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Orthogonality in Terms of 2-HH Norm and Bounded Linear Operators in Banach Spaces

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Abstract

In the present paper, the generalization of the Carlson orthogonality for functionals to operators in Banach spaces has been studied. We will also investigate various properties related to the Carlsson, Birkhoff-James, and Pythagorean orthogonality for operators. Kikianty and Dragomir (2010) mentioned in their paper by stating that Pythagorean and isosceles orthogonality through the medium of $2 - HH$ norm satisfies the non-degeneracy, symmetry and continuity properties without mentioning detailed proof. This paper provides the complete proof of these properties as well as the equivalency of additivity and homogeneity of the isosceles orthogonality with the help of $2 - HH$ norm. In the case of norm attaining bounded linear operators in a Hilbert space, we prove an equivalence relation between the Pythagorean and isosceles orthogonality including result that the Pythagorean implies Birkhoff-James, whereas the converse is not necessarily true. Furthermore, we will study a new particular case of Carlsson orthogonality and show that this orthogonality implies Birkhoff-James orthogonality, but the converse may not be true.

Keywords: Banach Spaces; Hilbert spaces; Carlsson orthogonality; Birkhoff orthogonality; Isosceles orthogonality; $p - HH$ norm; Pythagorean orthogonality

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1. Introduction

The notion of first orthogonality relation in a general normed linear space was introduced by Robert (1934). The Robert orthogonality implies both the Birkhoff orthogonality (introduced by Birkhoff in 1935) and the isosceles orthogonality (introduced by James in 1945). The study of Birkhoff orthogonality has been done in detail in the paper written by James, and therefore it is known as Birkhoff-James orthogonality or Blaschke Birkhoff-James orthogonality (James (1945)). Furthermore, James introduced the Pythagorean orthogonality in normed linear spaces which generalizes a result in an Euclidean space stating that two vectors are perpendicular if ad only if there is a right triangle having two vectors as legs (James (1945)). In the same way, the meaning of the isosceles orthogonality in an ordinary Euclidean space is that two vectors are perpendicular iff their sum and difference can be sides of isosceles triangle (James (1945)). It is interesting to investigate the properties like symmetry, homogeneity, additivity and existence of orthogonality in an Euclidean space as applied to normed space. The property of existence is satisfied by both the isosceles and Pythagorean orthogonalities, which are also clearly symmetric. However, these orthogonalities are not homogeneous or additive in general normed linear space (James (1945)). The most interesting fact here is that the isosceles and Pythagorean orthogonality are not equivalent in a normed linear space (James (1945)).

Carlsson in 1962 introduced a generalized concept of orthogonality in normed space with indicating that isosceles and Pythagorean orthogonality are special cases (James (1945)). In 2010, Kikianty and Dragomir played a vital role to introduce $p - HH$ norm on the Cartesian square of normed spaces by generalizing the previous definition of Carlsson orthogonality through the medium of $2 - HH$ norm, which also generalizes the Pythagorean and isosceles orthogonality through te medium of $2 - HH$ norm (Kikianty (2010)). The Carlsson orthogonality in terms of $2 - HH$ norm satisfies the non-degeneracy, simplification and continuity, yet it is not symmetric (Kikianty and Dragomir (2012)). Bhatia and Semrl came up with new concept of orthogonality in terms of matrices, specially in the case of the Birkhoff orthogonality (Bhatia and Semrl (1999)).

Sain and Paul linked the Bhatia-Semrl property with norm attaining operators in a finite dimensional normed spaces which attain its norm on connected closed subset of the unit sphere of X and proved that if two linear operators are orthogonal in the sense of Birkhoff-James, then there exist an element in the closed connected subset of the unit sphere at which the images of operators are also orthogonal in the sense of Birkhoff-James, Paul and Sain (2013). For the normed linear space X of dimension 2, the next research of Sain and Paul done in 2015 explores the converse of the previous result as obtained in 2013. They proved that a linear operator T satisfies Bhatia-Semrl property if and only if the set of unit vectors on which T attains its norm is connected in the corresponding projective space.

Paul, Sain and Jha introduced a concept of Strong Birkhoff-James orthogonality and proved that strong Birkhoff-James orthogonality implies the Birkhoff orthogonality, but the converse may not be true (Paul et al. (2013)). To study the difference of orthogonality in the complex case in comparison to real case Paul et al. in 2018 came up with a new concept of the Birkhoff-James orthogonality introducing new definitions on complex reflexive Banach spaces (Paul et al. (2018)). Recently, Bottazzi et al. have introduced a new generalization of earlier results on orthogonality of bounded linear operators. They discussed about Birkhoff-James, isosceles, and Robert orthogonality in Banach spaces in terms of bounded linear operators (Bottazzi et al. (2020)).

Motivated by the results of Bottazzi et al. (2020), we will introduce the Carlsson orthogonality for bounded linear operators in Banach Spaces. Furthermore, we verify some properties, like non-degeneracy, continuity, and homogeneity property of an inner-product space in the context of the Carlsson orthogonality for bounded linear operators. In the case of norm attaining bounded linear operators, when the norm is induced by an inner-product, we introduce a relation between the isosceles and Pythagorean orthogonalities by stating that the operator T_1 is isosceles orthogonal to T_2 if and only if T_1 is Pythagorean orthogonal to T_2. Furthermore, we prove an interesting relation that the Pythagorean orthogonality implies the Birkhoff-James orthogonality, but the converse may not be true. To disprove this statement, we take two dimensional matrix operators on a Hilbert space whose norm is obtained by taking trace of the product of the adjoint of an operator with the operator itself. Kikianty and Draagomir mentioned without proof in their paper that the Pythagorean and isosceles orthogonality via $2-HH$ norm satisfies non-degeneracy, symmetry and continuity property (Kikianty and Dragomir (2010)). In this paper, we will prove all these three properties when the norm on X is induced by an inner product. We also prove the equivalency of homogeneity and additivity of isosceles orthogonality via $2-HH$ norm, the proof of which also has been omitted in the paper by Kikianty and Dragomir (2010).

2. Preliminaries

Throughout this paper, $X = (X, \|\cdot\|)$ denotes a normed linear space and H is a real or complex Hilbert space. Let $B(X)$ be the set of all bounded linear operators defined on X, $S_X = \{x \in X : \|x\| = 1\}$ be the unit sphere on X with center at the origin and $M_T = \{x \in S_x : \|Tx\| = \|T\|\}$ is the norm attainment set of T. We write $x \perp_{2-HH-P} y$ and $x \perp_{2-HH-I} y$ to indicate that x is Pythagorean and isosceles orthogonal to y with respect to $2-HH$ norm respectively. For any bounded linear operators T_1 and T_2 on X, $T_1 \perp_C^O T_2$ denotes T_1 is Carlsson orthogonal to T_2. The norm of a matrix operator T in a two dimensional Hilbert space H is defined by

$$\|T\|^2 = \text{trace}(T^*T), \tag{1}$$

where the $trace$ of a matrix is the sum of all principal diagonal elements of a square matrix and T^* is the adjoint of a matrix operator T, and if for any $h \in M_T$, we have $\|Th\| = \|T\|$.

3. Main Results

Theorem 3.1.

Let X be a normed linear space with the norm induced by an inner-product. Then, the Pythagorean orthogonality via $2-HH$ norm satisfies the non-degeneracy, continuity and symmetry property.

Proof:

Non-degeneracy: If $x \perp_{2-HH-p} x$, then

$$\begin{aligned} & \int_0^1 \|(1-t)x + tx\|^2 dt = \frac{1}{3}(\|x\|^2 + \|x\|^2) \\ \Rightarrow \quad & \|x\|^2 = \frac{2}{3}\|x\|^2 \\ \Rightarrow \quad & \frac{1}{3}\|x\|^2 = 0 \\ \Rightarrow \quad & x = 0. \end{aligned} \tag{2}$$

Continuity: Let $x_n \to x, y_n \to y$ and $x_n \perp_{2-HH-p} y_n$. Then,

$$\begin{aligned} & \int_0^1 \|(1-t)x_n + ty_n\|^2 dt = \frac{1}{3}(\|x\|_n^2 + \|y_n\|^2) \\ \Rightarrow \quad & \lim_{n\to\infty} \int_0^1 \|(1-t)x_n + ty_n\|^2 dt = \lim_{n\to\infty} \frac{1}{3}(\|x_n\|^2 + \|y_n\|^2) \\ \Rightarrow \quad & \int_0^1 \|(1-t)x + ty\|^2 dt = \frac{1}{3}(\|x\|^2 + \|y\|^2) \\ \Rightarrow \quad & x \perp_{2-HH-P} y. \end{aligned} \tag{3}$$

Symmetry: If $x \perp_{2-HH-P} y$, then

$$\begin{aligned} \|(x,y)\|_{2-HH} &= \frac{1}{3}(\|x\|^2 + \|y\|^2) \\ &= \frac{1}{3}(\|y\|^2 + \|x\|^2) \\ &= \int_0^1 \|(1-t)y + tx\|^2 dt \\ &= \|(y,x)\|_{2-HH}. \end{aligned} \tag{4}$$

■

Theorem 3.2.

Let X be a normed linear space with the norm induced by an inner-product. Then, the isosceles orthogonality via $2-HH$ norm satisfies the non-degeneracy, continuity and symmetry property.

Proof:

Non-degeneracy: If $x \perp_{2-HH-I} y$, then

$$\begin{aligned} & \int_0^1 \|(1-t)x + tx\|^2 dt = \int_0^1 \|(1-t)x - tx\|^2 dt \\ \Rightarrow \quad & \int_0^1 \|x\|^2 dt = \int_0^1 \|x\|^2 (1-2t)^2 dt \\ \Rightarrow \quad & \|x\|^2 = \frac{2}{3}\|x\|^2 \\ \Rightarrow \quad & \frac{1}{3}\|x\|^2 = 0 \\ \Rightarrow \quad & x = 0. \end{aligned} \tag{5}$$

Continuity: Let $x_n \to x, y_n \to y$ for all n, and $x_n \perp_{2-HH-I} y_n$. Then,

$$\begin{aligned} & \int_0^1 \|(1-t)x_n + ty_n\|^2 dt = \int_0^1 \|(1-t)x_n - ty_n\|^2 dt \\ \Rightarrow \quad & \lim_{n\to\infty} \int_0^1 \|(1-t)x_n + ty_n\|^2 dt = \lim_{n\to\infty} \int_0^1 \|(1-t)x_n - ty_n\|^2 dt \\ \Rightarrow \quad & \int_0^1 \|(1-t)x + ty\|^2 dt = \int_0^1 \|(1-t)x - ty\|^2 dt \\ \Rightarrow \quad & x \perp_{2-HH-I} y. \end{aligned} \tag{6}$$

Symmetry: If $x \perp_{2-HH-I} y$, then

$$\begin{aligned} & \int_0^1 \|(1-t)x + ty\|^2 dt = \int_0^1 \|(1-t)x - ty\|^2 dt \\ \Rightarrow \quad & \frac{1}{3}(\|x\|^2 + \|y\|^2) = \frac{1}{3}(\|x\|^2 + \|y\|^2) \\ \Rightarrow \quad & \frac{1}{3}(\|y\|^2 + \|x\|^2) = \frac{1}{3}(\|y\|^2 + \|x\|^2) \\ \Rightarrow \quad & \int_0^1 \|(1-t)y + tx\|^2 dt = \int_0^1 \|(1-t)y - tx\|^2 dt \\ \Rightarrow \quad & y \perp_{2-HH-I} x. \end{aligned} \tag{7}$$

■

Kikianty and Dragomir (2010) proved that the homogeneity and additivity of the Pythagorean orthogonality via $2 - HH$ norm is equivalent; however, they also stated the similar result about the isosceles orthogonality with respect to $2 - HH$ norm, by omitting the proof. In the following theorem, we give complete proof regarding the equivalency of homogeneity and additivity of the $HH - I$ orthogonality.

Theorem 3.3.

Let $x \perp_{2-HH-I} y$. Then, the following are equivalent:

(1) Isosceles orthogonality via $2-HH$ norm is homogeneous.
(2) Isosceles orthogonality via $2-HH$ norm is additive.

Proof:

$(1) \Rightarrow (2)$. Assume that the isosceles orthogonality via $2-HH$ norm is homogeneous. We shall show that it is additive. As $2-HH-I$ orthogonality is homegeneous in a normed sapce X if and only if X is an inner product space, and therefore it is additive.
$(2) \Rightarrow (1)$. Conversely, assume that the additive property holds and $x \perp_{2-HH-I} y$. Since $2-HH-I$ orthogonality exists, for any $x, -y$ there exists a $\beta \in \mathbb{R} : x \perp_{2-HH-I} \beta x - y$, and by additivity property, we conclude that $x \perp_{2-HH-I} \beta x$. Hence, $\beta = 0$ whenever $x \neq 0$, and therefore, $x \perp_{2-HH-I} -y$. Again, by the symmetry and additivity property of isosceles orthogonality via $2-HH$ norm, we may conclude that $px \perp_{2-HH-I} qy$ for all integers p and q. When $p \neq 0$,

$$\int_0^1 \|(1-t)x + t(\frac{q}{p})y\|^2 dt = \frac{1}{3}(\|x\|^2 + \frac{q^2}{p^2}\|y\|^2) = \int_0^1 \|(1-t)x - t\frac{q}{p}y\|^2 dt. \tag{8}$$

This shows that $x \perp_{2-HH-I} ky$ for some $k \in \mathbb{Q}$, and by using the continuity of norm, $x \perp_{2-HH-I} ky$ for any real k. Again $2-HH-I$ orthogonality is symmetric, and therefore, we may conclude that it is homogeneous. ■

Definition 3.1.

Let T_1 and T_2 are bounded linear operators on X. Then, the operator T_1 is orthogonal to T_2 in the sense of Carlsson (denoted by $T_1 \perp_C^O T_2$) if for any $h \in M_T$,

$$\sum_{k=1}^{n} p_k \|(q_k T_1 + r_k T_2)(h)\|^2 = 0, \tag{9}$$

satisfying the conditions

$$\sum_{k=1}^{n} p_k q_k r_k = 1, \sum_{k=1}^{n} p_k q_k^2 = \sum_{k=1}^{n} p_k r_k^2 = 0. \tag{10}$$

Theorem 3.4.

Let T_1 and T_2 be norm attaining bounded linear operators on a Banach space X. If $T_1 = T_2 = T$, then

$$\sum_{k=1}^{n} p_k \|(q_k T_1 + r_k T_2)(h)\|^2 = 0 \Leftrightarrow T = 0.$$

Proof:

Let $T_1, T_2 \in B(X)$ and $h \in M_T$. Assume (9) under the condition (10). Since, $T_1 = T_2 = T$, we have

$$\begin{aligned} & \sum_{k=1}^{n} p_k \|(q_k T + r_k T)(h)\|^2 = 0 \\ \Rightarrow \quad & \sum_{k=1}^{n} p_k |q_k + r_k|^2 \|Th\|^2 = 0 \\ \Rightarrow \quad & \|Th\|^2 = 0 \\ \Rightarrow \quad & \|Th\| = 0. \end{aligned}$$

As $h \in M_T$, we can write $\|Th\| = \|T\|$. Therefore, we may conclude that

$$\|T\| = 0 \Rightarrow T = 0. \tag{11}$$

The converse part is obvious. ∎

Theorem 3.5.

Let $\{U_n\}$ and $\{V_n\}$ be sequences of norm attaining bounded linear operators on a Banach space X. Then,

$$U_n \perp_C^O V_n \Rightarrow U \perp_C^O V.$$

Proof:

In the case of linear operators, boundedness and continuity are equivalent. By the continuity of the U_n's and V_n's, we can write

$$\lim_{n\to\infty} U_n(h) = U(h) \quad \text{and} \quad \lim_{n\to\infty} V_n(h) = V(h).$$

Since, U_n is Carlsson orthogonal to V_n and $h \in M_T$, we have

$$\sum_{k=1}^{n} p_k \|(q_k U_n + r_k V_n)(h)\|^2 = 0, \tag{12}$$

under the condition (10). It follows that

$$\begin{aligned} & \lim_{n\to\infty} \sum_{k=1}^{n} p_k \|(q_k U_n + r_k V_n)(h\|^2 = 0 \\ \Rightarrow \quad & \sum_{k=1}^{n} p_k \|(q_k U + r_k V)(h)\|^2 = 0 \\ \Rightarrow \quad & U \perp_C^O V. \end{aligned}$$

Therefore, the Carlsson orthogonality satisfies continuity property of an inner-product space. ∎

Bottazzi et al. (2020) defined the disjoint support as follows: Let H be a real or complex Hilbert space and $T_1, T_2 \in B(H)$. Then, two operators T_1 and T_2 have disjoint support if and only if

$$T_1 T_2^* = T_2^* T_1 = 0. \tag{13}$$

Theorem 3.6.

Let T_1 and T_2 be norm attaining bounded linear operators on a Hilbert space H with disjoint support. Then, $T_1 \perp_C^O T_2$ implies that $T_2 \perp_C^O T_1$.

Proof:

Let $T_1, T_2 \in B(H)$ and $h \in M_T$. Then, $\|(T_1 + T_2)(h)\| = \|T_1 + T_2\|$. Since T_1 and T_2 have disjoint support then, equation (13) holds. Suppose $T_1 \perp_C^O T_2$. Then, for $h \in M_T$, Equation (9) under Condition (10) can be written as

$$\sum_{k=1}^{n} p_k[\langle q_k T_1 h + r_k T_2 h, q_k T_1 h + r_k T_2 h\rangle] = 0.$$

It follows that

$$\begin{aligned}
& \sum_{k=1}^{n} a_k[\|q_k T_1\|^2 + \|r_k T_2 h\|^2 + q_k r_k \langle T_1 h, T_2 h\rangle + q_k r_k \langle T_2 h, T_1 h\rangle] = 0 \\
\Rightarrow \quad & \sum_{k=1}^{n} p_k[\|q_k T_1 h\|^2 + \|r_k T_2 h\|^2 + 2 q_k r_k Re\, \langle T_2^* T_1 h, h\rangle] = 0 \\
\Rightarrow \quad & \sum_{k=1}^{n} p_k[\|q_k T_1 h\|^2 + \|r_k T_2 h\|^2 = 0 \\
\Rightarrow \quad & \sum_{k=1}^{n} p_k q_k^2 \|T_1\|^2 + a_k r_k^2 \|T_2\|^2 = 0. \qquad (14)
\end{aligned}$$

Similarly, if $T_2 \perp_C^O T_1$, then we have

$$\sum_{k=1}^{n} p_k q_k^2 \|T_2\|^2 + p_k r_k^2 \|T_1\|^2 = 0. \qquad (15)$$

Replacing the role of constants in Equations (14) and (15), we can conclude that $T_2 \perp_C^O T_1$. ∎

Bottazi et al. (2020) studied the isosceles orthogonality of bounded (positive) linear operators on Hilbert space with some of the related properties, including operators having disjoint support. Let T_1 and T_2 are norm attaining bounded linear operators in a Banach space X. Then, T_1 is said to be isosceles orthogonal to T_2 if for every $h \in M_T$,

$$\|(T_1 - T_2)(h)\| = \|(T_1 + T_2)(h)\|. \qquad (16)$$

Also, in the same paper, the Pythagorean orthogonal for operators was defined as follows: T_1 is said to be Pythagorean orthogonal to T_2 if for every $h \in M_T$,

$$\|(T_1 - T_2)(h)\|^2 = \|T_1\|^2 + \|T_2\|^2 \text{or} \|(T_1 + T_2)(h)\|^2 = \|T_1\|^2 + \|T_2\|^2. \qquad (17)$$

Theorem 3.7.

Let T_1 and T_2 be norm attaining bounded linear operators with the disjoint support in a Hilbert space H. Then, T_1 is isosceles orthogonal to T_2 if and only if T_1 is Pythagorean orthogonal to T_2.

Proof:

Let $T_1, T_2 \in B(H)$. Assume T_1 is isosceles orthogonal to T_2 and $h \in M_T$. Using the relation (16), we have

$$\begin{aligned}\|(T_1 - T_2)(h)\|^2 &= \|(T_1 + T_2)(h)\|^2 \\ &= \|T_1(h)\|^2 + \|T_2(h)\|^2 + 2Re\langle T_1 T_2^* h, h\rangle \\ &= \|T_1(h)\|^2 + \|T_2(h)\|^2 \\ &= \|T_1\|^2 + \|T_2\|^2. \end{aligned} \tag{18}$$

This shows that T_1 is Pythagorean orthogonal to T_2. Conversely assume that, for any $h \in M_T$,

$$\begin{aligned}\|(T_1 - T_2)(h)\|^2 &= \|T_1(h)\|^2 + \|T_2(h)|^2 \\ &= \|(T_1 + T_2)(h)\|^2. \end{aligned}$$

Therefore,

$$\|(T_1 - T_2)(h)\| = \|(T_1 + T_2)(h)\|.$$

That is,

$$\|T_1 - T_2\| = \|T_1 + T_2\|. \tag{19}$$

■

The following theorem gives the characterization of Birkhoff-James orthogonality for operators.

Theorem 3.8.

Let T_1 and T_2 be norm attaining bounded linear operators in a Hilbert space H. If T_1 is Pythagorean orthogonal to T_2, then T_1 is Birkhoff-James orthogonal to T_2, but the converse may not be true.

Proof:

Let $T_1, T_2 \in B(H)$ such that T_1 is Pythagorean orthogonal to T_2. Then, by using (17) with $h \in M_T$,

$$\begin{aligned} & \|(T_1 + \lambda T_2)(h)\|^2 = \|T_1(h)\|^2 + \|\lambda T_2(h)\|^2 \\ \Rightarrow \quad & \|(T_1 + \lambda T_2)(h)\|^2 \geq \|T_1(h)\|^2 \\ \Rightarrow \quad & \|(T_1 + \lambda T_2)(h)\| \geq \|T_1(h)\|^2 \\ \Rightarrow \quad & \|T_1 + \lambda T_2\| \geq \|T_1\|. \end{aligned} \tag{20}$$

This shows that T_1 is Birkhoff-James orthogonal to T_2. ■

The following example shows that the converse of above theorem may not be true.

Example 3.1.

Suppose that H is the two dimensional Hilbert space. Consider the Banach space $B(H)$. Let $T_1 =$

$\begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$, $T_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in B(H)$ and $\lambda \in [0, 1]$. We know that $\|T\|^2 = \text{trace}(T^*T)$. Then,

$$T_1 + \lambda T_2 = \begin{pmatrix} 4+\lambda & 0 \\ 0 & 3+\lambda \end{pmatrix},$$
$$(T_1 + \lambda T_2)^* = \begin{pmatrix} 4+\lambda & 0 \\ 0 & 3+\lambda \end{pmatrix},$$
$$(T_1 + \lambda T_2)^*(T_1 + \lambda T_2) = \begin{pmatrix} (4+\lambda)^2 & 0 \\ 0 & (3+\lambda)^2 \end{pmatrix}.$$

Thus,

$$\text{trace}[(T_1 + \lambda T_2)^*(T_1 + \lambda T_2)] = (4+\lambda)^2 + (3+\lambda)^2. \tag{21}$$

Since, $\lambda \in [0, 1]$,

$$\begin{aligned} & \min[\text{trace}((T_1 + \lambda T_2)^*(T_1 + \lambda T_2))] \geq 25 \\ \Rightarrow \quad & \|T_1 + \lambda T_2\|^2 \geq 25 \\ \Rightarrow \quad & \|T_1 + \lambda T_2\| \geq 5. \end{aligned} \tag{22}$$

Similarly we can find $\|T_1\| = 5$. Therefore, we may conclude that T_1 is Birkhoff-James orthogonal to T_2. On the other hand,

$$T_1 + T_2 = \begin{pmatrix} 5 & 0 \\ 0 & 4 \end{pmatrix} \text{and}[(T_1 + T_2)^*(T_1 + T_2)] = \begin{pmatrix} 25 & 0 \\ 0 & 16 \end{pmatrix}.$$

It follows that $\|T_1 + T_2\|^2 = 41$. However, $\|T_1|^2 + \|T_2\|^2 = 26$. Therefore, T_1 is not Pythagorean orthogonal to T_2.

Definition 3.2.

Let T_1 and T_2 be norm attaining bounded linear operators in a Banach space X. An operator $T_1 \in B(X)$ is said to be orthogonal to $T_2 \in B(X)$ if and only if for any $h \in M_T$,

$$\|(T_1 + \frac{1}{2}T_2)(h)\|^2 + \|(T_1 - \frac{1}{2}T_2)(h)\|^2 = \frac{1}{2}\|(\sqrt{2}T_1 + T_2)(h)\|^2 + \|T_1(h)\|^2. \tag{23}$$

Theorem 3.9.

Let T_1 and T_2 be bounded linear operators in a real Hilbert space H. Then, orthogonality relation (23) implies the Birkhoff-James orthogonality, but the converse may not be true.

Proof:

Let $T_1, T_2 \in B(H)$ and $h \in M_T$. Assume that T_1 is orthogonal to T_2. Then, we have

$$\begin{aligned} & \|(T_1 + \frac{1}{2}T_2)(h)\|^2 + \|(T_1 - \frac{1}{2}T_2)(h)\|^2 = \frac{1}{2}\|(\sqrt{2}T_1 + T_2)(h)\|^2 + \|T_1(h)\|^2 \\ \Rightarrow \quad & \|T_2(h)\|^2 \geq \|T_1(h)\|^2. \end{aligned}$$

Setting $T_2 = \frac{T_1}{1-\alpha}$, so that $T_2 = T_1 + \alpha T_2$ and we get

$$\|(T_1 + \alpha T_2)(h)\|^2 \geq \|T_1(h)\|^2.$$

This implies that

$$\|(T_1 + \alpha T_2)(h)\| \geq \|T_1(h). \tag{24}$$

Hence, T_1 is Birkhoff-James orthogonal to T_2. To disprove the above statement, we can take operators $T_1, T_2 \in B(H)$ as described in the example of Theorem 3.8, showing that T_1 is Birkhoff-James orthogonal to T_2. On the other hand,

$$\|(T_1 + \frac{1}{2}T_2)\|^2 + \|(T_1 - \frac{1}{2}T_2)\|^2 = 51. \tag{25}$$

However,

$$\frac{1}{2}\|(\sqrt{2}T_1 + T_2)\|^2 + \|T_1^2\| = 27 + 7\sqrt{2}, \tag{26}$$

showing that T_1 is not orthogonal to T_2. ■

4. Conclusion

We conclude that the Pythagorean and the isosceles orthogonalities via $2-HH$ norm on a normed linear space satisfies the non-degeneracy, continuity and symmetry property; moreover, the homogeneity and additivity of the isosceles orthogonality with respect to $2-HH$ norm are equivalent. We also found that the Carlsson orthogonality for norm attaining bounded linear operators is continuous; in addition, the Pythagorean orthogonalisty implies the Birkhoff-James orthogonality. In the case of norm attaining bounded linear operators on a Hilbert space H with disjoint support, the Carlsson orthogonality is symmetric and the isosceles orthogonality is equivalent to the Pythagorean orthogonality. Finally, the orthogonality of bounded linear operators in a real Hilbert space H implies Birkhoff-James orthogonality, but the converse may not be true.

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