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Anti-synchronization Scheme for the Stability Analysis of a Newly Designed Hamiltonian Chaotic System Based on Hénon-Heiles Model Using Adaptive Control Method

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Abstract

In this paper, a systematic approach to investigate the anti-synchronization among identical Hamiltonian chaotic systems has been proposed by using adaptive control method (ACM). Initially an adaptive controller and parameter update law are described to achieve asymptotical stability of state variables of given system with uncertain parameters using Lyapunov stability theory (LST). In addition, numerical simulations using MATLAB software are performed to validate the efficacy and effectiveness of the designed technique. Moreover, the proposed technique has numerous applications in encryption and secure communication.

Keywords: Adaptive control; Anti-synchronization; Chaotic system; Lyapunov stability theory; MATLAB

MSC 2010 No.: 34K23, 34K35, 37B25, 37N35

1. Introduction

Chaos theory is an intriguing field of applicable mathematics that focuses on the behaviour analysis of extremely irregular or disordered nonlinear dynamical systems mainly found in nature and plays a vital role in numerous fields, for instance, secure communication (Li and Liao (2004)), robotics (Patle et al. (2018)), neural networks (Bouallegue (2017)), biomedical engineering (Provata et al. (2012)), ecological models (Sahoo and Poria (2014)), weather models (Russell et al. (2017)), chemical reactions (Han et al. (1995)), finance models (Tong et al. (2015)), oscillations (Ghosh et al. (2018)), jerk systems (Wang et al. (2017)), encryption (Wu et al. (2016)), etc. Subsequently, chaos synchronization as well as control have sought significant attention in several research fields.

A significant characteristic of chaotic systems, described as "Butterfly Effect", is high sensitivity dependence on initial conditions. This property of chaotic systems was first reported in 1963 by E.N. Lorenz (1963) while analysing a weather prediction model. More importantly, Pecora and Carroll (1990) first introduced the notion of chaos synchronization in 1990. In chaos synchronization phenomenon, the state trajectories of two or more chaotic/ hyperchaotic systems are regulated to follow the similar dynamics. In recent years, chaos synchronization of chaotic systems using various control techniques has become a fascinating and an engaging area of study for researchers and scientists. Many significant techniques are introduced and studied to control, namely, Delavari and Mohadeszadeh (2018), Rasappan and Vaidyanathan (2012), Chen and Han (2003), Li and Zhang (2016), Khan and Chaudhary (2019), and synchronization, viz., Singh et al. (2017), Li and Zhang (2016), Sudheer and Sabir (2009), Khan and Chaudhary (2020a), Khan and Chaudhary (2020a), Zhou and Zhu (2011), Li and Liao (2004), Ma et al. (2017), Ding and Shen (2016), Khan and Chaudhary (2019), Khan and Chaudhary (2020a), Khan and Chaudhary (2020a), Khan and Chaudhary (2020), Kumar et al. (2020) of chaos occurring in dynamical systems.

Specifically, Hubler (1989) firstly introduced adaptive control method (ACM) in chaotic systems in 1989 . Since then, many researches have been conducted using ACM, for instance, Liao and Tsai (2000), Yassen, Li et al. (2012), Vaidyanathan (2015), Khan and Chaudhary (2019), Khan and Chaudhary (2020a), Khan and Chaudhary (2020c), Khan and Chaudhary (2020e), Khan and Chaudhary (2020), Kumar et al. (2020), Khan and Chaudhary (2021), Khan and Chaudhary (2021). Keeping the above discussions in view, our primal aim in this paper is to study anti-synchronization among identical newly described Hamiltonian chaotic systems by Vaidyanathan et al. (2018) based on Hénon-Heiles model using ACM. Basically, Henon and Heiles (1964) first modeled the Hénon-Heiles model in 1964 which describes the nonlinear motion of a star around a galactic centre with the motion restricted to a plane.

This paper is organized as follows. Section 2 consists of few preliminaries to be used throughout the paper. Section 3 describes the basic structured characteristics of the considered Hamiltonian chaotic system in detail. Section 4comprises of the anti-synchronization method for the given system via ACM. Section 5 contains the numerical simulations which are displayed graphically using MATLAB. Section 6 concludes the present paper.

2. Preliminaries

The master system and the corresponding slave system are defined as:

$$\dot{x}_m = F_1(x_m), \tag{1}$$

$$\dot{x}_s = F_2(x_s) + u, \tag{2}$$

where $x_m = (x_{m1}, x_{m2}, ..., x_{mn})^T$, $x_s = (x_{s1}, x_{s2}, ..., x_{sn})^T$ are the state variables of (1) and (2) respectively, $F_1, F_2 : R^n \to R^n$ are two nonlinear continuous vector functions and $u = (u_1, u_2, ..., u_n) \in R^n$ is the properly designed controller.

We define the anti-synchronization error as:

$$e = x_s + x_m.$$

Definition 2.1.

The master system (1) and the slave system (2) are said to be in anti-synchronization state if

$$lim_{t\to\infty}\|e(t)\| = lim_{t\to\infty}\|x_s(t) + x_m(t)\| = 0. \tag{3}$$

3. System Description

Introduced by Vaidyanathan et al. (2018), the considered chaotic system is given by

$$\begin{cases} \dot{x}_{m1} = x_{m2}, \\ \dot{x}_{m2} = -x_{m1} - 2x_{m1}x_{m3} + ax_{m1}^2, \\ \dot{x}_{m3} = x_{m4}, \\ \dot{x}_{m4} = -x_{m3} - x_{m1}^2 + x_{m3}^2 + bx_{m3}^4, \end{cases} \tag{4}$$

where $(x_{m1}, x_{m2}, x_{m3}, x_{m4})^T \in R^4$ is the state vector and a and b are positive parameters. When $a = 0.98$ and $b = 0.99$, the system (4) displays chaos phenomenon. Also, the Lyapunov exponents of system (4) are $LE_1 = 0.0015$, $LE_2 = 0$, $LE_3 = 0$, $LE_1 = -0.0015$. Furthermore, Figure 1(a-f) exhibit the phase plots of (4). In addition, Figure 2(a-d) display time-series of the given system. However, the detailed analytic study and numerical results for the system (4) can be found in Vaidyanathan et al. (2018).

4. Illustrative example

Here, we discuss anti-synchronization scheme to design the laws which estimate parameters with adaptive controllers in such a manner that the state vector x_{m1}, x_{m2}, x_{m3} and x_{m4} approaches to equilibrium points as t tends to infinity.

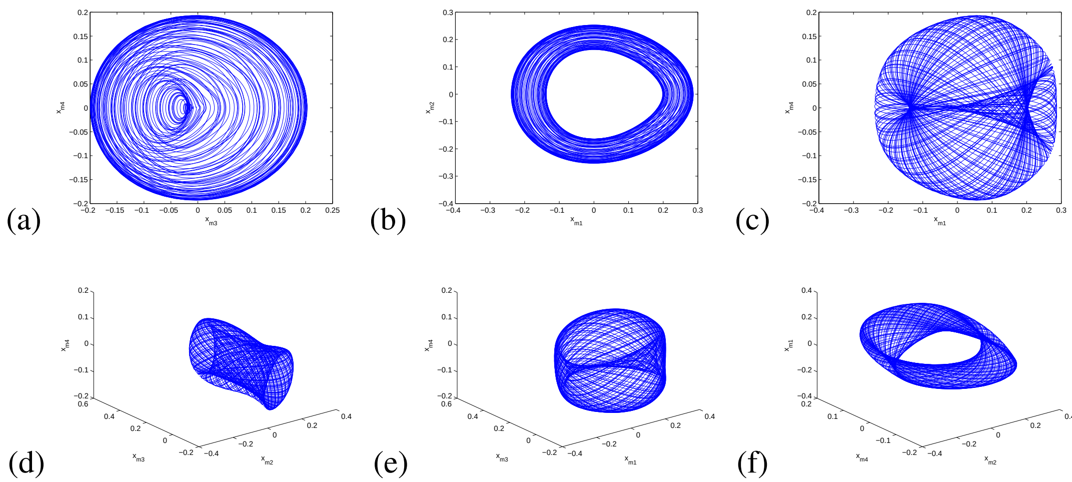

Figure 1. Phase plots of Hamiltonian chaotic system in (a) $x_{m3} - x_{m4}$ plane, (b)$x_{m1} - x_{m2}$ plane, (c) $x_{m1} - x_{m4}$ plane, (d) $x_{m2} - x_{m3} - x_{m4}$ space, (e) $x_{m1} - x_{m3} - x_{m4}$ space, (f) $x_{m2} - x_{m4} - x_{m1}$ space

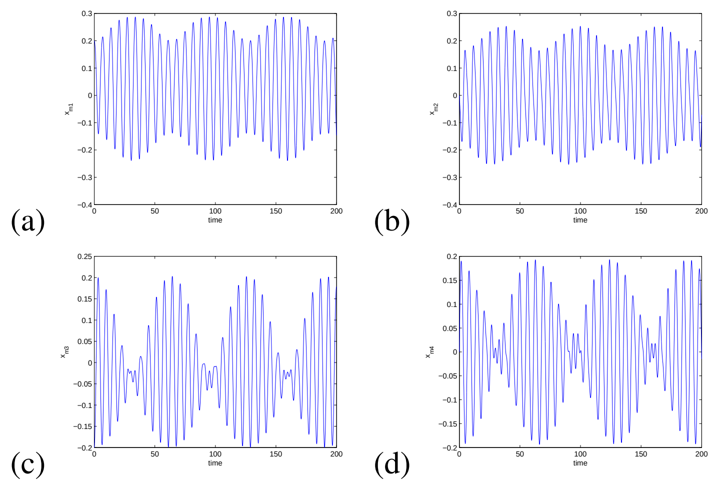

Figure 2. Time series of a new Hamiltonian chaotic system in (a) (t, x_{m1}), (b) (t, x_{m2}), (c) (t, x_{m3}), (d) (t, x_{m4})

Conveniently, the system (4) is taken as the master system and the corresponding slave system can be defined as:

$$\begin{cases} \dot{x}_{s1} = x_{s2} + u_1, \\ \dot{x}_{s2} = -x_{s1} - 2x_{s1}x_{s3} + ax_{s1}^2 + u_2, \\ \dot{x}_{s3} = x_{s4} + u_3, \\ \dot{x}_{s4} = -x_{s3} - x_{s1}^2 + x_{s3}^2 + bx_{s3}^4 + u_4, \end{cases} \tag{5}$$

where u_1, u_2, u_3 and u_4 are adaptive nonlinear controllers to be constructed so that anti-synchronization between two identical Hamiltonian chaotic systems will be attained.

Define the state errors as

$$\begin{cases} e_{11} = x_{s1} + x_{m1}, \\ e_{12} = x_{s2} + x_{m2}, \\ e_{13} = x_{s3} + x_{m3}, \\ e_{14} = x_{s4} + x_{m4}. \end{cases} \tag{6}$$

Our ultimate objective here is to design controllers $u_i, (i = 1, 2, 3, 4)$ so that the synchronization errors defined in (6) satisfy

$$lim_{t\to\infty} e_{1i}(t) = 0 \ \ for \ \ (i = 1, 2, 3, 4).$$

The transformed error dynamics is given by

$$\begin{cases} \dot{e}_{11} = e_{12} + u_1, \\ \dot{e}_{12} = -e_{11} - 2(x_{s1}x_{s3} + x_{m1}x_{m3}) + a(x_{s1}^2 + x_{m1}^2) + u_2, \\ \dot{e}_{13} = e_{14} + u_3, \\ \dot{e}_{14} = -e_{13} - (x_{s1}^2 + x_{m1}^2) + (x_{s3}^2 + x_{m3}^2) + b(x_{s3}^4 + x_{m3}^4) + u_4. \end{cases} \tag{7}$$

Now, the adaptive controllers are designed as:

$$\begin{cases} u_1 = -e_{12} - K_1 e_{11}, \\ u_2 = e_{11} + 2(x_{s1}x_{s3} + x_{m1}x_{m3}) - \hat{a}(x_{s1}^2 + x_{m1}^2) - K_2 e_{12}, \\ u_3 = -e_{14} - K_3 e_{13}, \\ u_4 = e_{13} + (x_{s1}^2 + x_{m1}^2) - (x_{s3}^2 + x_{m3}^2) - \hat{b}(x_{s3}^4 + x_{m3}^4) - K_4 e_{14}, \end{cases} \tag{8}$$

where $K_1 > 0$, $K_2 > 0$, $K_3 > 0$, $K_4 > 0$ are gain constants.

By substituting the controllers (8) in error dynamics (7), one finds that

$$\begin{cases} \dot{e}_{11} = -K_1 e_{11}, \\ \dot{e}_{12} = (a - \hat{a})(x_{s1}^2 + x_{m1}^2) - K_2 e_{12}, \\ \dot{e}_{13} = -K_3 e_{13}, \\ \dot{e}_{14} = (b - \hat{b})(x_{s3}^4 + x_{m3}^4) - K_4 e_{14}. \end{cases} \tag{9}$$

where $\hat{a}$, $\hat{b}$ are estimated quantities of unknown parameter a, b respectively.

Then, parameter estimation error is defined as:

$$\tilde{a} = a - \hat{a}, \tilde{b} = b - \hat{b}. \tag{10}$$

Using (10), the error dynamics (9) is written as:

$$\begin{cases} \dot{e}_{11} = -K_1 e_{11}, \\ \dot{e}_{12} = \tilde{a}(x_{s1}^2 + x_{m1}^2) - K_2 e_{12}, \\ \dot{e}_{13} = -K_3 e_{13}, \\ \dot{e}_{14} = \tilde{b}(x_{s3}^4 + x_{m3}^4) - K_4 e_{14}. \end{cases} \tag{11}$$

On differentiation of parameter estimation error (10), we get

$$\dot{\tilde{a}} = -\dot{\hat{a}}, \dot{\tilde{b}} = -\dot{\hat{b}}. \tag{12}$$

Considering the classic Lyapunov function as:

$$V = \frac{1}{2}[e_{11}^2 + e_{12}^2 + e_{13}^2 + e_{14}^2 + \tilde{a}^2 + \tilde{b}^2], \tag{13}$$

which imply that V is positive definite.

Derivative of Lyapunov function V is written as:

$$\dot{V} = e_{11}\dot{e}_{11} + e_{12}\dot{e}_{12} + e_{13}\dot{e}_{13} + e_{14}\dot{e}_{14} - \tilde{a}\dot{\hat{a}} - \tilde{b}\dot{\hat{b}}. \tag{14}$$

Keeping (14) in mind, we are defining the parameter estimates laws as:

$$\begin{cases} \dot{\hat{a}} = (x_{s1}^2 + x_{m1}^2)e_{12} + K_5\tilde{a}, \\ \dot{\hat{b}} = (x_{s3}^4 + x_{m3}^4)e_{14} + K_6\tilde{b}, \end{cases} \tag{15}$$

where $K_5 > 0$ and $K_6 > 0$ are gain constants.

Theorem 4.1.

The chaotic systems (4)-(5) are asymptotically anti-synchronized for all initial states $(x_{m1}(0), x_{m2}(0), x_{m3}(0), x_{m4}(0)) \in R^4$ by the designed adaptive controller (8) and the parameter update law (15).

Proof:

The Lyapunov function V as formulated in (13) is a positive definite function. On simplification, equations (14) and (15) give rise to

$$\dot{V} = -K_1e_{11}^2 - K_2e_{12}^2 - K_3e_{13}^2 - K_4e_{14}^2 - K_5\tilde{a}^2 - K_6\tilde{b} < 0,$$

ensuring that $\dot{V}$ is negative definite .

Thus, using Lyapunov stability theory, we conclude that synchronization error $e(t) \to 0$ exponentially as $t \to \infty$ for all initial conditions $e(0) \in R^4$. This completes the proof. ■

5. Numerical Simulation and Discussion

This section performs some numerical simulations to illustrate effectively the proposed anti-synchronization technique via ACM. The parameters of given system are selected as $a = 0.98$ and $b = 0.99$ to establish the chaoticity of considered system without control inputs. The initial states of the master (4) and slave systems (5) are $(x_{m1}(0) = 0.2, x_{m2}(0) = 0, x_{m3}(0) = -0.2, x_{m4}(0) = 0)$ and $(x_{s1}(0) = 0.2, x_{s2}(0) = 0.2, x_{s3}(0) = -0.2, x_{s4}(0) = 0)$, respectively. The control gains are taken as $K_i = 6$ for $i = 1, 2, ..., 6$. Further, simulation results concerning the state anti-synchronized trajectories of systems (4) and (5) are displaced in Figure 3 (a-d). Moreover, Figure 4 (a-e) show that the synchronization error $(e_{11}, e_{12}, e_{13}, e_{14}) = (0.4, 0.2, 0, 0)$ converging to zero as t tending to infinity. In Figure 5(a) and (b) it is noted that the estimated quantities $(\hat{a}, \hat{b})$ of unknown parameters converging to their original values asymptotical with time. Hence, the proposed anti-synchronization strategy among master and slave system is achieved computationally.

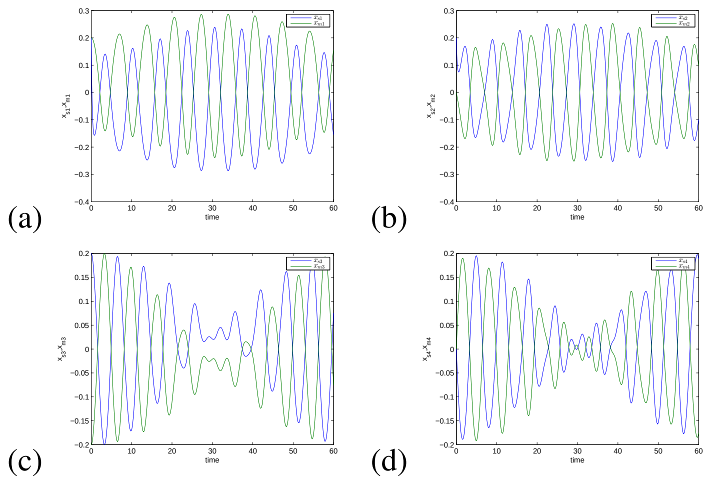

Figure 3. Anti-synchronization of Hamiltonian chaotic system (a) between $x_{m1}(t) - x_{s1}(t)$, (b) between $x_{m2}(t) - x_{s2}(t)$, (c) between $x_{m3}(t) - x_{s3}(t)$, (d) between $x_{m4}(t) - x_{s4}(t)$

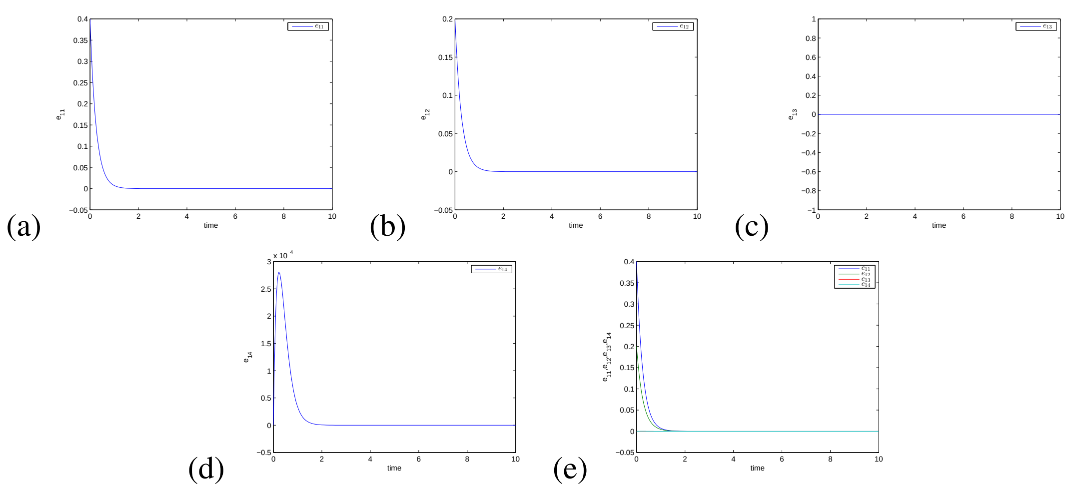

Figure 4. Dynamics in synchronization error states (a) (t, e_{11}), (b) (t, e_{12}), (c) (t, e_{13}), (d) (t, e_{14}), (e) $(t, e_{11}, e_{12}, e_{13}, e_{14})$

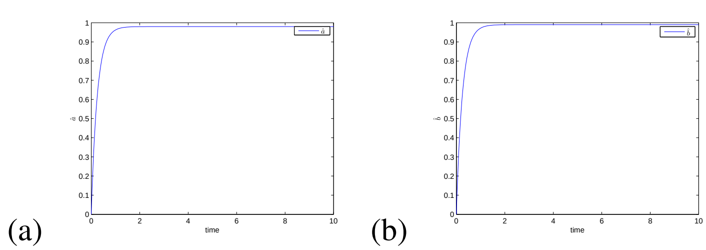

Figure 5. Time history of parameter estimates of Hamiltonian chaotic system (a) $(t, \hat{a})$, (b) $(t, \hat{b})$

6. Conclusion

In this paper, anti-synchronization among integer order Hamiltonian chaotic systems has been conducted using ACM keeping Lyapunav stability theory in view. Further numerical simulations in MATLAB are performed to validate the efficacy of the proposed technique. Remarkably, the theoretical results completely agree with the computational results. Such strategy may be used to control the nonlinear motion of a star around a galactic centre with motion restricted to a plane. Furthermore, the proposed scheme may find applications in the area of image encryption and secure communication.

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