, $a_1(X)$ and $a_0(X)$, we conclude that $\deg(g(X))=1$, only.

Suppose that $g(X) = A_1X + A_0$, then we find $a_0(X)$.

$$a_0(X) = c_1 + A_0X + \frac{1}{2}A_1X^2, \tag{46}$$

where c_1 is arbitrary integration constant. Substituting $a_0(X), a_1(X)$ and $g(X)$ in the last equation in (45) and setting all the coefficients of X to be zero, then we obtain a system of nonlinear equations and by solving it, we obtain

$$A_0 = 0, \quad A_1 = \frac{1}{k}\sqrt{\frac{2c}{bl^2-a}}, \quad \delta = b\beta^2 - a\alpha^2 + c_1k\sqrt{2c(bl^2-a)}, \tag{47}$$

$$A_0 = 0, \quad A_1 = -\frac{1}{k}\sqrt{\frac{2c}{bl^2-a}}, \quad \delta = b\beta^2 - a\alpha^2 - c_1k\sqrt{2c(bl^2-a)}, \tag{48}$$

where c_1, k, l, α, β are arbitrary constants.

Using the conditions (47) into Equation(41), we obtain

$$Y(\xi) = -c_1 - \frac{1}{2k}\sqrt{\frac{2c}{bl^2-a}}(X(\xi))^2. \tag{49}$$

Combining (49) with (39), we obtain the exact solution to equation (38) and then the exact solution to the (2 + 1)-dimensional nonlinear Schrödinger equation can be written as

$$\begin{aligned} u(x,y,t) = -\sqrt{\frac{c_1k(bl^2-a)\sqrt{2}}{c}\sqrt{\frac{c}{bl^2-a}}}\, e^{i(\alpha x+\beta y+(b\beta^2-a\alpha^2+c_1k\sqrt{2c(bl^2-a)})t)} \\ \times \tan[\frac{1}{k}\sqrt{\frac{c_1k}{2}\sqrt{\frac{2c}{bl^2-a}}}(k(x+ly-2(a\alpha-b\beta l)t)+\xi_0)], \end{aligned} \tag{50}$$

where ξ_0 is an arbitrary constant.

Similarly, in the case of (48), into Equation (41), we obtain

$$Y(\xi) = -c_1 + \frac{1}{2k}\sqrt{\frac{2c}{bl^2 - a}}(X(\xi))^2, \tag{51}$$

and then the exact solution of (2 + 1)-dimensional nonlinear Schrödinger equation can be written as

$$\begin{aligned} u(x,y,t) = -\sqrt{\frac{c_1 k(bl^2-a)\sqrt{2}}{c}\sqrt{\frac{c}{bl^2-a}}}\, e^{i(\alpha x + \beta y + (b\beta^2 - a\alpha^2 - c_1 k\sqrt{2c(bl^2-a)})t)} \\ \times \tanh[\frac{1}{k}\sqrt{\frac{c_1 k}{2}\sqrt{\frac{2c}{bl^2-a}}}(k(x+ly-2(a\alpha - b\beta l)t)+\xi_0)], \end{aligned} \tag{52}$$

where ξ_0 is an arbitrary constant.

4. Conclusion

In this work the first integral method was applied successfully for solving two nonlinear partial differential equations. Thus, we can say that the proposed method can be extended to solve the problems of nonlinear partial differential equations which arising in the theory of solitons and other areas.

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