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## Inverse Heat Conduction Problem in a Semi-Infinite Circular Plate and its Thermal Deflection by Quasi-Static Approach

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### Abstract

This paper concerns the inverse heat conduction problem in a semi-infinite thin circular plate subjected to an arbitrary known temperature under unsteady condition and the behavior of thermal deflection has been discussed on the outer curved surface with the help of mathematical modeling. The solutions are obtained in an analytical form by using the integral transform technique.

**Keywords:** Inverse thermoelastic problem, Thermal deflection, Circular plate

**AMS Classification:** 34B07, 34B40, 35G30, 35K05

## 1. Introduction

The inverse thermoelastic problem consists of determination of the temperature of the heating medium, the heat flux on the boundary surfaces of the solid when the conditions of the displacement and stresses are known at the some points of the solid under consideration. Sabherwal [(1965), (1966)], studied inverse problem in heat conduction. Cialkowski and Graya (1980); Graya and Cialkowski (1981); Graya and Kozlowski (1982), investigated one dimensional transient thermoelastic problem and derived the heating temperature and heat flux on the surface of isotropic infinite slab. Ashida et al. (2006) attempted an inverse thermoelastic problem in an isotropic plate associated with a piezoelectric ceramic plate. Deshmukh and Wankhede (1996) solved an inverse problem of thermo elasticity in a thin circular plate by determining the temperature on the curved surface of the plate, displacement and thermal stresses using quasi-static approach by employing integral transform techniques. Khobragade and Deshmukh (1998), studied an inverse axially symmetric quasi-static problem of thermo elasticity for a thin clamped circular plate in which a heat flux is prescribed on an internal cylindrical surface of the plate and suitable heat exchange conditions are met on the upper and lower surfaces of the plate is solved with the help of a generalized integral transform technique. Tikhe and Deshmukh (1968) studied the inverse heat conduction problem in a thin circular plate and its thermal deflection on the outer curved surface. Recently Deshmukh et al. (1998) studied a quasi-static thermal deflection of a thin clamped circular plate due to heat generation.

In this paper we consider two dimensional non-homogeneous boundary value problem of heat conduction and studied the thermal deflection of a semi-infinite circular plate on the outer curved surface for an infinite length. A circular plate is subjected to arbitrary known temperature under unsteady state condition. Initially the plate is at zero temperature and the lower surface is at zero temperature. The governing heat conduction equation has been solved by using integral transform method. The results are obtained in series form in terms of Bessel's functions. Mathematical model has been constructed of a semi-infinite circular plate with the help of numerical illustration. No one previously studied such type of problem. This is a new contribution to the field.

The inverse problem is very important in view of its relevance to various industrial mechanics subjected to heating such as the main shaft of lathe, turbines, and the role of the rolling mill. Also, arise the quenching studies, the analysis of experimental data and measurement of aerodynamic heating.

## 2. Formulation of the Problem

Consider a semi-infinite circular plate defined by  $0 \leq r \leq a, 0 \leq z < \infty$ . Let the plate be subjected to arbitrary known interior temperature  $f(z, t)$  within the region  $0 \leq r \leq a$ , with lower surface  $z = 0$  is at zero temperature. Under these more realistic prescribed conditions, the unknown

temperature on the outer curved surface of plate at  $r = a$  and quasi-static thermal deflection due to unknown temperature  $g(z, t)$  are required to determine.

The differential equation satisfying the deflection function  $\omega(r, t)$  as in 3, is given as

$$\nabla^4 \omega = -\frac{\nabla^2 M_T}{D(1-\nu)}, \quad (1)$$

where,  $M_T$  is the thermal moment of the plate defined as

$$M_T = a_t E \int_0^h T(r, z, t) z dz, \quad (2)$$

$D$  is the flexural rigidity of the plate denoted as

$$D = \frac{Eh^3}{12(1-\nu^2)}, \quad (3)$$

$a_t$ ,  $E$  and  $\nu$  are the coefficients of the linear thermal expansion, the Young's modulus and Poisson's ratio of the plate material respectively and

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r}. \quad (4)$$

Since, the edge of the circular plate is fixed and clamped;

$$\omega = \frac{\partial \omega}{\partial r} = 0 \text{ at } r = a. \quad (5)$$

Initially  $T = \omega = 0$ , when  $t = 0$ . (6)

The temperature of the semi-infinite circular plate satisfies the heat conduction equation, as in Deshmukh et al. (2009),

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{k} \frac{\partial T}{\partial t} \text{ in } 0 \leq r \leq a, 0 \leq z < \infty, t > 0, \quad (7)$$

with the conditions

$$T(r, z, t) = g(z, t) \text{ (unknown) at } r = a, 0 \leq z < \infty, \quad (8)$$

$$T(r, z, t) = 0 \text{ at } z = 0, 0 \leq r \leq a, \quad (9)$$

$$T(r, z, t) = 0 \text{ at } z = \infty, 0 \leq r \leq a, \quad (10)$$

$$T(r, z, t) = f(z, t) \text{ (known) at } r = \xi, 0 < \xi < a, 0 \leq z < \infty, \quad (11)$$

where  $k$  is thermal diffusivity of the semi-infinite circular plate.

Equations (1) - (11) constitute the mathematical formulation of the inverse thermoelastic problem in a semi-infinite circular plate.

### 3. The Solution

#### Unknown Temperature

By applying Fourier sine integral transform and its inversion over the heat conduction equation, one obtained the expression for temperature distribution function and unknown heating temperature as

$$T(r, z, t) = \frac{2k}{\xi} \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} \lambda_n \frac{J_0(\lambda_n r)}{J_1(\lambda_n \xi)} \int_{\beta=0}^{\infty} \sin(\beta z) \left[ \int_{t'=0}^t \bar{f}(\beta, t') e^{-k[\beta^2 + \lambda_n^2](t-t')} dt' \right] d\beta \quad (12)$$

$$g(z, t) = \frac{2k}{\xi} \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} \lambda_n \frac{J_0(\lambda_n a)}{J_1(\lambda_n \xi)} \int_{\beta=0}^{\infty} \sin(\beta z) \left[ \int_{t'=0}^t \bar{f}(\beta, t') e^{-k[\beta^2 + \lambda_n^2](t-t')} dt' \right] d\beta, \quad (13)$$

where

$$\bar{T}(r, \beta, t) = \int_{z=0}^{\infty} K(\beta, z) T(r, z, t) dz,$$

$$T(r, z, t) = \int_{\beta=0}^{\infty} K(\beta, z) \bar{T}(r, \beta, t) d\beta,$$

$$K(\beta, z) = \sqrt{\frac{2}{\pi}} \sin(\beta z),$$

and  $\lambda_1, \lambda_2, \dots$  are the positive roots of the transcendental equation  $J_0(\lambda_n \xi) = 0$ .

#### Thermal Deflection

Using equation (12) into equation (2), one obtains

$$M_T = a_t E \left( \frac{2k}{\xi} \right) \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} \lambda_n \frac{J_0(\lambda_n r)}{J_1(\lambda_n \xi)} \int_{z=0}^{\infty} z \left\{ \int_{\beta=0}^{\infty} \sin(\beta z) \left[ \int_{t'=0}^t \bar{f}(\beta, t') e^{-k[\beta^2 + \lambda_n^2](t-t')} dt' \right] d\beta \right\} dz. \quad (14)$$

Assume the solution of the equation (1) satisfy condition (5) as

$$\omega(r, t) = \sum_{n=1}^{\infty} C_n(t) \left[ 2aJ_0(\lambda_n r) + r^2 \lambda_n J_1(\lambda_n a) - 2aJ_0(\lambda_n a) - a^2 \lambda_n J_1(\lambda_n a) \right]. \quad (15)$$

It can be easily seen that the plate has been built-in-edge

$$\omega(r, t) = \frac{\partial \omega}{\partial r} = 0 \quad \text{at } r = a.$$

Using equation (15) and (16) in equation (1), one obtains

$$C_n(t) = \sqrt{\frac{2}{\pi}} \frac{a_t E}{2a D(1-\nu)} \left( \frac{2k}{\xi} \right) \frac{1}{\lambda_n} \frac{1}{J_1(\lambda_n \xi)} \int_{z=0}^{\infty} z \left\{ \int_{\beta=0}^{\infty} \sin(\beta z) \left[ \int_{t'=0}^t \bar{f}(\beta, t') e^{-k[\beta^2 + \lambda_n^2](t-t')} dt' \right] d\beta \right\} dz. \quad (16)$$

Substituting equation (16) into equation (15), one obtains the expression for thermal deflection as

$$\omega(r, t) = \sqrt{\frac{2}{\pi}} \frac{a_t E}{2a D(1-\nu)} \sum_{n=1}^{\infty} \left\{ \left( \frac{2k}{\xi} \right) \frac{1}{\lambda_n} \frac{1}{J_1(\lambda_n \xi)} \left[ 2aJ_0(\lambda_n r) + r^2 \lambda_n J_1(\lambda_n a) - 2aJ_0(\lambda_n a) - a^2 \lambda_n J_1(\lambda_n a) \right] \right. \\ \left. \times \left( \int_{z=0}^{\infty} z \left\{ \int_{\beta=0}^{\infty} \sin(\beta z) \left[ \int_{t'=0}^t \bar{f}(\beta, t') e^{-k[\beta^2 + \lambda_n^2](t-t')} dt' \right] d\beta \right\} dz \right) \right\}. \quad (17)$$

#### 4. Special Case and Numerical Calculations

To construct the mathematical thermoelastic behavior of a semi-infinite circular plate, we considered the following functions and parameters:

Set  $f(z, t) = (1 - e^{-\omega t}) z \cdot e^{-z}$  with  $\omega > 0$ ,  $t \rightarrow t' = 5$  sec.

##### Dimensions

Radius of the circular plate  $a = 1$  m. Thickness of a circular plate  $z = 0.2$  m.

## Material properties

The numerical calculation has been carried out for an aluminum (pure) circular plate with the material properties as

Thermal diffusivity  $k = 84.18 \times 10^{-6} (m^2 s^{-1})$ .

Density  $\rho = 2707 \text{ kg} / m^3$ .

Specific heat  $c_p = 896 \text{ J} / \text{kgK}$ ,

Poisson ratio  $\nu = 0.35$ ,

Coefficient of linear thermal expansion  $a_t = 22.2 \times 10^{-6} \frac{1}{K}$ , and

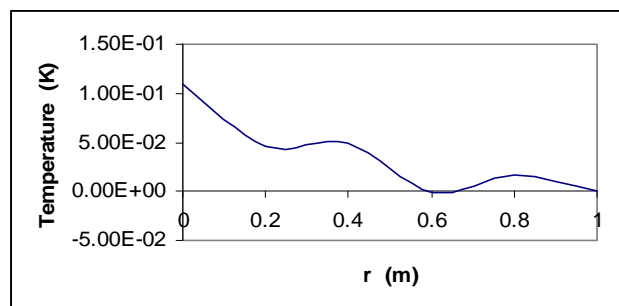
Lamé constant  $\mu = 26.67$ .

## Roots of the transcendental equation:

The  $\lambda_1 = 2.4048$ ,  $\lambda_2 = 5.5201$ ,  $\lambda_3 = 28.6537$ ,  $\lambda_4 = 11.7915$ ,  $\lambda_5 = 14.9309$ ,  $\lambda_6 = 18.0711$ ,  $\lambda_7 = 21.2116$ ,  $\lambda_8 = 24.3525$ ,  $\lambda_9 = 27.4935$ ,  $\lambda_{10} = 30.6346$  are the positive roots of the transcendental equation  $J_0(\lambda_n \xi) = 0$ .

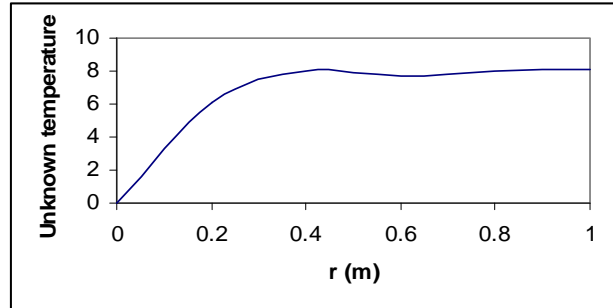
We set for convenience,  $X = 10^7$  and  $Y = \frac{10^5 a_t E h}{D(1-\nu)}$  which assume to be the constants.

The numerical calculation has been carried out with the help of computational mathematical software Mathcad-2000 and graphs are plotted with the help of Excel (MS Office-2003).



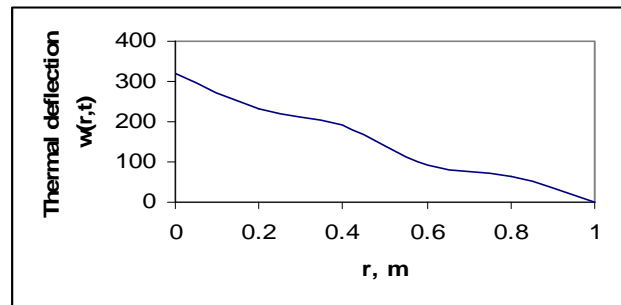
**Figure 1.** Temperature distribution

From the Figure 1, it can be observed that, the temperature decreases from the centre  $r = 0$  of a plate to the circular boundary  $r = 1$ . It is zero at the outer circular boundary  $r = 1$ .



**Figure 2.** Unknown Temperature distributions

From the Figure 2, it can be observed that the unknown temperature increases in the axial direction at  $z = 0.08$  and then remains constant thereafter.



**Figure 3.** Thermal deflection

From the Figure 3, it can be observed that, for aluminum circular plate the deflection is maximum at the centre and it is zero at the outer circular edge. Also, the variation in the deflection has shown when the plate is fixed and clamped.

## 5. Discussion and Results

In this paper we have extended the work of Deshmukh et al. (1996) for a two dimensional non-homogeneous boundary value problem of heat conduction and have determined the expressions of temperature, unknown temperature and thermal deflection.

As a special case the mathematical model is constructed for a semi-infinite thin circular plate made up of pure aluminum metal.



From the Figures 1, 2, and 3, it is observed that the temperature, unknown temperature and the thermal deflection respectively are negligible for an infinite length and also observed that the variations in temperature distribution, unknown heating temperature and thermal deflection take place for finite values of parameters and functions. This type of a problem has the applications in various industrial machines such as lathe machine and turbine. Also any particular case of special interest can be derived by assigning suitable values to the parameter and function in the expressions (12), (13) and (17).

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## **REFERENCES**

- Ashida, F., Choi, J., Noda, N. (1996). An inverse thermoelastic problem in an isotropic plate associated with a piezoelectric ceramic plate, *J. Thermal Stresses*, 19, 153-167.
- Deshmukh, K.C., and Wankhede, P.C. (1998). An inverse quasi-static transient thermoelastic problem in a thin circular plate, *Bull. Pure and Appl. Sci.*, 17, 209-207.
- Deshmukh, K. C., Warbhe, S.D., Kulkarni, V.S. (2009). Quasi-static thermal deflection of a thin clamped circular plate due to heat generation, *J. Thermal Stresses*, 32, 877-886.
- Grysa, K. and Cialkowski, M. J. (1980). On a certain inverse problem of temperature and thermal stress field, *Acta Mechanica*, 36, 169-185.
- Grysa, K., and Cialkowski, M.J. (1981). *Nuclear Engg.*, Dec. 64, pp. 169-184.
- Grysa, K., and Kozlowski, Z. (1982). One dimensional problems of temperature and heat flux determination at the surfaces of thermoelastic slab, *Nuclear Engg.*, Dec. 74, 1-14.
- Khobragade, N.L. and Deshmukh, K.C. (2005). An inverse quasi-static thermal deflection problem for a thin clamped circular plate, *J. Thermal Stresses*, 28, 353-361.
- Ozisik, M.N. (1968). *Boundary value problems of heat conduction*, International Text Book Company, Scranton, Pennsylvania.
- Sabherwal, K.C. (1965). *Indian Journal of Pure and Applied Physics*, vol.3, No.10; pp. 397-398.
- Sabherwal, K.C. (1966). *Indian Journal of Pure and Applied Physics*, Vol. XL, No.8, pp. 448-450, 1966.
- Tikhe, A.K. and Deshmukh, K.C. (2006). Inverse heat conduction problem in a thin circular plate and its thermal deflection, *J. Applied Mathematical Modelling*, 30, 554-560.