Applications and Applied Mathematics: An International Journal (AAM)

Volume 5 | Issue 2 | Article 10

12-2010

Reliability Measures of a Three-State Complex System: A Copula Approach

Mangey Ram
Graphic Era University

Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Vol. 05, Issue 2 (December 2010), pp. 386 – 395
(Previously, Vol. 05, Issue 10, pp. 1483 – 1492)

Applications and Applied Mathematics:
An International Journal
(AAM)

Reliability Measures of a Three-State Complex System: A Copula Approach

Mangey Ram
Department of Mathematics
Graphic Era University
Dehradun, India
drmrswami@yahoo.com

Received: April 12, 2010; Accepted: September 10, 2010

Abstract

Improvement in reliability and production play a very important role in system design. The two key factors, considered in predicting system reliability, are failure distribution of the component and system configuration. This research discusses the mathematical modeling of a highly reliable complex system, which is in three states i.e. normal, partial failed (degraded state) and complete failed state. The system, partial failed is due to the partial failure of internal components or redundancies and completely failed is due to catastrophic failure of the system. Repair rates are general functions of the time spent. All the transition rates are constant except for one transition where two types of repair namely exponential and general possible are tackled with the help of Gumbel-Hougaard family of copula. By employing the supplementary variable technique, Laplace transformation, various transition state probabilities, availability and cost analysis (predictable profit) are obtained along with steady-state behavior of the system. Inversions have also been carried out so as to obtain time dependent probabilities, which find out availability of the system at any instant. At last some special cases of the system have been taken.

Keywords: Reliability Prediction; Availability; Cost benefit analysis; Maintenance

MSC 2010 No.: 62N05, 68M15, 90B25

1. Introduction

It has been frequently observed that the data from repairable complex systems usually contain more information than just the failure times. For instance, there may be in addition to failure times some information on the identity of the failed component, type of failure, type of repair, etc. One of the reasons for this could be that if the system is repaired after failure and the system is put in operation, then it might again fail and so on.

Many researchers such as Gupta and Agarwal (1984), Garg and Goel (1985), Gupta and Sharma (1993), Cui and Li (2007), Pandey et al. (2008), and Ram and Singh (2008) considered availability and cost analysis of the repairable systems, with different types of failures and one type of repair; but, omitted to analyze the system of three states. Further, the system possibility of two different types of repair between adjacent states can't be ruled out, when this possibility exists, reliability of the system can be analyzed with the help of copula discussed by Nelsen (2006).

The paper demonstrates the system considered as a three state complex system which can be in partial failed and completely failed states. The partial failed system is repaired by general repair rates. When the system fails completely due to catastrophic failure, it is repaired in two ways namely exponential and general repair rates to reach its normal state directly.

So, in comparison to the former models, a model is well thought out where two different repair facilities, available between neighboring states (S_0 and S_c where S_0 is the normal state and S_c is the completely failed state due to catastrophic failure) has been addressed. Each component of the system has a constant failure rate and general repair rate. However, the repair from state S_c to S_0 is of two types, namely exponential and general. The system is studied by using the supplementary variable technique, Laplace transformation and Gumbel-Hougaard family copula to obtain various reliability measures such as transition state probabilities, steady state probabilities, availability and cost analysis. At last some particular cases of the system are taken to highlight the different possibilities. Transition diagram for this model is shown in Figure 4.1.

2. Brief Introduction of Gumbel-Hougaard Family Copula

The family of copulas has been studied extensively by a number of authors including Nelsen (2006). The Gumbel-Hougaard family copula is as:

$$C_\theta(u_1,u_2) = \exp(-((-\log u_1)^\theta + (-\log u_2)^\theta)^{1/\theta}), \quad 1 \le \theta \le \infty .$$

For θ = 1, the Gumbel-Hougaard copula models independence, for $\theta \to \infty$ it converges to comonotonicity.

3. Assumptions

(i) Initially the system is in normal state.

(ii) The system has three states: normal (S_0), partial failed (S_p) and completely failed (S_c) state.

(iii) Each component of the system has a constant failure rate and a general repair rate. A general repair is represented that the failure rate takes a value between 0 and 1 in repairable systems.

(iv) Transition from the completely failed state S_c to the normal state S_0 follows two different repair rates.

(v) After repairing system is as good as new. Repair never damages anything.

(vi) Joint probability distribution of repair from completely failed state S_c to the normal state S_0 by Gumbel-Hougaard family copula.

4. Nomenclatures

$\phi_p(x)$	Rates of partial repair.
$\phi_c(y), \xi_c(y)$	Rates of catastrophic repair and corresponding pdf of repair times respectively.
λ_p, λ_c	Partial and Catastrophic failure rates of transition from state S_0 to S_c and S_0 to S_p respectively.
$P_k(t)$	P[at epoch t the system is in state S_k]: k=0, c, p.
$P_k(h, t)dh$	P [system is in state S_k at epoch t and has sojourned in this state for duration between h and $h+dh$]: h=x, y
u_1, u_2	Marginal distribution of random variables.
$E_p(t)$	Expected profit during the interval (0, t].
K_1, K_2	Revenue per unit time and service cost per unit time respectively.
$u_1 = e^y$, $u_2 = \phi_c(y)$	The joint probability (completely failed state S_c to normal state S_0) according to Gumbel-Hougaard family is given as: $\exp[y^\theta + \{\log\phi_c(y)\}^\theta]^{1/\theta}$.

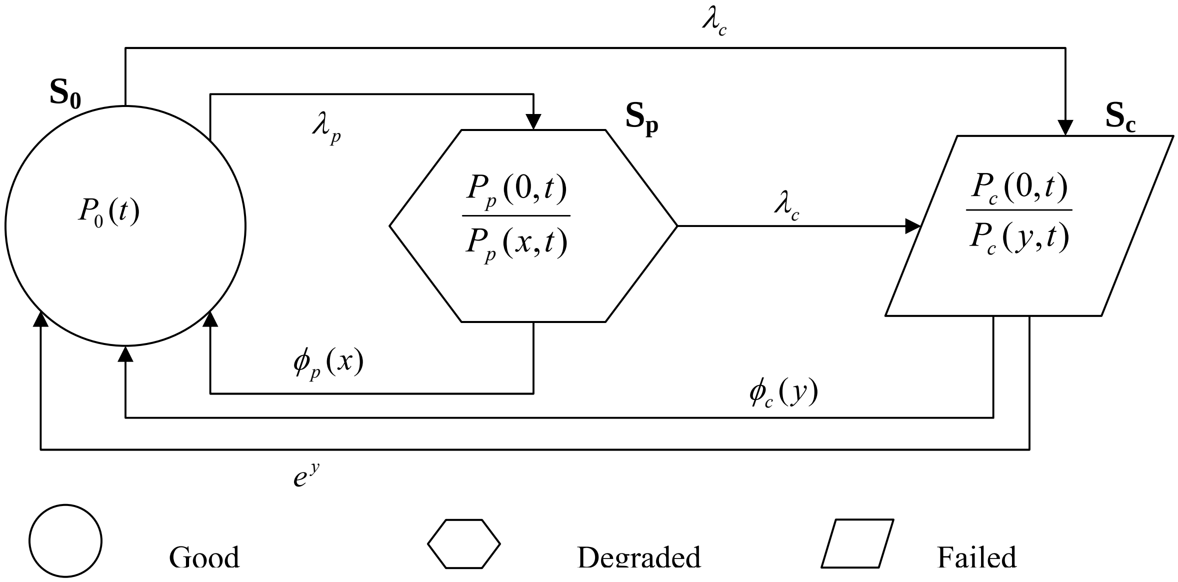

Figure 4.1: Transition State Diagram

5. Formulation and Solution of Mathematical Model

Probabilistic considerations and limiting procedures yield the following set of difference-differential equations governing the present mathematical model

$$\left[\frac{\partial}{\partial t}+\lambda_p+\lambda_c\right]P_0(t)=\int_0^\infty P_p(x,t)\phi_p(x)dx+\int_0^\infty P_c(y,t)\exp[y^\theta+\{\log\phi_c(y)\}^\theta]^{1/\theta}dy\ . \tag{1}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\phi_p(x)+\lambda_c\right]P_p(x,t)=0\,. \tag{2}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+\exp[y^\theta+\{\log\phi_c(y)\}^\theta]^{1/\theta}\right]P_c(y,t)=0\,. \tag{3}$$

Boundary conditions

$$P_p(0,t)=\lambda_p P_0(t)\ . \tag{4}$$

$$P_c(0,t)=\lambda_c P_0(t)+\lambda_c P_p(t)\,. \tag{5}$$

Initial condition

$P_0(0)=1$ and other state probabilities are zero at $t = 0$ (6)

Solving (1) to (3) with the help of (4)-(6) and Laplace transformation, one may get

$$\overline{P}_0(s)=\frac{1}{\alpha(s)-\beta(s)\gamma(s)}\ , \tag{7}$$

$$\overline{P}_p(s)=\frac{\gamma(s)}{\alpha(s)-\beta(s)\gamma(s)}, \tag{8}$$

$$\overline{P}_c(s)=\lambda_c\left[\frac{1-\overline{S}_c(s)}{s}\right][\overline{P}_0(s)+\overline{P}_p(s)], \tag{9}$$

where

$$\alpha(s)=s+\lambda_p+\lambda_c-\frac{\lambda_p\phi_p(x)}{s+\lambda_c+\phi_p(x)}-\frac{\lambda_c\exp[y^\theta+\{\log\phi_c(y)\}^\theta]^{1/\theta}}{s+\exp[y^\theta+\{\log\phi_c(y)\}^\theta]^{1/\theta}}, \tag{10}$$

$$\beta(s)=\frac{\lambda_c\exp[y^\theta+\{\log\phi_c(y)\}^\theta]^{1/\theta}}{s+\exp[y^\theta+\{\log\phi_c(y)\}^\theta]^{1/\theta}}, \text{ and} \tag{11}$$

$$\gamma(s)=\frac{\lambda_p}{s+\lambda_c+\phi_p(x)}. \tag{12}$$

6. Evaluation of Laplace Transformation of Up and Down State Probabilities

The Laplace transformations of the probabilities that the system is in up (i.e., either good or degraded state) and down state at any time are as follows:

$$\begin{aligned}\overline{P}_{\text{up}}(s)&=\overline{P}_0(s)+\overline{P}_p(s)\\&=\frac{s+\lambda_c+\lambda_p-\lambda_p\overline{S}_p(s+\lambda_c)}{(s+\lambda_c)\left[s+\lambda_c+\lambda_p-\lambda_p\overline{S}_p(s+\lambda_c)-\lambda_c\overline{S}_c(s)\right]-\left[\lambda_c\lambda_p\overline{S}_c(s)\{1-\overline{S}_p(s+\lambda_c)\}\right]}\end{aligned} \tag{13}$$

$$\overline{P}_{down}(s)=\lambda_c\left[\frac{1-\overline{S}_c(s)}{s}\right]\overline{P}_{up}(s). \tag{14}$$

7. Asymptotic Behavior of the System

By using Abel's Lemma in the Laplace transformation, that is to say:

$\underset{s\to 0}{Lim}\{s\overline{F}(s)\} = \underset{t\to\infty}{Lim} F(t) = F(say)$, where $F(t)$ is a function of time t and $\overline{F}(s)$ is the Laplace transform of $F(t)$, provided that the limit on right hand exists, the following time independent probabilities are obtained:

$$P_{\text{up}} = \frac{\lambda_c + \lambda_p - \lambda_p \overline{S}_p(\lambda_c)}{\lambda_c + \lambda_p - \lambda_p \overline{S}_p(\lambda_c) + \lambda^2{}_c M_c + \lambda_c\lambda_p M_c - \lambda_c\lambda_p M_c \overline{S}_p(\lambda_c)} \tag{15}$$

$$P_{\text{down}} = \frac{\lambda_c M_c\{\lambda_c + \lambda_p - \lambda_p \overline{S}_p(\lambda_c)\}}{\lambda_c + \lambda_p - \lambda_p \overline{S}_p(\lambda_c) + \lambda^2{}_c M_c + \lambda_c\lambda_p M_c - \lambda_c\lambda_p M_c \overline{S}_p(\lambda_c)}, \tag{16}$$

where M_c are expected duration of repair of the total failed unit i.e.,

$$M_c = \int_0^\infty y\xi_c(y)dy .$$

8. Particular Case

When repair follows exponential distribution, then setting $\overline{S}_p(s) = \frac{\phi_p(x)}{s+\phi_p(x)}$ and $S_c(s) = \frac{\exp[y^\theta + \{\log\phi_c(y)\}^\theta]^{1/\theta}}{s + \exp[y^\theta + \{\log\phi_c(y)\}^\theta]^{1/\theta}}$ in (13) through (14), the Laplace transformations of up and down state probabilities are as follows:

$$\overline{P}_{up}(s) = \frac{s + \exp[y^\theta + \{\log\phi_c(y)\}^\theta]^{1/\theta}}{s\{s + \lambda_c + \exp[y^\theta + \{\log\phi_c(y)\}^\theta]^{1/\theta}\}} \tag{17}$$

$$\overline{P}_{down}(s) = \frac{\lambda_c}{s\{s + \lambda_c + \exp[y^\theta + \{\log\phi_c(y)\}^\theta]^{1/\theta}\}} . \tag{18}$$

9. Numerical Computations

(i) Availability Analysis

Substituting $\lambda_c = 0.20, 0.40, 0.60, 0.80, \phi_c = y = \theta = 1$ in (17) and taking inverse Laplace transform, one can obtain

$$P_{up}(t) = e^{(-1.4591t)}[\cosh(1.4591t) + 0.8629292029 \sinh(1.4591t)], \quad (19)$$

$$P_{up}(t) = e^{(-1.5591t)}[\cosh(1.5591t) + 0.7434417292 \sinh(1.5591t)], \quad (20)$$

$$P_{up}(t) = e^{(-1.6591t)}[\cosh(1.6591t) + 0.6383581460 \sinh(1.6591t)], \text{and} \quad (21)$$

$$P_{up}(t) = e^{(-1.7591t)}[\cosh(1.7591t) + 0.5452219885 \sinh(1.7591t)], \quad (22)$$

Setting t =0, 1, 2, 3, 4, 5, 6, 7 units of time, in (19), (20), (21) and (22), to get the numerical values of availability and to have the Figure 9.1.

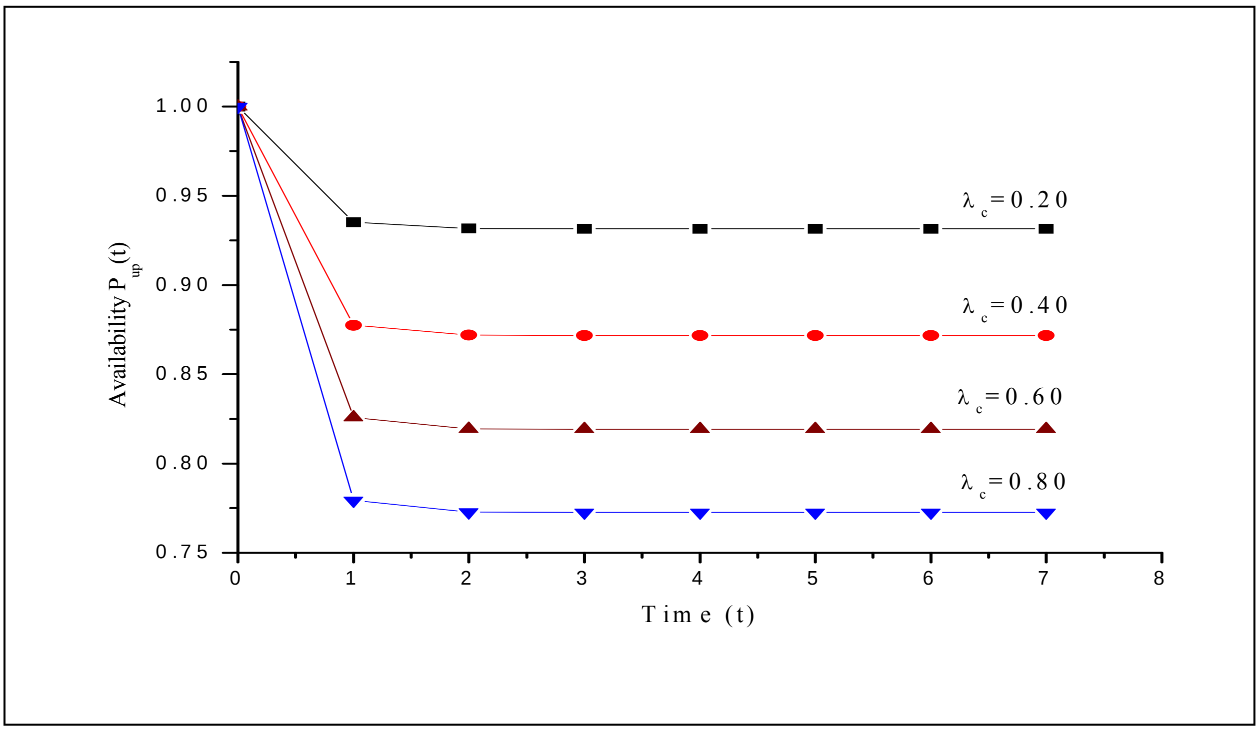

Figure 9.1. Changes of availability of the system with respect to time.

$$E_p(t) = K_1 \int_0^t P_{up}(t)dt - K_2 t \tag{23}$$

Using (19), (20), (21), and (22) for the same set of parameters in (23) respectively, one can obtain

$$E_p(t) = K_1[0.02348550427\sinh(2.9182t) - 0.2348550427\cosh(2.9182t) + 0.9314646014\,t + 0.02348550427] - K_2 t \tag{24}$$

$$E_p(t) = K_1[0.04113884145\sinh(3.1182t) - 0.04113884145\cosh(3.1182t) + 0.8717208646\,t + 0.04113884145] - K_2 t \tag{25}$$

$$E_p(t) = K_1[0.05449367940\sinh(3.3182t) - 0.05449367940\cosh(3.3182t) + 0.8191790730\,t + 0.05449367940] - K_2 t \tag{26}$$

$$E_p(t) = K_1[0.06463219992\sinh(3.5182t) - 0.06463219992\cosh(3.5182t) + 0.7726109942\,t + 0.06463219992] - K_2 t\,. \tag{27}$$

Taking K_1= 1; K_2= 0.10 and setting t =0, 1, 2, 3, 4, 5, 6, 7 units of time, in (24), (25), (26) and (27), to get the computed values of $E_p(t)$ and to have the Figure 9.2.

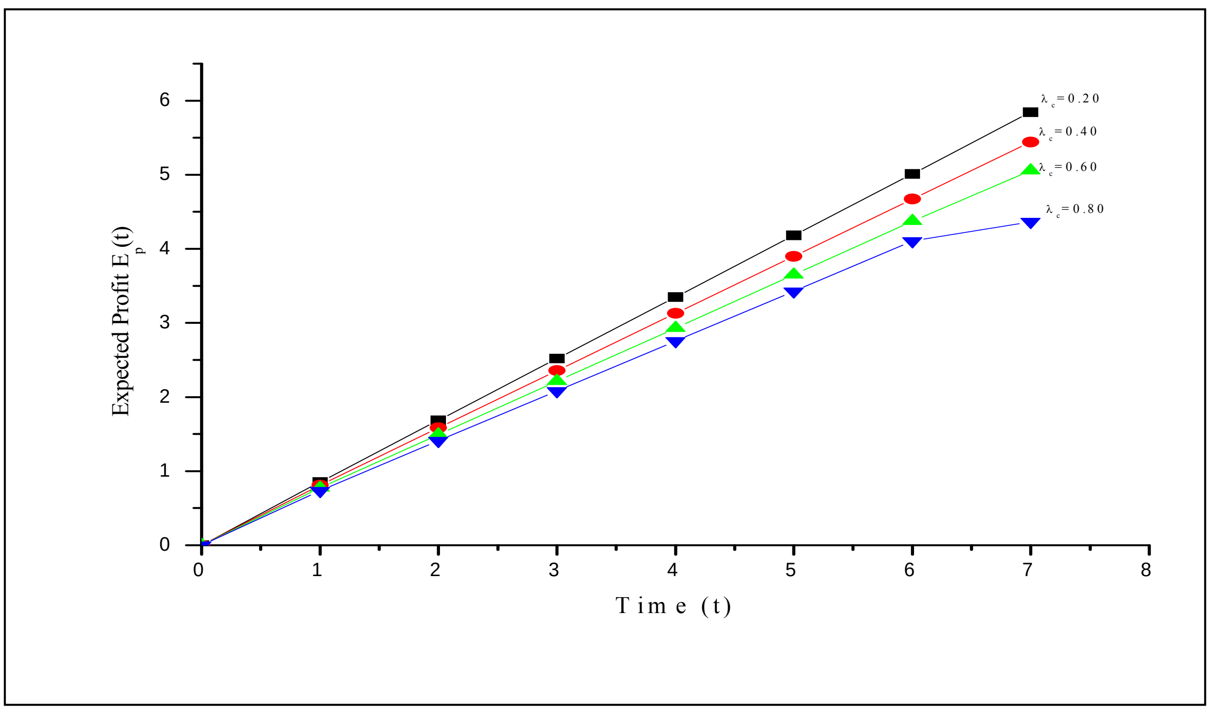

Figure 9.2. Expected profit as function of time

10. Conclusions

Fig 9.1 provides information on how availability of the complex repairable system changes with respect to time when catastrophic failure is fixed at different values. When λ_c fixed at 0.20 and taking repair makes the system hundred percent ok, the availability of the system decreases with respect to time but stabilizes at value 0.9314 in the long run. When failure rate is fixed at 0.40, the availability of the system decreases sharply during initial stage but later on stabilizes at 0.8717. Also when λ_c is fixed at 0.60 the availability of the system decreases sharply and stabilizes at value 0.8191. It is also interesting that availability of the system decreases more sharply and gives very low value with respect to other cases at λ_c=0.80 but stabilizes at value 0.7726 in the long run. The Figure 9.1 reveals that when the failure rate increases availability of the system decreases.

When the revenue cost per unit time K_1 is fixed at 1, service cost K_2 at 0.10 and failure rate λ_c varies from 0.20, 0.40, 0.60 and 0.80, one can obtain the numerical values of the expected profit for repairable system which is depicted by Fig 9.2. One can conclude by observing these graphs that as the failure rate increases, the expected profit decreases. For lower failure rates the expected profit is higher in comparison to higher failure rates.

The proposed model aims to study the reliability characteristics of a complex system with the incorporation of copula. The encouraging results presented in the paper confirm that incorporation of Gumbel-Hougaard copula improves significantly the reliability of a complex system when two adjacent states can be repaired in two different ways. Thus, in general with this study, behavior of such complex system can be analyzed and forecast in advance.

Acknowledgement

The author wishes to express his sincere thanks to Prof. A. M. Haghighi, the Editor-in-Chief and the referees of the Journal whose critical comments have significantly improved the paper in the present form. Author is also thankful to Prof. Kamal Ghanshala, Honorable President, Graphic Era University, Dehradun, India for his decent support and facilities provided for the research work.

REFERENCES

Cui, L. and Li, H. (2007). Analytical method for reliability and MTTF assessment of coherent systems with dependent components, Reliability Engineering & System Safety, Vol. 92, No. 3, pp. 300-307.

Garg, R. and Goel, L. R. (1985). Cost analysis of a system with common cause failure and two types of repair facilities. Microelectronics Reliability, Vol. 25, No. 2, pp. 281-284.

Gupta, P. P. and Agarwal, S. C. (1984). Cost function analysis of a 3-state reparable system. Microelectronics Reliability, Vol. 24, No. 1, pp. 51-53.

Gupta, P. P. and Sharma, M. K. (1993). Reliability and M.T.T.F evaluation of a two duplex-unit standby system with two types of repair. Microelectronics Reliability, Vol. 33, No. 3, pp. 291-295.

Nelsen, R. B. (2006). An Introduction to Copulas, 2nd edition. New York, NY: Springer.

Pandey, S. B., Singh, S. B. and Sharma, S. (2008). Reliability and Cost Analysis of a System with Multiple Components using Copula. Journal of Reliability and Statistical Studies, Vol. 1, No.1, pp. 25-32.

Ram, M. and Singh, S. B. (2008). Availability and Cost Analysis of a parallel redundant complex system with two types of failure under preemptive-resume repair discipline using Gumbel-Hougaard family copula in repair. International Journal of Reliability, Quality & Safety Engineering, Vol. 15, No. 4, pp. 341-365.

Biographical Note

Dr. Mangey Ram is an Assistant Professor in the Department of Mathematics, Graphic Era University, Dehradun, India. He received his PhD from the Department of Mathematics, Statistics and Computer Science, G. B. Pant University of Agriculture and Technology, Pantnagar, in 2008. His field of research is Reliability Theory/Engineering. Before he joined Graphic Era University, he was the Deputy Manager in Syndicate Bank. He has also presented his works at national and international conferences. Recently, he was awarded the “Young Scientist Award” in the 4th Uttarakhand State Science and Technology Congress-2009 by Uttarakhand State Council for Science and Technology, Dehradun.