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Approximate 2-dimensional Pexider Quadratic Functional Equations In Fuzzy Normed Spaces and Topological Vector Space

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Abstract

In this paper, we prove the Hyers-Ulam stability of the 2-dimensional Pexider quadratic functional equation in fuzzy normed spaces. Moreover, we prove the Hyers-Ulam stability of this functional equation, where f, g are functions defined on an abelian group with values in a topological vector space.

Keywords: Hyers-Ulam stability; 2-dimensional Pexider quadratic functional equation; Fuzzy normed space; Topological vector space

MSC 2010 No.: 39B82, 39B52, 54A40, 46S40, 47S40

1. Introduction

The theory of fuzzy sets was introduced by Zadeh (1965). Fuzzy set theory is a powerful set for modeling uncertainty and vagueness in various problems arising in the field of science and engineering. The fuzzy topology proves to be a very useful tool to deal with such situations where the use of classical theories breaks down. In 1984, Katsaras introduced an idea of a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. In the same year, Wu and Fang (1984) introduced a notion fuzzy normed space to give a generalization of the Kolmogoroff normalized theorem for fuzzy topological vector spaces. In 1992, Felbin introduced an alternative definition of a fuzzy norm on a vector space with an associated metric of Kaleva and Seikkala type (1984). Some mathematics have defined fuzzy norms on vector spaces from various point of view (Krishna and Sarma (1994), Park (2009), Xiao and Zhu (2003)). In particular, Cheng and Mordeson (1994), following Bag and Samanta (2003), gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric of Kramosil and Michalek type (1975). They established a decomposition theorem of fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces.

The first stability problem concerning group homomorphisms was raised by Ulam (1940). In 1941, Hyers gave a first affirmative answer to the question of Ulam in context of Banach spaces. Subsequently, the result of Hyers was generalized by Aoki (1950) for additive mapping and by Rassias (1978) for linear mapping by considering an unbounded Cauchy difference. Furthermore, in 1994, Găvruta provided a further generalization of Rassias' theorem in which he replaced the bound $\varepsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$. Recently, several stability results have been obtained for various equations and mappings with more general domains and ranges have been investigated by a number of authors and there are many interesting results concerning this problem (Abolfathi et al. (2014), Al-Faid and Mohiuddine (2013), Alotaibi and Mohiuddine (2012), Alotaibi and Mursaleen (2014), Ebadian et al. (2013), (2014), (2015), Park et al. (2013), Mirmostafaee et al. (2008), Mohiuddine (2009), Mohiuddine at al. (2011), (2012), Mursaleen and Ansari (2013), Mursaleen and Mohiuddine (2009)).

Recently, Adam and Czerwik (2007) investigated the problem of the Hyers-Ulam stability of a generalized quadratic functional equation in topological vector spaces.

In this paper, we prove the Hyers-Ulam stability of the following 2-dimensional Pexider quadratic functional equation

$$f(x+y, z+w) + f(x-y, z-w) = 2g(x, z) + 2g(y, w),$$

in fuzzy normed spaces and in topological vector spaces.

2. Approximatie 2-dimensional Pexider Quadratic Functional Equation in Fuzzy Normed Spaces

In this section, assume that X is a linear space, (Y, N) is a fuzzy Banach space and (Z, N') is a fuzzy normed space. We prove the Hyers-Ulam stability of the 2-dimensional Pexider quadratic functional equation

$$f(x+y, z+w) + f(x-y, z-w) = 2g(x, z) + 2g(y, w),$$

in fuzzy normed spaces.

It is easy to show the following lemma.

Lemma 2.1.

Let $\varphi : X \times X \times X \times X \to [0,\infty)$ and let $f, g : X \times X \to Y$ be mappings satisfying $f(0,0) = g(0,0) = 0$ and

$$N(f(x+y,z+w)+f(x-y,z-w)-2g(x,z)-2g(y,w),t) \geq N'(\varphi(x,y,z,w),t),$$

for all $x,y,z,w \in X$ and all $t > 0$. Then

$$\begin{aligned}N(f(x+y,z+w)+f(x-y,z-w)-2f(x,z)-2f(y,w),t)\\ \geq \min\{N'(\varphi(x,0,z,0),t), N'(\varphi(x,x,z,z),t), N'(\varphi(y,0,w,0),t)\},\end{aligned}$$

and

$$\begin{aligned}&N(g(x+y,z+w)+g(x-y,z-w)-2g(x,z)-2g(y,w),t)\\ &\geq \min\{N'(\varphi(x,y,z,w),t)N'(\varphi(x+y,0,z+w,0),2t), N'(\varphi(x-y,0,z-w,0),2t)\},\end{aligned}$$

for all $x,y,z,w \in X$ and all $t > 0$.

Theorem 2.1.

Let $\varphi : X \times X \times X \times X \to Z$ be a mapping such that, for some $0 < \alpha < 4$

$$N'(\varphi(2x,2y,2z,2w),t) \geq N'(\alpha\varphi(x,y,z,w),t), \tag{1}$$

for all $x,y,z,w \in X$ and all $t > 0$. Let $f, g : X \times X \to Y$ be mappings satisfying $f(0,0) = g(0,0) = 0$ and

$$N(f(x+y,z+w)+f(x-y,z-w)-2g(x,z)-2g(y,w),t) \geq N'(\varphi(x,y,z,w),t), \tag{2}$$

for all $x,y,z,w \in X$ and all $t > 0$. Then, there exists a unique 2-dimensional quadratic mapping $\mathcal{Q} : X \times X \to Y$ such that

$$N(\mathcal{Q}(x,z)-f(x,z),t) \geq \min\{N'(2\varphi(x,0,z,0),(4-\alpha)t), N'(\varphi(x,x,z,z),(4-\alpha)t)\}, \tag{3}$$

and

$$\begin{aligned}&N(\mathcal{Q}(x,z)-g(x,z),t)\\ &\geq \min\{N'(4\varphi(x,0,z,0),(4-\alpha)t), N'(2\varphi(x,x,z,z),(4-\alpha)t), N'(\varphi(x,0,z,0),t)\},\end{aligned} \tag{4}$$

for all $x,z \in X$ and all $t > 0$.

Proof:

Putting $y = 0$ and $w = 0$ in (2), we get

$$N\left(f(x,z)-g(x,z),\frac{t}{2}\right) \geq N'(\varphi(x,0,z,0),t). \tag{5}$$

Replacing y and w by x and z in (2), respectively, we get

$$N\left(\frac{1}{4}f(2x,2z)-g(x,z),\frac{t}{4}\right) \geq N'(\varphi(x,x,z,z),t).$$

Hence,

$$N\left(\frac{1}{4}f(2x,2z)-f(x,z),t\right) \geq \min\{N'(\varphi(x,0,z,0),2t), N'(\varphi(x,x,z,z),4t)\}, \tag{6}$$

for all $x, z \in X$ and all $t > 0$. Replacing x, z by $2^n x$ and $2^n z$ in (6), respectively, and dividing both sides by 4^n and using (1), we get

$$\begin{aligned} &N\left(\frac{1}{4^{(n+1)}}f(2^{n+1}x, 2^{n+1}z) - \frac{1}{4^n}f(2^n x, 2^n z), \frac{t}{4^n}\right) \\ &\geq \min\{N'(\varphi(2^n x, 0, 2^n z, 0), 2t), N'(\varphi(2^n x, 2^n x, 2^n z, 2^n z), 4t)\} \\ &\geq \min\{N'(\varphi(x, 0, z, 0), \frac{2t}{\alpha^n}), N'(\varphi(x, x, z, z), \frac{4t}{\alpha^n})\}, \end{aligned} \tag{7}$$

for all $x, z \in X$ and all $t > 0$. Replacing t by $\alpha^n t$ in (7), we get

$$\begin{aligned} &N\left(\frac{1}{4^{n+1}}f(2^{n+1}x, 2^{n+1}z) - \frac{1}{4^n}f(2^n x, 2^n z), \frac{\alpha^n t}{4^{n+1}}\right) \\ &\geq \min\{N'(\varphi(x, 0, z, 0), \frac{t}{2}), N'(\varphi(x, x, z, z), t)\}, \end{aligned}$$

for all $x, z \in X$ and all $t > 0$. So

$$\begin{aligned} &N\left(\frac{1}{4^n}f(2^n x, 2^n z) - f(x, z), \sum_{i=0}^{n-1}\frac{\alpha^i}{4^{i+1}}t\right) \\ &= N\left(\sum_{i=0}^{n-1}[\frac{1}{4^{i+1}}f(2^{i+1}x, 2^{i+1}z) - \frac{1}{4^i}f(2^i x, 2^i z)], \sum_{i=0}^{n-1}\frac{\alpha^i}{4^{i+1}}t\right) \\ &\geq \min\{N'(\varphi(x, 0, z, 0), \frac{t}{2}), N'(\varphi(x, x, z, z), t)\}, \end{aligned} \tag{8}$$

for all $x, z \in X$ and all $t > 0$. Replacing x by $2^p x$ and z by $2^p z$ in (8), we have

$$\begin{aligned} &N\left(\frac{1}{4^{n+p}}f(2^{n+p}x, 2^{n+p}z) - \frac{1}{4^p}f(2^p x, 2^p z), \sum_{i=0}^{n-1}\frac{\alpha^i}{4^{i+p+1}}t\right) \\ &\geq \min\{N'(\varphi(2^p x, 0, 2^p z, 0), \frac{t}{2}), N'(\varphi(2^p x, 2^p x, 2^p z, 2^p z), t)\} \\ &\geq \min\{N'(\varphi(x, 0, z, 0), \frac{t}{2\alpha^p}), N'(\varphi(x, x, z, z), \frac{t}{\alpha^p})\}, \end{aligned}$$

for all $x, z \in X$, all $t > 0$, all $p \geq 0$ and all $n \in \mathbb{N}$. So

$$\begin{aligned} &N\left(\frac{1}{4^{n+p}}f(2^{n+p}x, 2^{n+p}z) - \frac{1}{4^p}f(2^p x, 2^p z), t\right) \\ &\geq \min\{N'(\varphi(x, 0, z, 0), \frac{t}{2\sum_{i=p}^{p+n-1}\frac{\alpha^i}{4^{i+1}}}), N'(\varphi(x, x, z, z), \frac{t}{\sum_{i=p}^{p+n-1}\frac{\alpha^i}{4^{i+1}}})\}, \end{aligned} \tag{9}$$

for all $x, z \in X$, all $t > 0$ all $p \geq 0$ and all $n \in \mathbb{N}$. Since $0 < \alpha < 4$ and $\sum_{i=0}^{\infty}(\frac{\alpha}{4})^i < \infty$ $(\sum_{i=p}^{p+n-1}\frac{\alpha^i}{4^{i+1}} \to 0$ as $p \to \infty$ for all $x, z \in X)$, the Cauchy criterion for convergence show that is a Cauchy sequence in (Y, N). Since (Y, N) is a fuzzy Banach space, this sequence converges to some point $\mathcal{Q}(x, y) \in Y$ for all $x, z \in X$.

Therefore,

$$\lim_{n\to\infty} N\left(\mathcal{Q}(x,z) - \frac{1}{4^n} f(2^n x, 2^n z), t\right) = 1.$$

In addition, by putting $p = 0$ in (9), we get

$$N\left(\frac{1}{4^n} f(2^n x, 2^n z) - f(x,z), t\right)$$
$$\geq \min\left\{N'\left(\varphi(x,0,z,0), \frac{t}{2\sum_{i=0}^{n-1} \frac{\alpha^i}{4^{i+1}}}\right), N'\left(\varphi(x,x,z,z), \frac{t}{\sum_{i=0}^{n-1} \frac{\alpha^i}{4^{i+1}}}\right)\right\},$$

for all $x, z \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$, we have

$$N(\mathcal{Q}(x,z) - f(x,z), t)$$
$$\geq \min\{N'(2\varphi(x,0,z,0), (4-\alpha)t), N'(\varphi(x,x,z,z), (4-\alpha)t)\}.$$

By (2.1) and (1),

$$N\left(\frac{1}{4^n} f(2^n x + 2^n y, 2^n z + 2^n w) + \frac{1}{4^n} f(2^n x - 2^n y, 2^n z - 2^n w)\right.$$
$$\left.-2\frac{1}{4^n} f(2^n x, 2^n z) - 2\frac{1}{4^n} f(2^n y, 2^n w), t\right)$$
$$\geq \min\{N'(\varphi(x,0,z,0), (\frac{4}{\alpha})^n t), N'(\varphi(x,x,z,z), (\frac{4}{\alpha})^n t), N'(\varphi(y,0,w,0), (\frac{4}{\alpha})^n t)\},$$

for all $x, z \in X$, all $t > 0$ and $n \in \mathbb{N}$.

Since

$$\lim_{n\to\infty} N'\left(\varphi(x,y,z,w), (\frac{4}{\alpha})^n t\right) = 1,$$
$$\lim_{n\to\infty} N'\left(\varphi(x,0,z,0), (\frac{4}{\alpha})^n t\right) = 1,$$
$$\lim_{n\to\infty} N'\left(\varphi(y,0,w,0), (\frac{4}{\alpha})^n t\right) = 1,$$

we observe that $\mathcal{Q}$ is 2-dimensional quadratic mapping.

On the other hand, by (3) and (5), we get

$$N(\mathcal{Q}(x,z) - g(x,z), t) \geq \min\{N(\mathcal{Q}(x,z) - f(x,z), \frac{t}{2}), N(f(x,z) - g(x,z), \frac{t}{2})\}$$
$$\geq \{N'(4\varphi(x,0,z,0), (4-\alpha)t), N'(2\varphi(x,x,z,z), (4-\alpha)t), N'(\varphi(x,0,z,0), t)\}.$$

To prove the uniqueness, assume that there exists another mapping $\mathcal{Q}' : X \times X \to Y$ which also satisfies (3) and (4). Then we get

$$N(\mathcal{Q}(x,z)-\mathcal{Q}'(x,z),t) = N\left(\frac{1}{4^n}\mathcal{Q}(2^nx,2^nz)-\frac{1}{4^n}\mathcal{Q}'(2^nx,2^nz),t\right)$$
$$\geq \min\{N(\frac{1}{4^n}\mathcal{Q}(2^nx,2^nz)-\frac{1}{4^n}f(2^nx,2^nz),\frac{t}{2})\},$$
$$N(\frac{1}{4^n}\mathcal{Q}'(2^nx,2^nz)-\frac{1}{4^n}f(2^nx,2^nz),\frac{t}{2})$$
$$\geq \min\{N'(\varphi(2^nx,0,2^nz,0),\frac{4^n(4-\alpha)}{2}t),N'(\varphi(2^nx,2^nx,2^nz,2^nz),\frac{4^n(4-\alpha)}{2}t)\}$$
$$\geq \min\{N'(\varphi(x,0,z,0),\frac{4^n(4-\alpha)}{2\alpha^n}t),N'(\varphi(x,x,z,z),\frac{4^n(4-\alpha)}{2\alpha^n}t)\}.$$

Since $\lim\limits_{n\to\infty}\dfrac{4^n(4-\alpha)}{2\alpha^n}t=\infty$ for all $t>0$, we have

$$\lim_{n\to\infty}N'\left(\varphi(x,y,z,w),\frac{4^n(4-\alpha)}{2\alpha^n}t\right)=1,$$
$$\lim_{n\to\infty}N'\left(\varphi(x,0,z,0),\frac{4^n(4-\alpha)}{2\alpha^n}t\right)=1,$$

for all $t>0$.

Therefore, $N(\mathcal{Q}(x,z)-\mathcal{Q}'(x,z),t)=1$ for all $t>0$. Hence

$$\mathcal{Q}(x,z)=\mathcal{Q}'(x,z),$$

for all $x,z\in X$. This completes the proof. ■

Corollary 2.1.

Let X be a normed linear space with norm $\|\cdot\|$, (Y,N) be a fuzzy Banach space and N' be fuzzy norm defined in Example 1.2. Assume that $\delta>0$ and p is a real number with $0<p<2$. Let $f,g:X\times X\to Y$ be mappings satisfying $f(0,0)=g(0,0)=0$ and

$$N(f(x+y,z+w)+f(x-y,z-w)-2g(x,z)-2g(y,w),t)$$
$$\geq N'(\delta(\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p),t),$$

for all $x,y,z,w\in X$ and all $t>0$. Then, there exists a unique 2-dimensional quadratic mapping $\mathcal{Q}:X\times X\to Y$ such that

$$N(\mathcal{Q}(x,z)-f(x,z),t)\geq\frac{(4-2^p)t}{(4-2^p)t+2\delta(\|x\|^p+\|z\|^p)},$$

and

$$N(\mathcal{Q}(x,z)-g(x,z),t)\geq\min\left\{\frac{(4-2^p)t}{(4-2^p)t+4\delta(\|x\|^p+\|z\|^p)},\frac{t}{t+\delta(\|x\|^p+\|z\|^p)}\right\},$$

for all $x,y,z,w\in X$ and all $t>0$.

Proof:

Let $\varphi(x,y,z,w)=\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p$ for all $x,y,z,w\in X$. Choosing $\alpha=2^p$ in Theorem 2.1, we get the desired result. ■

In the following theorem, we consider the case $\alpha > 4$.

Theorem 2.2.

Let $\varphi : X \times X \times X \times X \to Z$ be a mapping such that, for some $\alpha > 4$,

$$N'\left(\varphi(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, \frac{w}{2}), t\right) \geq N'(\varphi(x, y, z, w), \alpha t),$$

for all $x, y, z, w \in X$ and all $t > 0$. Let $f, g : X \times X \to Y$ be mappings satisfying (2) and $f(0, 0) = g(0, 0) = 0$. Then there exists a unique 2-dimensional quadratic mapping $\mathcal{Q} : X \times X \to Y$ such that

$$N(\mathcal{Q}(x, z) - f(x, z), t) \geq \min\{N'(2\varphi(x, 0, z, 0), (\alpha - 4)t), N'(\varphi(x, x, z, z), (\alpha - 4)t)\},$$

and

$$N(\mathcal{Q}(x, z) - g(x, z), t) \\ \geq \min\{N'(4\varphi(x, 0, z, 0), (\alpha - 4)t), N'(2\varphi(x, x, z, z), (\alpha - 4)t), N'(\varphi(x, 0, z, 0), t)\},$$

for all $x, z \in X$ and all $t > 0$.

Proof:

The technique is similar to that of Theorem 2.1. Replacing x and z by $\frac{x}{2}$ and $\frac{z}{2}$ in (6), respectively,

$$N\left(f(x, z) - 4f\left(\frac{x}{2}, \frac{z}{2}\right), t\right) \geq \min\{N'(2\varphi(x, 0, z, 0), \alpha t), N'(\varphi(x, x, z, z), \alpha t)\},$$

for all $x, z \in X$ and all $t > 0$. We can deduce

$$N\left(4^p f\left(\frac{x}{2^p}, \frac{z}{2^p}\right) - 4^{n+p} f\left(\frac{x}{2^{n+p}}, \frac{z}{2^{n+p}}\right), t\right) \tag{10}$$
$$\geq \min\left\{N'\left(\varphi(x, 0, z, 0), \frac{t}{2\sum_{i=p}^{p+n-1} \frac{4^i}{\alpha^{i+1}}}\right), N'\left(\varphi(x, x, z, z), \frac{t}{\sum_{i=p}^{p+n-1} \frac{4^i}{\alpha^{i+1}}}\right)\right\},$$

for all $x, z \in X$, all $t > 0$, all $p \geq 0$ and all $n \in \mathbb{N}$. Hence, the sequence $\{4^n f(\frac{x}{2^n}, \frac{z}{2^n})\}$ is a Cauchy sequence in the fuzzy Banach space Y. Therefore, there is a mapping $\mathcal{Q} : X \times X \to Y$ defined by $\mathcal{Q}(x, z) = \lim_{n\to\infty} 4^n f(\frac{x}{2^n}, \frac{z}{2^n})$ for all $x, z \in X$. Let $p = 0$ in (10). Then we have

$$N(\mathcal{Q}(x, z) - f(x, z), t) \geq \min\{N'(2\varphi(x, 0, z, 0), (\alpha - 4)t), N'(\varphi(x, x, z, z), (\alpha - 4)t)\},$$

for all $x, z \in X$ and all $t > 0$.

The rest of the proof is similar to the proof of Theorem 2.1. ■

Corollary 2.2.

Let X be a normed linear space with norm $\|\cdot\|$, (Y, N) be a fuzzy Banach space and

$$N'(x, t) = \begin{cases} \dfrac{\alpha t}{\alpha t + \beta\|x\|}, & t > 0, \quad x \in X, \\ 0, & t \leq 0, \quad x \in X. \end{cases}$$

Assume that $\delta > 0$ and p is a real number with $p > 2$. Let $f, g : X \times X \to Y$ be mappings satisfying $f(0,0) = g(0,0) = 0$ and

$$N(f(x+y,z+w)+f(x-y,z-w)-2g(x,z)-2g(y,w),t) \geq N'(\delta(\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p),t),$$

for all $x, y, z, w \in X$ and all $t > 0$. Then, there exists a unique 2-dimensional quadratic mapping $\mathcal{Q} : X \times X \to Y$ such that

$$N(\mathcal{Q}(x,z)-f(x,z),t) \geq \frac{(2^p-4)t}{(2^p-4)t+2\delta(\|x\|^p+\|z\|^p)},$$

and

$$N(\mathcal{Q}(x,z)-g(x,z),t) \geq \min\left\{\frac{(2^p-4)t}{(2^p-4)t+4\delta(\|x\|^p+\|z\|^p)}, \frac{t}{t+\delta(\|x\|^p+\|z\|^p)}\right\},$$

for all $x, y, z, w \in X$ and all $t > 0$.

Proof:

Let $\varphi(x,y,z,w) = \|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p$ for all $x, y, z, w \in X$. Choosing $\alpha = 2^p$ in Theorem 2.2, we get the desired result. ■

3. Approximatie 2-dimensional Pexider Quadratic Functional Equation in Topological Vector Spaces

In this section, we prove the Hyers-Ulam stability of the 2-dimensional Pexider quadratic functional equation,

$$f(x+y,z+w)+f(x-y,z-w) = 2g(x,z)+2g(y,w),$$

where f, g are mappings defined on an abelian group with values in a topological vector space.

We denote the convex hull of a set $U \subseteq X$ by $conv(U)$ and the sequential closer of U by $\overline{U}$. Moreover it is well-known that, if $A, B \subseteq X$ and real numbers α, β, then $\alpha \cdot conv(A) + \beta \cdot conv(B) = conv(\alpha A + \beta B)$.

Remark 3.1.

A trivial observation is that $0 \in conv(B-B)$, which will play an essential role in Section 3.

We start with the following lemma.

Lemma 3.1.

Let G be an abelian group and let $B \subseteq X$ be a nonempty subset. If mappings $f, g : G \times G \to X$ satisfy

$$f(x+y,z+w)+f(x-y,z-w)-2g(x,z)-2g(y,w) \in B, \tag{11}$$

then

$$f(x+y,z+w)+f(x-y,z-w)+2f(0,0)-2f(x,z)-2f(y,w)\in 2\cdot conv(B-B), \quad (12)$$

$$g(x+y,z+w)+g(x-y,z-w)+2g(0,0)-2g(x,z)-2g(y,w)\in conv(B-B), \quad (13)$$

for all $x,y,z,w\in G$.

Proof:

Putting $y=x=z=w=0$ in (11), we get

$$2f(0,0)-4g(0,0)\in B. \quad (14)$$

Setting $y=w=0$ in (11), we get

$$2f(x,z)-2g(x,z)-2g(0,0)\in B, \quad (15)$$

for all $x,z\in G$. Putting $x=y$ and $z=w$ in (15), we get

$$2f(y,w)-2g(y,w)-2g(0,0)\in B, \quad (16)$$

for all $y,w\in G$.

It follows from (11), (14), (15) and (16) that

$$\begin{aligned}&f(x+y,z+w)+f(x-y,z-w)+2f(0,0)-2f(x,z)-2f(y,w)\\&=[f(x+y,z+w)+f(x-y,z-w)-2g(x,z)-2g(y,w)]\\&-[2f(x,z)-2g(x,z)-2g(0,0)]-[2f(y,w)-2g(y,w)-2g(0,0)]\\&+[2f(0,0)-4g(0,0)]\\&\in B-B-B+B\subset 2\cdot convB+2\cdot conv(-B)=2\cdot conv(B-B),\end{aligned}$$

for all $x,y,z,w\in G$. If we replace x by $x+y$ and z by $z+w$ in (15), then we get

$$2f(x+y,z+w)-2g(x+y,z+w)-2g(0,0)\in B,$$

for all $x,y,z,w\in G$. On the other hand, if we replace x by $x-y$ and z by $z-w$ in (15), then

$$2f(x-y,z-w)-2g(x-y,z-w)-2g(0,0)\in B,$$

for all $x,y,z,w\in G$. Therefore,

$$\begin{aligned}&g(x+y,z+w)+g(x-y,z-w)+2g(0,0)-2g(x,z)-2g(y,w)\\&=[f(x+y,z+w)+f(x-y,z-w)-2g(x,z)-2g(y,w)]\\&-[f(x+y,z+w)-g(x+y,z+w)-g(0,0)]\\&-[f(x-y,z-w)-g(x-y,z-w)-g(0,0)]\\&\in B-\frac{1}{2}B-\frac{1}{2}B\subset convB+conv(-B)=conv(B-B),\end{aligned}$$

as desired. ■

Theorem 3.1.

Let G be an abelian group and let $B \subseteq X$ be a nonempty bounded subset. Suppose that mappings $f, g : G \times G \to X$ satisfy (3.1). Then there exists exactly one 2-dimensional quadratic mapping $\mathcal{Q} : G \times G \to X$ such that

$$\mathcal{Q}(x,y) - f(x,y) + f(0,0) \in \frac{2}{3}\overline{conv(B-B)}, \qquad \mathcal{Q}(x,y) - g(x,y) + g(0,0) \in \frac{1}{3}\overline{conv(B-B)}, \tag{17}$$

for all $x, y \in G$. Moreover, the mapping $\mathcal{Q}$ is given by

$$\mathcal{Q}(x,y) = \lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^n y) = \lim_{n\to\infty} \frac{1}{4^n} g(2^n x, 2^n y),$$

for all $x, y \in G$, and the convergence is uniform on $G \times G$.

Proof:

Setting $y = x, w = z$ in (12), we get

$$f(2x, 2z) - 4f(x,z) \in 2 \cdot conv(B-B) - 3f(0,0), \tag{18}$$

for all $x, z \in G$. Replacing z by y in (18), we have

$$f(2x, 2y) - 4f(x,y) \in 2 \cdot conv(B-B) - 3f(0,0), \tag{19}$$

for all $x, y \in G$. Replacing x by $2^n x$ and y by $2^n y$ in (19), we have

$$\frac{1}{4^{(n+1)}} f(2^{(n+1)}x, 2^{(n+1)}y) - \frac{1}{4^n} f(2^n x, 2^n y) \in \frac{1}{4^{(n+1)}}(2conv(B-B) - 3f(0,0)),$$

for all $x, y \in G$ and all integers n. Therefore,

$$\begin{aligned} \frac{1}{4^n} f(2^n x, 2^n y) - \frac{1}{4^m} f(2^m x, 2^m y) &= \Sigma_{k=m}^{k=n-1}\left(\frac{1}{4^{(k+1)}} f(2^{(k+1)}x, 2^{(k+1)}y) - \frac{1}{4^k} f(2^k x, 2^k y)\right) \\ &\in \Sigma_{k=m}^{k=n-1} \frac{1}{4^{(k+1)}}(2 \cdot conv(B-B) - 3f(0,0)) \\ &\subseteq \Sigma_{k=m}^{k=n-1} - \frac{3}{4^{(k+1)}} f(0,0) + \frac{2}{3}\frac{1}{4^m} conv(B-B), \end{aligned} \tag{20}$$

for all $x, y \in G$ and all integer $n > m \geq 0$. Since B is bounded, we conclude that $conv(B - B)$ is bounded. Therefore, $\{\frac{1}{4^n} f(2^n x, 2^n y)\}$ is a Cauchy sequence in X. Since X is a sequential complete topological vector space, the sequence $\{\frac{1}{4^n} f(2^n x, 2^n y)\}$ is convergent for $x, y \in G$, and the convergence is uniform on $G \times G$.

Define

$$\mathcal{Q}_1(x,y) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^n y),$$

for all $x, y \in G$. Letting $m = 0$ and $n \to \infty$ in (20), we get

$$\mathcal{Q}_1(x,y) - f(x,y) + f(0,0) \in \frac{2}{3}\overline{conv(B-B)}. \tag{21}$$

Replacing x, y, z by $2^n x, 2^n y, 2^n z$ and $2^n w$ in (12), respectively, we obtain

$$\begin{aligned}&\frac{1}{4^n} f(2^n(x+y), 2^n(z+w)) + \frac{1}{4^n} f(2^n(x-y), 2^n(z-w))\\ &- 2\frac{1}{4^n} f(2^n x, 2^n z) - 2\frac{1}{4^n} f(2^n y, 2^n w)\\ &\in \frac{1}{2n}(2 \cdot conv(B-B) - 2f(0,0)),\end{aligned}$$

for all $x, y, z, w \in G$. Since $conv(B-B)$ is bounded, letting $n \to \infty$, we get

$$\mathcal{Q}_1(x+y, z+w) + \mathcal{Q}_1(x-y, z-w) - 2\mathcal{Q}_1(x,z) - 2\mathcal{Q}_1(y,w) = 0,$$

for all $x, y, z, w \in G$, i.e., $\mathcal{Q}_1$ is a 2-dimensional quadratic mapping.

Similarly, applying (13), we have a 2-dimensional quadratic mapping $\mathcal{Q}_2 : G \times G \to X$ defined by $\mathcal{Q}_2(x,y) := \lim_{n\to\infty} \frac{1}{4^n} g(2^n x, 2^{2n} y)$ satisfying

$$\mathcal{Q}_2(x,y) - g(x,y) + g(0,0) \in \frac{1}{3}\overline{conv(B-B)}, \tag{22}$$

for all $x, y \in G$. Since B is bounded, it follows from (15) that $\mathcal{Q}_1 = \mathcal{Q}_2$. Letting $\mathcal{Q} := \mathcal{Q}_1$, we obtain (17) from (21) and (22).

To prove the uniqueness of $\mathcal{Q}$, suppose that there exists another 2-dimensional quadratic mapping $\mathcal{Q}' : G \times G \to X$ satisfying (17). Then

$$\mathcal{Q}'(x,y) - \mathcal{Q}(x,y) = [\mathcal{Q}'(x,y) - f(x,y) + f(0,0)] + [f(x,y) - f(0,0) - \mathcal{Q}(x,y)] \in \frac{4}{3}\overline{conv(B-B)},$$

for all $x, y \in G$. Since $\mathcal{Q}$ and $\mathcal{Q}'$ are 2-dimensional quadratic mappings, replacing x and y by $2^n x$ and $2^n y$, respectively, we get

$$\mathcal{Q}'(x,y) - \mathcal{Q}(x,y) = \frac{1}{4^n}\mathcal{Q}'(2^n x, 2^n y) - \frac{1}{4^n}\mathcal{Q}(2^n x, 2^n y) \in \frac{1}{3}\frac{1}{4^{(n-1)}}(\overline{conv(B-B)}),$$

for all $x, y \in G$ and all integers n. Since $conv(B-B)$ is bounded, we obtain $\mathcal{Q}' = \mathcal{Q}$. This completes the proof. ■

4. Conclusion

In the real world, there are many problems which have fuzzy cases. In this work, stability for functional equations in fuzzy spaces has been studied by using 2-dimensional Pexider quadratic functional equation. We showed that there is an approximate solution for this functional equation in fuzzy spaces. In addition, by using another type of control function, w generalized the stability of functional equations in vector topological spaces.

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