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Numerical Technique for Solving Fractional-order of IVGTT Glucose-insulin Interaction

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Abstract

In this paper, we use a numerical technique to solve fractional nonlinear differential equations systems that arise in Bergman's minimal model that describe blood glucose and insulin metabolism, after intravenous tolerance testing. Shifted Jacobi -Gauss-Radau collocation (SJ-GR-C) method is developed for approximating the proposed model. The principal target in our technique is to transform the proposed model to a system of algebraic equations. Finally, numerical simulation is introduced to illustrate the analytical results.

Keywords: Bergman's minimal model; Jacobi-Gauss-Radau collocation method; Fractional order of IVGTT glucose-insulin interaction

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1. Introduction

Mathematical modeling is major into infectious disease epidemiology. They describe the dynamical evolution of infectious diseases which improve our understanding and predictive ability. Using

this approach, we can convert a real world problem into a mathematical model and then analyze it in a better manner. The dynamic relationship between glucose and its controlling hormone insulin has been mathematically modeled and studied by many researchers. One of the mostly used models is generally known as the minimal model which was first published in 1979 and modified in 1986. The "Minimal Model" plays an important role in glucose regulatory modeling and is still used in the clinical settings, but it has been criticized in the recent decades (see De Gaetano and Arino (2000); Panunzi et al. (2007); Panunzi et al. (2010)). In recent decades, several researchers have studied and modeled intravenous glucose tolerance test (IVGTT). Some of these models are widely used, (see, e.g., Bergman et al. (1979); De Gaetano and Arino (2000); Li et al. (2020); Mukhopadhyay et al. (2004); Panunzi et al. (2007)). This minimal model has been introduced by De Gaetano and Arino (2000) from both physiological and modeling aspects. In this model, a body is described as a compartment with a basal concentration of glucose and insulin. Bergman's minimal model has two variations, the first describes glucose kinetics, and the second describes insulin kinetics. The two models have mostly been used to understand the kinetics during IVGTT test (Glucose Tolerance Test) (see De Gaetano and Arino (2000); Gatewood et al. (1968)).

Large amounts of work on modelling biological systems has been restricted to fractional order ordinary differential equations. Therefore, the urgent necessity to find the exact solutions or merely the approximate ones to these problems has emerged. Since the finding of the exact solutions is not possible for most of these fractional differential equations, the numerical methods have been developed to obtain the approximate solutions to them. Some local numerical techniques are introduced for solving fractional order biological systems, and this method may become computationally heavy, due to the non-local property of fractional differential operator. Moreover, the local methods listed the approximate solution at specific points, while the global methods give the approximate solution in whole the mentioned interval. Hence, the global behavior of the solution can be naturally taken into account.

The spectral collocation method (Youssri et al. (2015); Abdelkawy (2020)), global numerical technique, is a particular kind of famous spectral method that is widely applicable for almost types of differential equations. The convergence speed is one of the major advantages of spectral method. Spectral methods have exponential convergence rates as will as a high accuracy level. Thus, the spectral methods are more reliable, suitable and accurate technique for treating such problems. The spectral method is classified into four kinds, namely, collocation (see Abdelkawy et al. (2020)), tau (see Ortiz and Samara (1981)), Galerkin (see Hafez et al. (2020)), and Petrov Galerkin (see Doha et al. (2011)) methods. Here, SJ-GR-C method is developed to approximate the system of a fractional IVGTT glucose-insulin interaction model.

Our work is ordered as follows. In Section 2, we list some mathematical fundamentals. In Section 3, a system of a fractional IVGTT glucose-insulin interaction model is numerically treated by the suggested algorithm. A few remarks are mentioned in the last section.

2. Preliminary Results

We consider the Jacobi polynomials $\mathcal{G}_k^{(\rho,\sigma)}(x)$, which satisfy the following properties:

$$\begin{aligned}
&\mathcal{G}_{k+1}^{(\rho,\sigma)}(x) = (a_k^{(\rho,\sigma)}x - b_k^{(\rho,\sigma)})\mathcal{G}_k^{(\rho,\sigma)}(x) - c_k^{(\rho,\sigma)}\mathcal{G}_{k-1}^{(\rho,\sigma)}(x), \quad k \geq 1,\\
&\mathcal{G}_0^{(\rho,\sigma)}(x) = 1, \quad \mathcal{G}_1^{(\rho,\sigma)}(x) = \frac{1}{2}(\rho+\sigma+2)x + \frac{1}{2}(\rho-\sigma),\\
&\mathcal{G}_k^{(\rho,\sigma)}(-x) = (-1)^k\mathcal{G}_k^{(\rho,\sigma)}(x), \quad \mathcal{G}_k^{(\rho,\sigma)}(-1) = \frac{(-1)^k\Gamma(k+\sigma+1)}{k!\Gamma(\sigma+1)},
\end{aligned} \tag{1}$$

where $\rho,\ \sigma > -1$, $x \in [-1,1]$ and

$$\begin{aligned}
a_k^{(\rho,\sigma)} &= \frac{(2k+\rho+\sigma+1)(2k+\rho+\sigma+2)}{2(k+1)(k+\rho+\sigma+1)},\\
b_k^{(\rho,\sigma)} &= \frac{(\sigma^2-\rho^2)(2k+\rho+\sigma+1)}{2(k+1)(k+\rho+\sigma+1)(2k+\rho+\sigma)},\\
c_k^{(\rho,\sigma)} &= \frac{(k+\rho)(k+\sigma)(2k+\rho+\sigma+2)}{(k+1)(k+\rho+\sigma+1)(2k+\rho+\sigma)}.
\end{aligned}$$

The rth derivative of $\mathcal{G}_j^{(\rho,\sigma)}(x)$, is computed as

$$D^r\mathcal{G}_j^{(\rho,\sigma)}(x) = \frac{\Gamma(j+\rho+\sigma+q+1)}{2^r\Gamma(j+\rho+\sigma+1)}\mathcal{G}_{j-r}^{(\rho+r,\sigma+r)}(x), \tag{2}$$

where r is an integer. For the shifted Jacobi polynomial $\mathcal{G}_{\mathcal{L},k}^{(\rho,\sigma)}(x) = \mathcal{G}_k^{(\rho,\sigma)}(\frac{2x}{\mathcal{L}} - 1),\ \mathcal{L} > 0$, the explicit analytic form is written as

$$\begin{aligned}
\mathcal{G}_{\mathcal{L},k}^{(\rho,\sigma)}(x) &= \sum_{j=0}^{k}(-1)^{k-j}\frac{\Gamma(k+\sigma+1)\Gamma(j+k+\rho+\sigma+1)}{\Gamma(j+\sigma+1)\Gamma(k+\rho+\sigma+1)(k-j)!j!\mathcal{L}^j}x^j\\
&= \sum_{j=0}^{k}\frac{\Gamma(k+\rho+1)\Gamma(k+j+\rho+\sigma+1)}{j!(k-j)!\Gamma(j+\rho+1)\Gamma(k+\rho+\sigma+1)\mathcal{L}^j}(x-\mathcal{L})^j.
\end{aligned} \tag{3}$$

Thereby, we conclude the next

$$\begin{aligned}
\mathcal{G}_{\mathcal{L},k}^{(\rho,\sigma)}(0) &= (-1)^k\frac{\Gamma(k+\sigma+1)}{\Gamma(\sigma+1)\ k!},\\
\mathcal{G}_{\mathcal{L},k}^{(\rho,\sigma)}(\mathcal{L}) &= \frac{\Gamma(k+\rho+1)}{\Gamma(\rho+1)\ k!},
\end{aligned} \tag{4}$$

$$D^r\mathcal{G}_{\mathcal{L},k}^{(\rho,\sigma)}(0) = \frac{(-1)^{k-r}\Gamma(k+\sigma+1)(k+\rho+\sigma+1)_r}{L^r\Gamma(k-r+1)\Gamma(r+\sigma+1)}, \tag{5}$$

$$D^r\mathcal{G}_{\mathcal{L},k}^{(\rho,\sigma)}(\mathcal{L}) = \frac{\Gamma(k+\rho+1)(k+\rho+\sigma+1)_r}{L^r\Gamma(k-r+1)\Gamma(r+\rho+1)}, \tag{6}$$

$$D^r \mathcal{G}_{\mathcal{L},k}^{(\rho,\sigma)}(x) = \frac{\Gamma(r+k+\rho+\sigma+1)}{\mathcal{L}^r \Gamma(k+\rho+\sigma+1)} \mathcal{G}_{\mathcal{L},k-r}^{(\rho+r,\sigma+r)}(x). \tag{7}$$

3. Numerical simulations

The most widely used model in physiological research on the metabolism of glucose is the so-called minimal model, which describes IVGTT experimental data well with the smallest set of identifiable and meaningful parameters (Bergman et al. (1979); Toffolo et al. (1980); Pacini and Bergman (1986)). After incorporating the insulin dynamics, it takes the form (De Gaetano and Arino (2000)),

$$\begin{aligned} \frac{dG(t)}{dt} &= -[p_1 + X(t)]G(t) + p_1 G_b, \\ \frac{dX(t)}{dt} &= -p_2 X(t) + p_3[I(t) - I_b], \\ \frac{dI(t)}{dt} &= p_4[G(t) - p_5]^+ t - p_6[I(t) - I_b], \end{aligned} \tag{8}$$

where

$$[G(t) - p_5]^+ = \begin{cases} G(t) - p_5, & \text{if } G(t) > p_5, \\ 0, & \text{if } G(t) \leq p_5, \end{cases}$$

subject to initial conditions $G(0) = b_0, X(0) = 0, I(0) = I_b$, where $G(t)$ [mg/dL] and $I(t)$ [mU/L] are the plasma glucose and insulin concentration at time t [min], respectively, and $X(t)$ [1/min] is an auxiliary function representing insulin excitable tissue glucose uptake activity, roughly proportional to insulin concentration in a "distant" compartment, G_b [mg/dL], and I_b [mU/L] are concentration of Basal blood glucose and insulin, respectively, p_0 [mg/dL] is the theoretical glycemia at time 0 after the instantaneous glucose bolus intake, p_1 [1/min] is the insulin-independent glucose clearance rate, p_2 [1/min] is the active insulin clearance rate (upt. decrease), p_3 [L/(min2mU)] is the increase in uptake ability caused by insulin, p_4 [1/min], is the decay rate of blood insulin, p_5 [mg/dL] is the target glucose level, p_6 [mUdL/Lmgmin] is the Pancreatic release rate after glucose bolus, p_7 (mg/dl)[1/min] is the plasma insulin concentration at time 0, above basal insulinemia, immediately after the glucose bolus intake.

The concept of fractional calculus has great importance in many branches and is also important for modeling real world problems (Khader et al. (2020); Khader and Kumar (2019); Atangana and Alabaraoye (2013); Dubey et al. (2014); Belgacem (2001); El-Sayed et al. (2007); Agarwal et al. (2013); Khan et al. (2019); Alkahtani et al. (2017)). In this paper, we are interested in the discrete version fractional order of the minimal model (9) of the glucose insulin. More precisely, we analyze the dynamical behavior of the following fractional order:

$$\begin{aligned} D^{\nu_1} G(t) &= -[p_1 + X(t)]G(t) + p_1 G_b, \\ D^{\nu_2} X(t) &= -p_2 X(t) + p_3[I(t) - I_b], \\ D^{\nu_3} I(t) &= p_4[G(t) - p_5]^+ t - p_6[I(t) - I_b], \end{aligned} \tag{9}$$

where

$$[G(t)-p_5]^+=\begin{cases}G(t)-p_5, & \text{if } G(t)>p_5,\\ 0, & \text{if } G(t)\le p_5,\end{cases}$$

subject to initial conditions $G(0)=b_0$, $X(0)=0$, $I(0)=I_b$.

This model can be used to describe the pancreas as the source of insulin (see Alkahtani et al. (2017)). In a healthy individual, a small amount of insulin is always created and cleared (see Cobelli et al. (1998)). This helps to keep the basal concentration I_b. The glucose-independent production and clearance of insulin is proportional to the blood insulin concentration. If the insulin level is above basal concentration, clearance increases. On the other hand, if the insulin level is below basal concentration, production increases. When the glucose level gets high, the pancreas reacts by releasing more insulin at a given rate. To explain this mathematically, one has to derive a function describing the reaction of the pancreas. This function was derived by Bergman et al. and adjusted by Gaetano et al. (De Gaetano and Arino (2000); Gatewood et al. (1968)) to become Pancreas$(t)=[G(t)-p_5]^+t$, where $[G(t)-p_5]^+=\max\left([G(t)-p_5],0\right)$.

3.1. Jacobi spectral collocation scheme

In this work, we want to numerically solve the fractional-order of IVGTT glucose-insulin interaction

$$\begin{aligned}
D^{\nu_1}G(t)&=-(p_1+X(t))G(t)+p_1G_b, && G(0)=b_0,\\
D^{\nu_2}X(t)&=-p_2X(t)+p_3(I(t)-I_b), && X(0)=X_0,\\
D^{\nu_3}I(t)&=p_4[G(t)-p_5]^+t-p_6(I(t)-I_b), && I(0)=p_7+I_b.
\end{aligned}\tag{10}$$

Firstly, we rewrite the system (10) as

$$\frac{d^\nu}{dt^\nu}\Psi(t)=F(t,\Psi(t)),\tag{11}$$

where

$$\begin{aligned}
\Psi(t)&=\begin{pmatrix}\psi_1(t)\\ \psi_2(t)\\ \psi_3(t)\end{pmatrix}=\begin{pmatrix}G(t)\\ X(t)\\ I(t)\end{pmatrix},\quad \frac{d^\nu}{dt^\nu}\Psi=\begin{pmatrix}D^{\nu_1}G(t)\\ D^{\nu_1}X(t)\\ D^{\nu_1}I(t)\end{pmatrix},\\
F(t,\Psi(t))&=\begin{pmatrix}-(p_1+X(t))G(t)+p_1G_b\\ -p_2X(t)+p_3(I(t)-I_b)\\ p_4[G(t)-p_5]^+t-p_6(I(t)-I_b)\end{pmatrix}.
\end{aligned}\tag{12}$$

We approximate the independent variable using SJ-GR-C method at $t^{(\theta,\vartheta)}_{\mathcal{L},\mathcal{N},j}$ nodes. The nodes are the set of points in a specified domain where the dependent variable values are approximated. In general, the choice of the location of the nodes are optional but taking $t^{(\theta,\vartheta)}_{\mathcal{L},\mathcal{N},i}$ as a Jacobi-Gauss-Radau collocation nodes.

The solution of Equation (11) is approximated as,

$$\psi^{(\theta,\vartheta)}_{\mathcal{L},\mathcal{L},i}(t)=\sum_{j=0}^{N}a_{ij}\mathcal{G}^{(\theta,\vartheta)}_{\mathcal{L},j}(t),\quad i=1,2,3, \tag{13}$$

where $\psi^{(\theta,\vartheta)}_{\mathcal{L},\mathcal{L},1}(t)=G^{(\theta,\vartheta)}_{\mathcal{L},\mathcal{L}}(t)$, $\psi^{(\theta,\vartheta)}_{\mathcal{L},\mathcal{L},2}(t)=X^{(\theta,\vartheta)}_{\mathcal{L},\mathcal{L}}(t)$, $\psi^{(\theta,\vartheta)}_{\mathcal{L},\mathcal{L},2}(t)=I^{(\theta,\vartheta)}_{\mathcal{L},\mathcal{L}}(t)$, the fractional derivative order ν of the approximate solution $\psi^{(\theta,\vartheta)}_{\mathcal{L},\mathcal{L},i}(t)$ is estimated as

$$\frac{d^{\nu}}{dt^{\nu}}\psi^{(\theta,\vartheta)}_{\mathcal{L},\mathcal{L},i}(t)=\sum_{j=0}^{N}a_{ij}\frac{d^{\nu}}{dt^{\nu}}\big(\mathcal{G}^{(\theta,\vartheta)}_{\mathcal{L},j}(t)\big),\quad i=1,2,3,4. \tag{14}$$

Using the analytical form of shifted Jacobi polynomial, we find

$$\begin{aligned}\frac{d^{\nu}}{dt^{\nu}}\mathcal{G}^{(\theta,\vartheta)}_{\mathcal{L},j}(x)=&\mathcal{G}^{(\theta,\vartheta,\nu)}_{\mathcal{L},j}(t)\\&=\sum_{k=0}^{j}(-1)^{j-k}\frac{\Gamma(j+\vartheta+1)\Gamma(k+j+\theta+\vartheta+1)}{\Gamma(k+\vartheta+1)\Gamma(j+\theta+\vartheta+1)(j-k)!k!\mathcal{L}^{k}}\frac{d^{\nu}}{dt^{\nu}}t^{k}\\&=\sum_{k=0}^{j}(-1)^{j-k}\frac{(\Gamma(k+1)\Gamma(j+\vartheta+1)\Gamma(j+k+\theta+\vartheta+1))}{k!\mathcal{L}^{k}(j-k)!\Gamma(k+\vartheta+1)\Gamma(k+\nu)\Gamma(j+\theta+\vartheta+1)}t^{k+\nu-1}.\end{aligned} \tag{15}$$

Then,

$$\frac{d^{\nu}}{dt^{\nu}}\psi^{(\theta,\vartheta)}_{\mathcal{L},\mathcal{L},i}(t)=\sum_{j=m}^{N}a_{ij}\frac{d^{\nu}}{dt^{\nu}}\big(\mathcal{G}^{(\theta,\vartheta)}_{\mathcal{L},j}(t)\big)=\sum_{j=m}^{N}a_{ij}\mathcal{G}^{(\theta,\vartheta,\nu)}_{\mathcal{L},j}(t). \tag{16}$$

Then, we can write Equation (11) as

$$\Psi^{\nu}(t)=\mathcal{F}(t), \tag{17}$$

where

$$\begin{aligned}\Psi^{\nu}(t)&=\begin{pmatrix}\sum\limits_{j=m}^{N}a_{1j}\mathcal{G}^{(\theta,\vartheta,\nu_1)}_{\mathcal{L},j}(t)\\ \sum\limits_{j=m}^{N}a_{2j}\mathcal{G}^{(\theta,\vartheta,\nu_2)}_{\mathcal{L},j}(t)\\ \sum\limits_{j=m}^{N}a_{3j}\mathcal{G}^{(\theta,\vartheta,\nu_3)}_{\mathcal{L},j}(t)\end{pmatrix},\\ \mathcal{F}(t)&=\begin{pmatrix}-(p_1+\sum\limits_{j=0}^{N}a_{2j}\mathcal{G}^{(\theta,\vartheta)}_{\mathcal{L},j}(t))\sum\limits_{j=0}^{N}a_{1j}\mathcal{G}^{(\theta,\vartheta)}_{\mathcal{L},j}(t)+p_1G_b\\ -p_2\sum\limits_{j=0}^{N}a_{2j}\mathcal{G}^{(\theta,\vartheta)}_{\mathcal{L},j}(t)+p_3(\sum\limits_{j=0}^{N}a_{3j}\mathcal{G}^{(\theta,\vartheta)}_{\mathcal{L},j}(t)-I_b)\\ p_4[\sum\limits_{j=0}^{N}a_{1j}\mathcal{G}^{(\theta,\vartheta)}_{\mathcal{L},j}(t)-p_5]^{+}t-p_6(\sum\limits_{j=0}^{N}a_{3j}\mathcal{G}^{(\theta,\vartheta)}_{\mathcal{L},j}(t)-I_b)\end{pmatrix}.\end{aligned} \tag{18}$$

In the proposed method, the residual of (17) is set to zero at the $\mathcal{N}$ points

$$\Psi^{\nu}(t^{(\theta,\vartheta)}_{\mathcal{L},\mathcal{N},r})=\mathcal{F}(t^{(\theta,\vartheta)}_{\mathcal{L},\mathcal{N},r}),\quad r=1,2,\ldots,\mathcal{N}, \tag{19}$$

using the initial conditions, we obtain

$$\sum_{j=0}^{N} a_{1j}\mathcal{G}_{\mathcal{L},j}^{(\theta,\vartheta)}(0) = G_0, \qquad \sum_{j=0}^{N} a_{2j}\mathcal{G}_{\mathcal{L},j}^{(\theta,\vartheta)}(0) = X_0, \qquad \sum_{j=0}^{N} a_{3j}\mathcal{G}_{\mathcal{L},j}^{(\theta,\vartheta)}(0) = I_0. \tag{20}$$

Finally, from Equations (19) and (20), we get a system of $(3N+3)$ algebraic equations which can be solved for the unknown coefficients $a_{ij}, \qquad i = 1, 2, 3,\ j = 0, \ldots, N$.

3.2. Numerical Results

Using the algorithm presented in the previous subsection, we give in this subsection some numerical results.

Example 3.1.

Let us consider the parameter values $\theta = \frac{1}{2}$, $\vartheta = \frac{1}{2}$, $\nu_1 = \nu_2 = \nu_3 = 0.5$, $G_b = 92$, $I_b = 7.3$, $p_1 = 0.03082$, $p_2 = 0.02093$, $p_3 = 1.062 \times 10^{-5}$, $p_4 = 0.3$, $p_5 = 89.5$, $p_6 = 0.3349 \times 10^{-2}$, $G_0 = 287$, $X_0 = 0$, $I_0 = 403.4$. We obtain the numerical solutions of the fractional-order of IVGTT glucose-insulin, interaction

$$\begin{aligned} G_{100,20}^{(\frac{1}{2},\frac{1}{2})}(t) =& 287 + 6.353638803494439 \times 10^{-32}t^{20} - 6.842137960774486 \times 10^{-29}t^{19} \\ &+ 0.0434921t^4 + 3.4298760758123567 \times 10^{-26}t^{18} - 0.0000191035t^7 \\ &- 1.0627085796028098 \times 10^{-23}t^{17} + 2.278993215555972 \times 10^{-21}t^{16} \\ &- 3.589626986754096 \times 10^{-19}t^{15} + 4.3007058597086106 \times 10^{-17}t^{14} \\ &- 4.004947297410018 \times 10^{-15}t^{13} + 2.937162795212789 \times 10^{-13}t^{12} \\ &- 1.7086250081509372 \times 10^{-11}t^{11} + 7.903804657163513 \times 10^{-10}t^{10} \\ &- 2.902847791136348 \times 10^{-8}t^9 + 8.419200001597479 \times 10^{-7}t^8 \\ &+ 1.5411t^2 - 6.86471t - 0.306864t^3 - 0.00443568t^5 + 0.00033449t^6, \end{aligned} \tag{21}$$

$$\begin{aligned} X_{100,20}^{(\frac{1}{2},\frac{1}{2})}(t) =& - 6.671336352056591 \times 10^{-35}t^{20} + 7.17027488417351 \times 10^{-32}t^{19} \\ &- 3.5875287519050936 \times 10^{-29}t^{18} + 1.1094775128029478 \times 10^{-26}t^{17} \\ &- 2.3748863503232645 \times 10^{-24}t^{16} + 3.733745049412373 \times 10^{-22}t^{15} \\ &- 4.464985109857724 \times 10^{-20}t^{14} + 4.149893776646388 \times 10^{-18}t^{13} \\ &- 3.037267190592385 \times 10^{-16}t^{12} + 1.7629611275643226 \times 10^{-14}t^{11} \\ &- 8.134987059605977 \times 10^{-13}t^{10} + 2.979119298872092 \times 10^{-11}t^9 \\ &- 8.60974178273425 \times 10^{-10}t^8 + 1.9445496704263148 \times 10^{-8}t^7 \\ &- 3.382811615347342 \times 10^{-7}t^6 + 4.44191246801422 \times 10^{-6}t^5 \\ &- 0.0000428267t^4 + 0.000292215t^3 - 0.00134931t^2 + 0.00423648t, \end{aligned} \tag{22}$$

$$\begin{aligned} I_{100,20}^{(\frac{1}{2},\frac{1}{2})}(t) =& 403.4 + 1.8393201667252017 \times 10^{-30}t^{20} - 1.979264004256827 \times 10^{-27}t^{19} \\ &+ 9.914247055262861 \times 10^{-25}t^{18} - 3.0693951395454833 \times 10^{-22}t^{17} \\ &+ 6.576898867360193 \times 10^{-20}t^{16} - 1.035003295202461 \times 10^{-17}t^{15} \\ &+ 1.2388286577160279 \times 10^{-15}t^{14} - 1.1523877441573528 \times 10^{-13}t^{13} \\ &+ 8.440970952543153 \times 10^{-12}t^{12} - 4.903189114376244 \times 10^{-10}t^{11} \\ &+ 2.264122829451964 \times 10^{-8}t^{10} - 8.297014176242745 \times 10^{-7}t^{9} \\ &+ 0.000023994t^{8} - 0.000542259t^{7} + 0.00943954t^{6} - 0.124042t^{5} \\ &+ 1.19716t^{4} - 8.18309t^{3} + 37.9968t^{2} - 120.017t. \end{aligned} \tag{23}$$

Example 3.2.

Let us consider the parameter values $\theta = 0$, $\vartheta = 0$, $\nu_1 = \nu_2 = \nu_3 = 1$, $G_b = 92$, $I_b = 7.3$, $p_1 = 0.03082$, $p_2 = 0.02093$, $p_3 = 1.062 \times 10^{-5}$, $p_4 = 0.3$, $p_5 = 89.5$, $p_6 = 0.3349 \times 10^{-2}$, $G_0 = 287$, $X_0 = 0$, $I_0 = 403.4$. We obtain the numerical solutions of the fractional-order of IVGTT glucose-insulin interaction,

$$\begin{aligned} G_{100,20}^{(0,0)}(t) =& 287 + 1.1359114365296436 \times 10^{-31}t^{20} - 1.1307430687165715 \times 10^{-28}t^{19} \\ &+ 5.214818641620502 \times 10^{-26}t^{18} - 1.4785265470882445 \times 10^{-23}t^{17} \\ &+ 2.8836836796638653 \times 10^{-21}t^{16} - 4.1017985782464915 \times 10^{-19}t^{15} \\ &+ 4.401604896113486 \times 10^{-17}t^{14} - 3.635779393401938 \times 10^{-15}t^{13} \\ &+ 2.3377736965150437 \times 10^{-13}t^{12} - 1.1754681829924774 \times 10^{-11}t^{11} \\ &+ 4.6166621690403816 \times 10^{-10}t^{10} - 1.4065712558638919 \times 10^{-8}t^{9} \\ &+ 3.2788332130205134 \times 10^{-7}t^{8} - 5.709627637577579 \times 10^{-6}t^{7} \\ &+ 0.0000711513t^{6} - 0.000577913t^{5} + 0.00214642t^{4} + 0.0117359t^{3} \\ &- 0.249088t^{2} - 6.16685t, \end{aligned} \tag{24}$$

$$\begin{aligned} X_{100,20}^{(0,0)}(t) =& 4.397533394747653 \times 10^{-35}t^{20} - 4.3457855077793764 \times 10^{-32}t^{19} \\ &+ 1.9870050915756846 \times 10^{-29}t^{18} - 5.575728471275728 \times 10^{-27}t^{17} \\ &+ 1.0739204014764541 \times 10^{-24}t^{16} - 1.5041192521922484 \times 10^{-22}t^{15} \\ &+ 1.5830190038175788 \times 10^{-20}t^{14} - 1.2754031337058079 \times 10^{-18}t^{13} \\ &+ 7.935372741365426 \times 10^{-17}t^{12} - 3.814546146211714 \times 10^{-15}t^{11} \\ &+ 1.404574139163027 \times 10^{-13}t^{10} - 3.875713048037366 \times 10^{-12}t^{9} \\ &+ 7.625615974047054 \times 10^{-11}t^{8} - 9.294181243647421 \times 10^{-10}t^{7} \\ &+ 2.4090790436567306 \times 10^{-9}t^{6} + 1.555346131525155 \times 10^{-7}t^{5} \\ &- 3.5474394390627843 \times 10^{-6}t^{4} - 1.3877787807814457 \times 10^{-17} \\ &+ 0.0000441746t^{3} - 0.000447097t^{2} + 0.00416538t, \end{aligned} \tag{25}$$

$$\begin{aligned} I_{100,20}^{(0,0)}(t) =& 403.4 - 1.8515266293032804 \times 10^{-30} t^{20} + 1.8369846514170685 \times 10^{-27} t^{19} \\ & - 8.438740305112501 \times 10^{-25} t^{18} + 2.3814272925241386 \times 10^{-22} t^{17} \\ & - 4.618559584807879 \times 10^{-20} t^{16} + 6.524347469434951 \times 10^{-18} t^{15} \\ & - 6.941419760887544 \times 10^{-16} t^{14} + 5.671675401059743 \times 10^{-14} t^{13} \\ & - 3.595705384715099 \times 10^{-12} t^{12} + 1.774158203543195 \times 10^{-10} t^{11} \\ & - 6.7876273000445516 \times 10^{-9} t^{10} + 1.9901856797929638 \times 10^{-7} t^{9} \\ & - 4.36754697466888 \times 10^{-6} t^{8} + 0.0000683657 t^{7} + 11.562 t^{2} - 116.931 t \\ & - 0.000674812 t^{6} + 0.0020955 t^{5} + 70.049759 t^{4} - 0.964515 t^{3}. \end{aligned} \tag{26}$$

Example 3.3.

Let us consider the parameter values $\theta = \frac{1}{2}$, $\vartheta = \frac{1}{2}$, $\nu_1 = \nu_2 = \nu_3 = 0.75$, $G_b = 92$, $I_b = 7.3$, $p_1 = 0.03082$, $p_2 = 0.02093$, $p_3 = 1.062 \times 10^{-5}$, $p_4 = 0.3$, $p_5 = 94$, $p_6 = 0.3349 \times 10^{-2}$, $G_0 = 287$, $X_0 = 0$, $I_0 = 407.73$. We obtain the numerical solutions of the fractional-order of IVGTT glucose-insulin interaction,

$$\begin{aligned} G_{100,20}^{(\frac{1}{2},\frac{1}{2})}(t) =& 287 + 2.118903608902899 \times 10^{-32} t^{20} - 2.1977157369969697 \times 10^{-29} t^{19} \\ & + 1.0622128723994101 \times 10^{-26} t^{18} - 3.1773099536131463 \times 10^{-24} t^{17} \\ & + 6.588263231260877 \times 10^{-22} t^{16} - 1.0052146750521798 \times 10^{-19} t^{15} \\ & + 1.1691739546926136 \times 10^{-17} t^{14} - 1.0597123077730821 \times 10^{-15} t^{13} \\ & + 7.587456777101064 \times 10^{-14} t^{12} - 4.324722948575461 \times 10^{-12} t^{11} \\ & + 1.9685573267836345 \times 10^{-10} t^{10} - 7.150781345833736 \times 10^{-9} t^{9} \\ & + 2.0639426550081019 \times 10^{-7} t^{8} - 4.696193251466701 \times 10^{-6} t^{7} \\ & + 0.0000832644 t^{6} - 0.00113302 t^{5} + 0.011631 t^{4} - 0.0891915 t^{3} \\ & + 0.542545 t^{2} - 6.36438 t, \end{aligned} \tag{27}$$

$$\begin{aligned} X_{100,20}^{(\frac{1}{2},\frac{1}{2})}(t) =& - 2.0649724660056455 \times 10^{-35} t^{20} + 2.252228648614916 \times 10^{-32} t^{19} \\ & - 1.1443277682375657 \times 10^{-29} t^{18} + 3.5966435248563045 \times 10^{-27} t^{17} \\ & - 7.831442413127942 \times 10^{-25} t^{16} + 1.253779984881355 \times 10^{-22} t^{15} \\ & - 1.5286491609634486 \times 10^{-20} t^{14} + 1.4506542858094594 \times 10^{-18} t^{13} \\ & - 1.0859193866456728 \times 10^{-16} t^{12} + 6.460264584822663 \times 10^{-15} t^{11} \\ & - 3.063177527766398 \times 10^{-13} t^{10} + 1.1564158300966819 \times 10^{-11} t^{9} \\ & - 3.4597318202765723 \times 10^{-10} t^{8} + 8.1344939420075 \times 10^{-9} t^{7} \\ & - 1.4847871796491968 \times 10^{-7} t^{6} + 2.069851221142708 \times 10^{-6} t^{5} \\ & - 0.0000215984 t^{4} + 0.00016535 t^{3} - 0.000932726 t^{2} \\ & + 0.0043218 t - 3.469446951953614 \times 10^{-18}, \end{aligned} \tag{28}$$

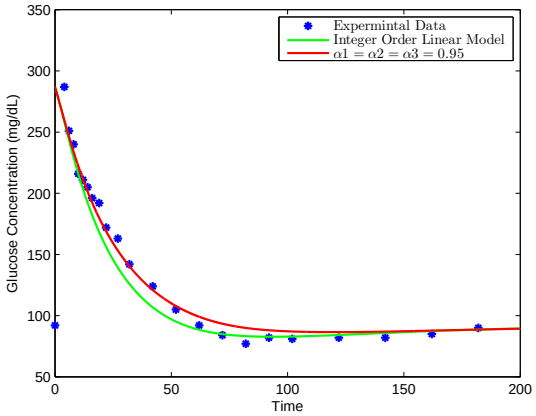

Figure 1. Glucose concentration versus time

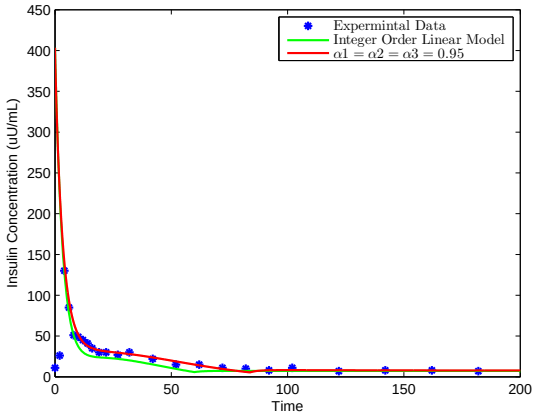

Figure 2. Insulin concentration versus time

$$
\begin{aligned}
I_{100,20}^{(\frac{1}{2},\frac{1}{2})}(t) =& 407.73 + 4.731708908197222 \times 10^{-31} t^{20} - 5.261753688466492 \times 10^{-28} t^{19} \\
&+ 2.721910353251266 \times 10^{-25} t^{18} - 8.698694732418227 \times 10^{-23} t^{17} \\
&+ 1.9235016050844932 \times 10^{-20} t^{16} - 3.1236062904995628 \times 10^{-18} t^{15} \\
&+ 3.858719413668909 \times 10^{-16} t^{14} - 3.706290872765347 \times 10^{-14} t^{13} \\
&+ 2.805267752160471 \times 10^{-12} t^{12} - 1.6858088925176994 \times 10^{-10} t^{11} \\
&+ 8.066946220987033 \times 10^{-9} t^{10} - 3.0707458338962214 \times 10^{-7} t^{9} \\
&+ 9.25538597574769 \times 10^{-6} t^{8} - 0.000219053 t^{7} - 122.032 t \\
&+ 0.00402175 t^{6} - 0.0563525 t^{5} + 0.590707 t^{4} - 4.54287 t^{3} + 25.8959 t^{2}.
\end{aligned}
\tag{29}
$$

In Figures 1-4, the curves of the approximate solutions related to glucose concentration insulin concentration are displayed, with values of parameters listed in their caption. There are different linear and nonlinear mathematical models to describe the system of glucose and insulin in blood. The most famous nonlinear Bergman's minimal model is modified to fractional order model and on the basis of simulations and comparison with the experimental data. The experimental data in this work taken from the reference Pacini and Bergman (1986). It is concluded that the fractional order version of Bergman's minimal model is better representative of the system than its integer order form.

4. Conclusions

In this paper, we obtain the approximate solution of the proposed model and a numerical solution of the system which shows that effect of time on the concentrations $G(t)$, $X(t)$, and $I(t)$. Chaos and bifurcation of the resulting discrete system were numerically investigated by varying the system parameters and the fractional-order parameter ν. Also we provided numerical simulations exhibiting dynamical behavior and stability around equilibria of the system, and on the basis

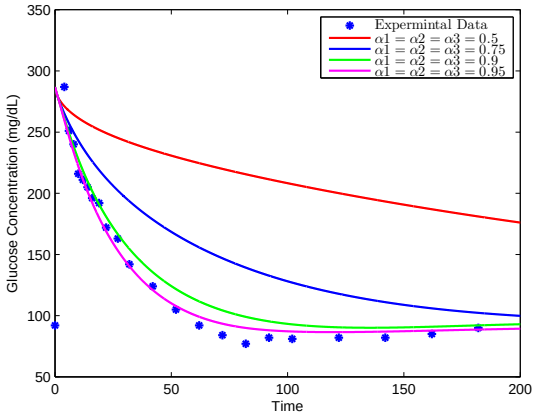

Figure 3. Glucose concentration versus time

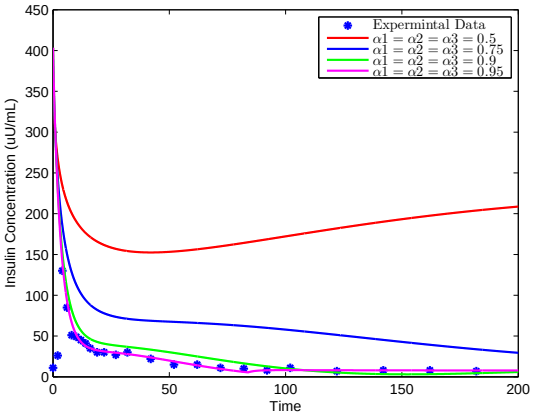

Figure 4. Insulin concentration versus time

of simulations and comparison with the experimental data, it is concluded that the fractional order version of Bergman's minimal model is better representative of the system than its integer order form.

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