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On the Generalization of Interval Valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy Generalized Bi-ideals in Ordered Semigroups

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Abstract

In this paper, a new general form than interval valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy generalized bi-ideals in ordered semigroups is introduced. The concept of interval valued $\left(\in, \in \vee q_{\tilde{k}}^{\tilde{\delta}}\right)$-fuzzy generalized bi-ideals is initiated and several properties and characterizations are provided. A condition for an interval valued $\left(\in, \in \vee q_{\tilde{k}}^{\tilde{\delta}}\right)$-fuzzy generalized bi-ideal to be an interval valued fuzzy generalized bi-ideal is obtained. Using implication operators and the notion of implication-based an interval valued fuzzy generalized bi-ideal, characterizations of an interval valued fuzzy generalized bi-ideal and an interval valued $\left(\in, \in \vee q_{\tilde{k}}^{\tilde{\delta}}\right)$-fuzzy generalized bi-ideal are considered.

Keywords: Interval valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy generalized; Interval valued $\left(\in, \in \vee q_{\tilde{k}}^{\tilde{\delta}}\right)$-fuzzy generalized; Implication based interval valued $\left(\in, \in \vee q_{\tilde{k}}^{\tilde{\delta}}\right)$-fuzzy generalized

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1. Introduction

The fundamental notion of a fuzzy set, proposed by Zadeh (1965), provides a natural framework for generalizing several basic notions of algebra. The study of fuzzy sets in semigroups was introduced by Kuroki (1980, 1981, 1991). A systematic exposition of fuzzy semigroups was given by Mordeson et al. (2003), where one can find theoretical results on fuzzy semigroups and their use in fuzzy coding, fuzzy finite state machines and fuzzy languages. The monograph by Mordeson and Malik (2002) deals with the application of fuzzy approach to the concepts of automata and formal languages. Murali (2004) proposed the definition of a fuzzy point belonging to a fuzzy subset under a natural equivalence on fuzzy subset. The concept of a fuzzy set in topological structure has been studied by Pu and Liu (1980). The idea of quasicoincidence of a fuzzy point with a fuzzy set played a vital role to generate some different types of fuzzy subgroups. Bhakat and Das (1992, 1996) gave the concepts of (α, β)-fuzzy subgroups by using the belongs to relation $(\in)$ and quasi-coincident with relation (q) between a fuzzy point and a fuzzy subgroup and introduced the concept of an $(\in, \in \vee q)$-fuzzy subgroup.

Many researchers have used the idea of generalized fuzzy sets and gave several results in different branches of algebra. Jun and Song (2006) initiated the study of (α, β)-fuzzy interior ideals of a semigroup. Kazanci and Yamak (2008) studied $(\in, \in \vee q)$-fuzzy bi-ideals of a semigroup. Shabir et al. (2010) studied characterization of regular semigroups by (α, β)-fuzzy ideals. Jun et al. (2010) discussed a generalization of an $(\in, \in \vee q)$-fuzzy ideals of a BCK/BCI-algebra. Jan et al. (2016) introduced the notions of $(\in, \in \vee q)$-intuitionistic fuzzy BCI-subalgebras of BCI-algebras and investigated some of their related properties. For further study on generalized fuzzy sets in ordered semigroups, we refer the reader to Davvaz and Khan (2011), Jun et al. (2009), and Khan et al. (2012).

In mathematics, an ordered semigroup is a semigroup together with a partial order that is compatible with the semigroup operation. Ordered semigroups have many applications in the theory of sequential machines, formal languages, computer arithmetic and error-correcting codes. The concept of a fuzzy bi-ideal in ordered semigroups was first introduced by Kehayopulu and Tsingelis (2002), where some basic properties of fuzzy bi-ideals were discussed. A theory of fuzzy generalized sets on ordered semigroups can be developed. The notion of general form of the quasi-coincidence of a fuzzy point with a fuzzy set is initiated by Jun (submitted). Kang (2016) generalized the concept of $(\in, \in \vee q_k^\delta)$-fuzzy subsemigroup and initiated the notion of $(\in, \in \vee q_k)$-fuzzy-fuzzy subsemigroup of semigroups.

Interval valued fuzzy sets were introduced independently by Zadeh (1975), Grattan-Guiness (1975), John (1975), in the same year, where the value of the membership functions are intervals of numbers in place of the numbers. Interval-valued fuzzy sets were initiated as a natural extension of fuzzy sets. Thillaigovindan and Chinnadurai (2009) initiated the concept of interval valued fuzzy ideals (bi-ideals, interior ideals and quasi-ideals) in semigroups. Khan et al. (2013) introduced a new sort of interval valued fuzzy generalized bi-ideals in ordered semigroup called an interval valued $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of an ordered semigroup. In this paper, we attempt to have a new general form of an interval valued $(\in, \in \vee q)$-fuzzy generalized bi-ideal and interval valued $\left(\in, \in \vee q_{\widetilde{k}}\right)$-fuzzy generalized bi-ideal of an ordered semigroup.

The concept of an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy generalized bi-ideal of an ordered semigroup is a generalization of the concepts studied in Khan et al. (2013). If we take $\delta = 1$, then we get an interval valued $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of an ordered semigroup. If we take $\delta = 1$ and $k = 0$, then we get an interval valued $(\in, \in \vee q)$-fuzzy generalized bi-ideal. This means that these fuzzy substructures become a special case of an interval valued $(\in, \in \vee q_k^{\delta})$-fuzzy generalized bi-ideal. Due to the motivation and inspiration of the concept, we introduce the notion of an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy generalized bi-ideal of an ordered semigroup and give examples which are interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy generalized bi-ideals but not interval valued $(\in, \in \vee q)$-fuzzy generalized bi-ideals and not an interval valued $\left(\in, \in \vee q_{\widetilde{k}}\right)$-fuzzy generalized bi-ideal. Some characterizations of interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy generalized bi-ideals in ordered semigroups are discussed. We provide a condition for an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy generalized bi-ideal to be an interval valued fuzzy bi-ideal. We finally consider characterizations of an interval valued fuzzy generalized bi-ideal and an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy generalized bi-ideals by using implication operators and the notion of implication-based interval valued generalized fuzzy bi-ideals. The important achievement of the study with an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy generalized bi-ideal is that the notion of interval valued $(\in, \in \vee q)$-fuzzy generalized bi-ideal and interval valued $\left(\in, \in \vee q_{\widetilde{k}}\right)$-fuzzy generalized bi-ideal are special cases of an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy generalized bi-ideal.

2. Preliminaries

By an ordered semigroup (or po-semigroup) (Fuchs (1963)), we mean a structure $(S, \cdot, \leq)$ in which the followings are satisfied:

$(OS1)$ $(S, \cdot)$ is a semigroup,

$(OS2)$ $(S, \leq)$ is a poset,

$(OS3)$ $(\forall x, y, a \in S)\,(x \leq y \Rightarrow a \cdot x \leq a \cdot y,\ x \cdot a \leq a \cdot y)$.

A non-empty subset A of an ordered semigroup S is said to be a subsemigroup (Fuchs (1963)) of S if $A^2 \subseteq A$.

A non-empty subset F of an ordered semigroup S is called a left (respectively right) ideal (Kehayopulu (1990)) of S if it satisfies:

(1) $(\forall x \in S)\,(y \in F)\,(x \leq y \Rightarrow x \in A)$,

(2) $SA \subseteq A$ (respectively $AS \subseteq A$).

A non-empty subset A of S is called an ideal (Kehayopulu (1990)) if it is both a left and a right ideal of S.

A non-empty subset A of an ordered semigroup S is called a generalized bi-ideal of S if

(1) $(\forall x \in S)$ $(y \in F)$ $(x \leq y \Rightarrow x \in A)$,

(2) $ASA \subseteq A$.

A non-empty subset A of an ordered semigroup S is called a bi-ideal of S if

(1) $(\forall x \in S)$ $(y \in F)$ $(x \leq y \Rightarrow x \in A)$,

(2) $A^2 \subseteq A$,

(3) $ASA \subseteq A$.

A fuzzy subset λ (Zadeh (1965)) of a universe X is a function from X into the unit closed interval $[0, 1]$, that is, $\lambda : X \to [0, 1]$.

Let λ be a fuzzy set of a semigroup S and $t \in [0, 1]$. The set $U(\lambda; t) = \{x \in S | \lambda(x) \geq t\}$ is called a level subset of the fuzzy set λ.

Let λ be a fuzzy subset of S and $t \in [0, 1]$. Then, the set $U(\lambda; t) = \{x \in S : \lambda(x) \geq t\}$ is called the level subset of S.

Let A be a non-empty subset of S. We denote by λ_A, the characteristic function of A, that is, the mapping of S into $[0, 1]$ defined by

$$\lambda_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Obviously, λ_A is a fuzzy subset of S.

Let $(S, \cdot, \leq)$ be an ordered semigroup and λ a fuzzy subset of S. Then, λ of S is said to be a fuzzy subsemigroup (Kehayopulu and Tsingelis (2002)) of S if $(\forall u, v \in S)(\lambda(xy) \geq \lambda(x) \wedge \lambda(y))$.

A fuzzy subset λ is said to be a fuzzy left (right) ideal (Kehayopulu and Tsingelis (2002)) of S if,

1) $(\forall x, y \in S)(x \leq y \Rightarrow \lambda(x) \geq \lambda(y))$, and

2) $(\forall x, y \in S)(\lambda(xy) \geq \lambda(y)(\lambda(xy) \geq \lambda(x)))$.

A fuzzy subset λ of S is said to be a fuzzy ideal (Kehayopulu and Tsingelis (2002)) of S if it is both a fuzzy left and a fuzzy right ideal of S.

A fuzzy subsemigroup λ is said to be a fuzzy bi-ideal (Kehayopulu and Tsingelis (2005)) of S if:

1) $(\forall x, y \in S)(x \leq y \Rightarrow \lambda(x) \geq \lambda(y))$,

2) $(\forall x, y, z \in S)(\lambda(xyz) \geq \lambda(x) \wedge \lambda(z))$.

A fuzzy set λ in a set S of the form

$$\lambda(y) := \begin{cases} t \in (0,1], & \text{if } y = x, \\ 0, & \text{if } y \neq x, \end{cases} \tag{1}$$

is said to be a *fuzzy point* (Pu and Liu (1980)) with support x and value t and is denoted by (x, t).

For a fuzzy subset λ in S, a fuzzy point (x, t) said to be

• contained in λ denoted $(x, t) \in \lambda$, if $\lambda(x) \geq t$,

• quasi-coincident with λ, denoted by $(x, t)\, q\lambda$, if $\lambda(x) + t > 1$.

For a fuzzy subset λ and fuzzy point (x, t) in a set S, we say that

• $(x, t) \in \wedge q\lambda$ if $(x, t) \in \lambda$ or $(x, t)\, q\lambda$.

For a fuzzy subset λ and fuzzy point (x, t) in a set S and $k \in [0, 1)$, we say that $(x, t)\, q_k\lambda$ if $\lambda(x) + t + k > 1$ and $(x, t) \in \vee q_k\lambda$ if $(x, t) \in \lambda$ or$(x, t)\, q_k\lambda$. Jun and Xin considered the general form of the symbol $(x, t)\, q_k\lambda$ and $(x, t) \in \vee q_k\lambda$ as follows: for a fuzzy point (x, t) and fuzzy subset λ in a set X, we say that

$i)$ $(x, t)\, q^\delta\lambda$ if $\lambda(x) + t > \delta$,

$ii)$ $(x, t)\, q_k^\delta\lambda$ if $\lambda(x) + t + k > \delta$,

$iii)$ $(x, t) \in \vee q_k^\delta\lambda$ if $(x, t) \in \lambda$ or $(x, t)\, q_k^\delta\lambda$,

$iv)$ $(x, t)\, \overline{\alpha}\lambda$ if $(x, t)\, \alpha\lambda$ does not hold, for $\alpha \in \{\in, q, \in \vee q, \in \vee q_k, q_k^\delta, \in \vee q_k^\delta\}$,

where $k \in [0, 1)$ and $k < \delta$ in $[0, 1]$. Obviously, $(x, t)\, q^\delta\lambda$ implies $(x, t)\, q_k^\delta\lambda$.

By an interval number $\widetilde{x}$ (Zadeh (1975)) we mean an interval $[x^-, x^+]$ where $0 \leq x^- < x^+ \leq 1$. The set of all interval numbers is denoted by $D[0, 1]$. The interval $[x, x]$ can be simply identified by the number $x \in [0, 1]$. We define the following for the interval number $\widetilde{x}_i = [x_i^-, x_i^+]$, $\widetilde{y}_i = [y_i^-, y_i^+]$ for all $i \in I$:

$i)$ $r\max\{\widetilde{x}_i, \widetilde{y}_i\} = \left[\max\left(x_i^-, y_i^-\right), \max\left(x_i^+, y_i^+\right)\right]$,

$ii)$ $r\min\{\widetilde{x}_i, \widetilde{y}_i\} = \left[\min\left(x_i^-, y_i^-\right), \min\left(x_i^+, y_i^+\right)\right]$,

$iii)$ $r\inf \widetilde{x}_i = \left[\bigwedge_{i \in I} x_i^-, \bigwedge_{i \in I} x_i^+\right]$, $r\sup \widetilde{x}_i = \left[\bigvee_{i \in I} x_i^-, \bigvee_{i \in I} x_i^+\right]$,

$iv)$ $\widetilde{x}_1 \leq \widetilde{x}_2 \Leftrightarrow x_1^- \leq x_2^-$ and $x_1^+ \leq x_2^+$,

$v)$ $\widetilde{x}_2 = \widetilde{x}_2 \Leftrightarrow x_1^- = x_2^-$ and $x_1^+ = x_2^+$,

$vi)$ $\widetilde{x}_2 < \widetilde{x}_2 \Leftrightarrow x_1^- < x_2^-$ and $x_1^+ < x_2^+$,

$vii)$ $k\widetilde{x}_2 = [kx_i^-, kx_i^+]$.

Clearly $(D[0,1], \leq, \wedge, \vee)$ forms a complete lattice where $\widetilde{0} = [0,0]$ is its least element and $\widetilde{1} = [1,1]$ is its greatest element. The interval-valued fuzzy subsets deliver a more suitable explanation of uncertainty than the traditional fuzzy subsets. Therefore, it is significant to use interval-valued fuzzy subsets in applications. One of the key applications of fuzzy subsets is fuzzy control, and one of the greatest computationally concentrated parts of fuzzy control is the defuzzification. As a conversion to interval-valued fuzzy subsets typically increase the extent of computations, it is vitally important to design faster algorithms for the corresponding defuzzification. An interval-valued fuzzy subset $\widetilde{\lambda} : X \rightarrow D[0,1]$ is the set $\widetilde{\lambda} = \{x \in X | \left[\lambda^-(x), \lambda^+(x)\right] \in D[0,1]\}$, where $\lambda^- : X \rightarrow [0,1]$ and $\lambda^+ : X \rightarrow [0,1]$ are fuzzy subsets such that $0 \leq \lambda^-(x) < \lambda^+(x) \leq 1$ for all $x \in X$ and $\left[\lambda^-(x), \lambda^+(x)\right]$ is the interval degree of membership function of an element x to the set $\widetilde{\lambda}$.

Let $\widetilde{\lambda}$ be an interval-valued fuzzy subset of X. Then, for every $\widetilde{0} \leq \widetilde{t} \leq \widetilde{1}$, the crisp set $U\left(\widetilde{\lambda}; \widetilde{t}\right) = \left\{x \in X | \widetilde{\lambda}(x) \geq \widetilde{t}\right\}$ is said to be the level set of $\widetilde{\lambda}$.

Note that every $x \in [0,1]$ is in correspondence with the interval $[x,x] \in D[0,1]$. Therefore, a fuzzy set is a particular case of the interval valued fuzzy sets.

For any $\widetilde{\lambda} = \left[\lambda^-, \lambda^+\right]$ and $\widetilde{t} = \left[t^-, t^+\right]$, we define $\widetilde{\lambda}(x) + \widetilde{t} = \left[\lambda^-(x) + t^-, \lambda^+(x) + t^+\right]$, for all $x \in X$. In particular, if $\lambda^-(x) + t^- > 1$ and $\lambda^+(x) + t^+ > 1$, we write $\widetilde{\lambda}(x) + \widetilde{t} > [1,1]$.

An interval valued fuzzy subset $\widetilde{\lambda}$ of a set S of the form

$$\widetilde{\lambda}(y) := \begin{cases} \widetilde{t} \in D[0,1], & \text{if } y = x, \\ [0,0], & \text{if } y \neq x, \end{cases}$$

is said to be an interval valued fuzzy point (Tang et al. (2014)) with support x and value $\widetilde{t}$ and is denoted by $\left(x, \widetilde{t}\right)$.

For an interval valued fuzzy subset $\widetilde{\lambda}$ of a set S, we say that an interval valued fuzzy point $\left(x, \widetilde{t}\right)$ is

- contained in $\widetilde{\lambda}$ denoted by $\left(x, \widetilde{t}\right) \in \widetilde{\lambda}$, if $\widetilde{\lambda}(x) \geq \widetilde{t}$,

- quasi-coincident with $\widetilde{\lambda}$ denoted by $\left(x, \widetilde{t}\right) q\widetilde{\lambda}$ if $\widetilde{\lambda}(x) + \widetilde{t} > \widetilde{1}$, where $\lambda^-(x) + t^- > 1$ and $\lambda^+(x) + t^+ > 1$.

For an interval valued fuzzy point $\left(x, \widetilde{t}\right)$ and an interval valued fuzzy subset $\widetilde{\lambda}$ of a set S, we say that

- $\left(x,\widetilde{t}\right) \in \vee q\widetilde{\lambda}$ if $\left(x,\widetilde{t}\right) \in \widetilde{\lambda}$ or $\left(x,\widetilde{t}\right) q\widetilde{\lambda}$,
- $\left(x,\widetilde{t}\right) \overline{\alpha}\widetilde{\lambda}$ if $\left(x,\widetilde{t}\right) \alpha\widetilde{\lambda}$ does not hold for $\alpha \in \{\in, q, \in\vee q\}$.

For an interval valued fuzzy subset $\widetilde{\lambda}$ of a set S, we say that an interval valued fuzzy point $\left(x,\widetilde{t}\right)$ is

- contained in $\widetilde{\lambda}$ denoted by $\left(x,\widetilde{t}\right) \in \widetilde{\lambda}$, if $\widetilde{\lambda}(x) \geq \widetilde{t}$,
- quasi-coincident with $\widetilde{\lambda}$ denoted by $\left(x,\widetilde{t}\right) q_{\widetilde{k}}\widetilde{\lambda}$ if $\widetilde{\lambda}(x) + \widetilde{t} + \widetilde{k} > \widetilde{1}$, where $\lambda^{-}(x) + t^{-} + k^{-} > 1$ and $\lambda^{+}(x) + t^{+} + k^{+} > 1$.

For an interval valued fuzzy point $\left(x,\widetilde{t}\right)$ and an interval valued fuzzy subset $\widetilde{\lambda}$ of a set S, we say that

- $\left(x,\widetilde{t}\right) \in \vee q_{\widetilde{k}}\widetilde{\lambda}$ if $\left(x,\widetilde{t}\right) \in \widetilde{\lambda}$ or $\left(x,\widetilde{t}\right) q_{\widetilde{k}}\widetilde{\lambda}$,
- $\left(x,\widetilde{t}\right) \overline{\alpha}\widetilde{\lambda}$ if $\left(x,\widetilde{t}\right) \alpha\widetilde{\lambda}$ does not hold for $\alpha \in \left\{q_{\widetilde{k}}, \in\vee q_{\widetilde{k}}\right\}$.

3. Interval V $\left(\in, \in\vee q_{\widetilde{k}}\right)$-fuzzy Generalized Bi-ideals

In what follows, let S be an ordered semigroup and let $\widetilde{k} = [k^{-}, k^{+}]$ denotes an arbitrary element of $D[0,1)$, $\widetilde{\delta} = \left[\delta^{-}, \delta^{+}\right]$ denotes an arbitrary element of $D(0,1]$ where $\widetilde{0} \leq \widetilde{k} < \widetilde{\delta}$ in $D(0,1]$ unless otherwise specified. For an interval valued fuzzy point $\left(x,\widetilde{t}\right)$ and an interval valued fuzzy subset $\widetilde{\lambda}$ of S, we say that

- $\left(x,\widetilde{t}\right) q_{\widetilde{k}}^{\widetilde{\delta}}\widetilde{\lambda}$ if $\widetilde{\lambda}(x) + \widetilde{t} + \widetilde{k} > \widetilde{1}$, where $\lambda^{-}(x) + t^{-} + k^{-} > \delta^{-}$ and $\lambda^{+}(x) + t^{+} + k^{+} > \delta^{+}$,
- $\left(x,\widetilde{t}\right) \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\widetilde{\lambda}$ if $\left(x,\widetilde{t}\right) \in \widetilde{\lambda}$ or $\left(x,\widetilde{t}\right) q_{\widetilde{k}}^{\widetilde{\delta}}\widetilde{\lambda}$,
- $\left(x,\widetilde{t}\right) \overline{\alpha}\widetilde{\lambda}$ if $\left(x,\widetilde{t}\right) \alpha\widetilde{\lambda}$ does not hold for $\alpha \in \left\{q_{\widetilde{k}}^{\widetilde{\delta}}, \in\vee q_{\widetilde{k}}^{\widetilde{\delta}}\right\}$.

Theorem 3.1.

Let $\widetilde{\lambda}$ be an interval valued fuzzy subset of S. Then, the following statements are equivalent:

(i) $\left(\forall\widetilde{t} \in D(\frac{\delta - k}{2}, 1]\right) \left(U\left(\widetilde{\lambda};\widetilde{t}\right) \neq \emptyset \text{ is a generalized bi-ideal of } S\right)$,

(ii) $\widetilde{\lambda}$ satisfies the following conditions:

(iia) $(\forall x, y \in S) \left(x \leq y \Rightarrow \widetilde{\lambda}(x) \leq r\max\left\{\widetilde{\lambda}(y), \left[\frac{\delta^{-} - k^{-}}{2}, \frac{\delta^{+} - k^{+}}{2}\right]\right\}\right)$,

(iib) $(\forall x, y \in S) \left(r\min\left\{ \widetilde{\lambda}(x), \widetilde{\lambda}(z) \right\} \leq r\max\left\{ \widetilde{\lambda}(xyz), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2} \right] \right\} \right)$.

Proof:

Suppose that $U\left(\widetilde{\lambda};\widetilde{t}\right) \neq \emptyset$ is a generalized bi-ideal of S for all $\widetilde{t} \in (\frac{\delta - k}{2}, 1]$. If there exist $x, y \in S$ such that the condition (iia) is not satisfied, that is, there exist $x, y \in S$ with $x \leq y$ such that $\widetilde{\lambda}(y) > r\max\left\{ \widetilde{\lambda}(x), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2} \right] \right\}$, then $\widetilde{\lambda}(y) \in D(\frac{\delta - k}{2}, 1]$ and $y \in U\left(\widetilde{\lambda}; \widetilde{\lambda}(x)\right)$. But $\widetilde{\lambda}(x) < \widetilde{\lambda}(y)$ which follows that $y \notin U\left(\widetilde{\lambda}; \widetilde{\lambda}(x)\right)$, which is a contradiction. Hence, condition (iia) is satisfied. Assume that (iib) is not true, that is, $\widetilde{t} = r\min\left\{ \widetilde{\lambda}(x), \widetilde{\lambda}(z) \right\} > r\max\left\{ \widetilde{\lambda}(xyz), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2} \right] \right\}$ for some $x, y \in S$. Then, $\widetilde{t} \in D(\frac{\delta - k}{2}, 1]$ and $x, z \in U\left(\widetilde{\lambda};\widetilde{t}\right)$. But $xyz \notin U\left(\widetilde{\lambda};\widetilde{t}\right)$, since $\widetilde{\lambda}(xyz) < \widetilde{t}$, which is a contradiction and hence (iib) satisfied.

Conversely, assume that $\widetilde{\lambda}$ satisfies the condition (iia) and (iib). Assume that $U\left(\widetilde{\lambda};\widetilde{t}\right) \neq \emptyset$ for all $\widetilde{t} \in D(\frac{\delta - k}{2}, 1]$. Let $x, y \in S$ be such that $x \leq y$ and $y \in U\left(\widetilde{\lambda};\widetilde{t}\right)$. Then, $\widetilde{\lambda}(y) \geq \widetilde{t}$ and so $r\max\left\{ \widetilde{\lambda}(x), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2} \right] \right\} \geq \widetilde{\lambda}(y) \geq \widetilde{t} > \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2} \right]$. Hence, $\widetilde{\lambda}(x) \geq \widetilde{t}$, that is, $x \in U\left(\widetilde{\lambda};\widetilde{t}\right)$. If $x, y, z \in U\left(\widetilde{\lambda};\widetilde{t}\right)$, then from (iib) it follows that $r\max\left\{ \widetilde{\lambda}(xyz), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2} \right] \right\} \geq r\min\left\{ \widetilde{\lambda}(x), \widetilde{\lambda}(z) \right\} \geq \widetilde{t} > \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2} \right]$. Hence, $\widetilde{\lambda}(xyz) \geq \widetilde{t}$. That is, $xyz \in U\left(\widetilde{\lambda};\widetilde{t}\right)$. Therefore, $U\left(\widetilde{\lambda};\widetilde{t}\right)$ is a generalized bi-ideal of S for all $\widetilde{t} \in D(\frac{\delta - k}{2}, 1]$ with $U\left(\widetilde{\lambda};\widetilde{t}\right) \neq \emptyset$. ■

Corollary 3.1.

Let $\widetilde{\lambda}$ be an interval valued fuzzy subset of S. Then, the following statements are equivalent:

(i) $\left(\forall \widetilde{t} \in D(\frac{1 - k}{2}, 1] \right) \left(U\left(\widetilde{\lambda};\widetilde{t}\right) \neq \emptyset \text{ is a generalized bi-ideal of } S \right)$,

(ii) $\widetilde{\lambda}$ satisfies the following conditions:

(iia) $(\forall x, y \in S) \left(x \leq y \Rightarrow \widetilde{\lambda}(x) \leq r\max\left\{ \widetilde{\lambda}(y), \left[\frac{1 - k^-}{2}, \frac{1 - k^+}{2} \right] \right\} \right)$,

(iib) $(\forall x, y \in S) \left(r\min\left\{ \widetilde{\lambda}(x), \widetilde{\lambda}(z) \right\} \leq r\max\left\{ \widetilde{\lambda}(xyz), \left[\frac{1 - k^-}{2}, \frac{1 - k^+}{2} \right] \right\} \right)$.

Proof:

By Theorem 3.1, if we take $\widetilde{\delta} = [1, 1]$, then we get the equivalence of $(i), (ii), (iii)$. ■

Corollary 3.2.

Let $\widetilde{\lambda}$ be an interval valued fuzzy subset of S. Then, the following statements are equivalent:

(i) $\left(\forall \widetilde{t} \in D(\frac{1}{2}, 1]\right) \left(U\left(\widetilde{\lambda}; \widetilde{t}\right) \neq \emptyset \text{ is a generalized bi-ideal of } S\right)$,

(ii) $\widetilde{\lambda}$ satisfies the following conditions:

(iia) $(\forall x, y \in S) \left(x \leq y \Rightarrow \widetilde{\lambda}(x) \leq r\max\left\{\widetilde{\lambda}(y), \left[\frac{1}{2}, \frac{1}{2}\right]\right\}\right)$,

(iib) $(\forall x, y \in S) \left(r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\} \leq r\max\left\{\widetilde{\lambda}(xyz), \left[\frac{1}{2}, \frac{1}{2}\right]\right\}\right)$.

Proof:

By Theorem 3.1, if we take $\widetilde{\delta} = [1, 1]$ and $\widetilde{k} = [0, 0]$, then we get the equivalence of $(i), (ii), (iii)$. ■

Definition 3.1.

An interval valued fuzzy subset $\widetilde{\lambda}$ of S is said to be an $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy generalized bi-ideal of S, if is satisfies the following conditions:

(i) $x \leq y$, $\left(y, \widetilde{t}\right) \in \widetilde{\lambda} \Rightarrow \left(x, \widetilde{t}\right) \in \vee q_{\widetilde{k}}^{\widetilde{\delta}} \widetilde{\lambda}$,

(ii) $\left(x, \widetilde{t_1}\right) \in \widetilde{\lambda}$, $\left(z, \widetilde{t_2}\right) \in \widetilde{\lambda} \Rightarrow \left(xyz, r\min\left\{\widetilde{t_1}, \widetilde{t_2}\right\}\right) \in \vee q_{\widetilde{k}}^{\widetilde{\delta}} \widetilde{\lambda}$,

for all $x, y \in S$ and $\widetilde{t}, \widetilde{t_1}, \widetilde{t_2} \in D(0, 1]$.

Example 3.1.

Consider the ordered semigroup $S = \{1, 2, 3, 4\}$ with the following multiplication table and ordered relation "$\leq$":

$\cdot$	a	b	c	d
a	a	b	b	d
b	b	b	b	b
c	c	b	b	b
d	d	d	d	b

$\leq := \{(a, a), (b, b), (c, c), (d, d), (a, d), (a, c), (c, d), (b, c), (b, d), (c, d), (d, c)\}$.

Let $\widetilde{\lambda}$ be an interval valued fuzzy subset defined by

$$\widetilde{\lambda}(x) = \begin{cases} [0.6, 0.7], & \text{if } x \in \{a, c\}, \\ [0.2, 0.3], & \text{if } x = b, \\ [0.3, 0.4], & \text{if } x = d. \end{cases}$$

Let $\widetilde{t_1}, \widetilde{t_2} \in D(0,1]$ such that $\widetilde{t_1} = [0.1, 0.2], \widetilde{t_1} = [0.2, 0.3]$. Then, $\widetilde{\lambda}$ is an interval valued $\left(\in, \in \vee q_{[0.2,0.3]}^{[0.3,0.4]}\right)$-fuzzy generalized bi-ideal of S.

Theorem 3.2.

An interval valued fuzzy subset $\widetilde{\lambda}$ of S is an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy generalized bi-ideal of S if and only if:

(i) $\left(\widetilde{\lambda}(y) \geq r\min\left\{\widetilde{\lambda}(x), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \text{ with } x \leq y\right)$,

(ii) $\widetilde{\lambda}(xyz) \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$,

for all $x, y, z \in S$.

Proof:

Assume that $\widetilde{\lambda}$ is an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy generalized bi-ideal of S. Let $x, y \in S$ be such that $x \leq y$. If $\widetilde{\lambda}(y) < \widetilde{\lambda}(x)$, then $\widetilde{\lambda}(y) < \widetilde{t} \leq \widetilde{\lambda}(x)$ for some $\widetilde{t} \in D\left(0, \frac{\delta - k}{2}\right)$. It follows that $\left(x, \widetilde{t}\right) \in \widetilde{\lambda}$, but $\left(y, \widetilde{t}\right) \overline{\in} \widetilde{\lambda}$. Since $\widetilde{\lambda}(y) + \widetilde{t} < 2\widetilde{t} < \widetilde{\delta} - \widetilde{k}$, then we have $\left(y, \widetilde{t}\right) \overline{q_{\widetilde{k}}^{\widetilde{\delta}}} \widetilde{\lambda}$. Therefore, $\left(y, \widetilde{t}\right) \overline{\in \vee q_{\widetilde{k}}^{\widetilde{\delta}}} \widetilde{\lambda}$, which is a contradiction. Hence, $\widetilde{\lambda}(y) \geq \widetilde{\lambda}(x)$. Now if $\widetilde{\lambda}(x) \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$, then $\left(x, \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right) \in \widetilde{\lambda}$ and so $\left(y, \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right) \in \vee q_{\widetilde{k}}^{\widetilde{\delta}} \widetilde{\lambda}$. It follows that $\widetilde{\lambda}(y) \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$ or $\widetilde{\lambda}(y) + \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] > \widetilde{\delta} - \widetilde{k}$. Hence, $\widetilde{\lambda}(y) \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. Otherwise, $\widetilde{\lambda}(y) + \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] < \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] + \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] = \widetilde{\delta} - \widetilde{k}$, which is a contradiction. Therefore, $\widetilde{\lambda}(y) \geq r\min\left\{\widetilde{\lambda}(x), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$ for all $x, y \in S$ with $x \leq y$. Let $x, y, z \in S$ be such that $r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\} < \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. We suppose that $\widetilde{\lambda}(xyz) \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\}$. If not, then we choose $\widetilde{t} \in D\left(0, \frac{\delta - k}{2}\right)$ such that $\widetilde{\lambda}(xyz) < \widetilde{t} \leq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right\}$. It implies that $\left(x,\widetilde{t}\right) \in \widetilde{\lambda}$ and $\left(z,\widetilde{t}\right) \in \widetilde{\lambda}$, but $\left(xyz,\widetilde{t}\right)\overline{\in}\widetilde{\lambda}$ and $\widetilde{\lambda}\left(xyz\right)+\widetilde{t} < 2\widetilde{t} < \widetilde{\delta}-\widetilde{k}$ i.e, $\overline{\left(xy,\widetilde{t}\right) \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}}\widetilde{\lambda}$, which is a contradiction. Hence, $\widetilde{\lambda}\left(xyz\right) \geq r\min\left\{\widetilde{\lambda}\left(x\right),\widetilde{\lambda}\left(z\right)\right\}$ for all $x,y\in S$. If $r\min\left\{\widetilde{\lambda}\left(x\right),\widetilde{\lambda}\left(z\right)\right\} \geq \left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]$, then $\left(x,\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]\right) \in \widetilde{\lambda}$ and so $\left(z,\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]\right) \in \widetilde{\lambda}$. Thus, $\left(xyz,\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]\right) = \left(xyz, r\min\left\{\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right],\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]\right\}\right) \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\widetilde{\lambda}$, and so $\widetilde{\lambda}\left(xyz\right) \geq \left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]$ or $\widetilde{\lambda}\left(xyz\right)+\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right] > \widetilde{\delta}-\widetilde{k}$. If $\widetilde{\lambda}\left(xyz\right) < \left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]$, then $\widetilde{\lambda}\left(xyz\right)+\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right] < \left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]+\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right] = \widetilde{\delta}-\widetilde{k}$, which is a contradiction. Hence, $\widetilde{\lambda}\left(xyz\right) \geq \left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]$. Therefore, $\widetilde{\lambda}\left(xy\right) \geq r\min\left\{\widetilde{\lambda}\left(x\right),\widetilde{\lambda}\left(y\right),\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]\right\}$.

Conversely, let $\widetilde{\lambda}$ be an interval valued fuzzy subset of S that satisfies the three conditions (i) and (ii). Let $x,y\in S$ and $\widetilde{t}\in D(0,1]$ be such that $x\leq y$ and $\left(x,\widetilde{t}\right)\in\widetilde{\lambda}$. Then, $\widetilde{\lambda}\left(x\right)\geq\widetilde{t}$, and so

$$\widetilde{\lambda}\left(x\right) \geq r\min\left\{\widetilde{\lambda}\left(y\right),\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]\right\} \geq r\min\left\{\widetilde{t},\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]\right\}$$
$$= \begin{cases} \widetilde{t}, & \text{if } \widetilde{t} \leq \left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right], \\ \left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right], & \text{if } \widetilde{t} > \left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]. \end{cases}$$

It implies that $\left(x,\widetilde{t}\right)\in\widetilde{\lambda}$ or $\widetilde{\lambda}\left(x\right)+\widetilde{t} \geq \left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]+\widetilde{t} > \widetilde{\delta}-\widetilde{k}$, that is, $\left(x,\widetilde{t}\right) q_{\widetilde{k}}^{\widetilde{\delta}}\widetilde{\lambda}$. Hence, $\left(x,\widetilde{t}\right)\in\vee q_{\widetilde{k}}^{\widetilde{\delta}}\widetilde{\lambda}$. Let $x,y,z\in S$ and $\widetilde{t_x},\widetilde{t_z}\in D(0,1]$ be such that $\left(x,\widetilde{t_x}\right)\in\widetilde{\lambda}$ and $\left(z,\widetilde{t_z}\right)\in\widetilde{\lambda}$. Then, $\widetilde{\lambda}\left(x\right)\geq\widetilde{t_x}$ and $\widetilde{\lambda}\left(z\right)\geq\widetilde{t_z}$. It follows from (ii) that

$$\begin{aligned}\widetilde{\lambda}\left(xyz\right) &\geq r\min\left\{\widetilde{\lambda}\left(x\right),\widetilde{\lambda}\left(z\right),\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]\right\} \\ &\geq r\min\left\{\widetilde{t_x},\widetilde{t_z},\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]\right\} \\ &= \begin{cases} r\min\left\{\widetilde{t_x},\widetilde{t_z}\right\}, & \text{if } r\min\left\{\widetilde{t_x},\widetilde{t_z}\right\} \leq \left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right], \\ \left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right], & \text{if } r\min\left\{\widetilde{t_x},\widetilde{t_z}\right\} > \left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]. \end{cases}\end{aligned}$$

Thus, $\left(xyz, r\min\left\{\widetilde{t_1}, \widetilde{t_2}\right\}\right) \in \widetilde{\lambda}$ or $\widetilde{\lambda}(xyz) + r\min\left\{\widetilde{t_x}, \widetilde{t_z}\right\} \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] + r\min\left\{\widetilde{t_x}, \widetilde{t_z}\right\} > \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] + \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] = \widetilde{\delta} - \widetilde{k}$, that is, $\left(xyz, r\min\left\{\widetilde{t_x}, \widetilde{t_z}\right\}\right) q_{\widetilde{k}}^{\widetilde{\delta}} \widetilde{\lambda}$. Hence, $\left(xyz, r\min\left\{\widetilde{t_x}, \widetilde{t_z}\right\}\right) \in \vee q_{\widetilde{k}}^{\widetilde{\delta}} \widetilde{\lambda}$. Therefore, $\widetilde{\lambda}$ is an $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy generalized bi-ideal of S. ■

Corollary 3.3.

An interval valued fuzzy subset $\widetilde{\lambda}$ of S is an interval valued $\left(\in, \in \vee q_{\widetilde{k}}\right)$-fuzzy generalized bi-ideal of S if and only if:

(i) $\left(\widetilde{\lambda}(y) \geq r\min\left\{\widetilde{\lambda}(x), \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]\right\} \text{ with } x \leq y\right)$,

(ii) $\widetilde{\lambda}(xyz) \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]\right\}$,

for all $x, y, z \in S$.

Proof:

By Theorem 3.2, if we take $\widetilde{\delta} = [1, 1]$, then we get the equivalence of (i) and (ii). ■

Corollary 3.4.

An interval valued fuzzy subset $\widetilde{\lambda}$ of S is an interval valued $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S if and only if:

(i) $\left(\widetilde{\lambda}(y) \geq r\min\left\{\widetilde{\lambda}(x), \left[\frac{1}{2}, \frac{1}{2}\right]\right\} \text{ with } x \leq y\right)$,

(ii) $\widetilde{\lambda}(xy) \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \left[\frac{1}{2}, \frac{1}{2}\right]\right\}$,

for all $x, y, z \in S$.

Proof:

By Theorem 3.2, if we take $\widetilde{\delta} = [1, 1]$ and $\widetilde{k} = [0, 0]$, then we get the equivalence of (i) and (ii). ■

Obviously, every interval valued fuzzy generalized bi-ideal, an interval valued $(\in, \in \vee q)$-fuzzy generalized bi-ideal and interval valued $\left(\in, \in \vee q_{\widetilde{k}}\right)$-fuzzy generalized bi-ideal of S is an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy generalized bi-ideal of S. But the converse is not true. For this we have the following example.

Example 3.2.

The interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy generalized bi-ideal of S is not an interval valued fuzzy generalized bi-ideal, is not an interval valued $(\in, \in \vee q)$-fuzzy generalized bi-ideal and is not interval valued $\left(\in, \in \vee q_{\widetilde{k}}\right)$-fuzzy generalized bi-ideal of S. Consider an ordered semigroup in Example 3.1. Since $b < d$, then

$$\widetilde{\lambda}(b) = [0.2, 0.3] \not\geq [0.3, 0.4] = \widetilde{\lambda}(d).$$

This shows that $\widetilde{\lambda}$ is not an interval valued fuzzy generalized bi-ideal of S.

Also,

$$\widetilde{\lambda}(d \cdot d \cdot d) = \widetilde{\lambda}(b) = [0.2, 0.3] \not\geq [0.3, 0.4] = r\min\left\{\widetilde{\lambda}(d), \widetilde{\lambda}(d), \left[\frac{1}{2}, \frac{1}{2}\right]\right\}.$$

This shows that $\widetilde{\lambda}$ is not an interval valued $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S.

Moreover,

$$\widetilde{\lambda}(d \cdot d \cdot d) = \widetilde{\lambda}(b) = [0.2, 0.3] \not\geq [0.25, 0.35] = r\min\left\{\widetilde{\lambda}(d), \widetilde{\lambda}(d), \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]\right\}.$$

Thus $\widetilde{\lambda}$ is not an interval valued $\left(\in, \in \vee q_{\widetilde{k}}\right)$-fuzzy generalized bi-ideal of S.

In the following Theorem, we give a condition for interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy generalized bi-ideal of S to be an interval valued fuzzy generalized bi-ideal of S.

Theorem 3.3.

Let $\widetilde{\lambda}$ be an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy generalized bi-ideal of S. If $\widetilde{\lambda}(x) < \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$ for all $x \in S$, then $\widetilde{\lambda}$ is an interval valued fuzzy generalized bi-ideal of S.

Proof:

Proof follows from Theorem 3.2. ■

Corollary 3.5.

Let $\widetilde{\lambda}$ be an interval valued $\left(\in, \in \vee q_{\widetilde{k}}\right)$-fuzzy generalized bi-ideal of S. If $\widetilde{\lambda}(x) < \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]$ for all $x \in S$, then $\widetilde{\lambda}$ is an interval valued fuzzy generalized bi-ideal of S.

Proof:

In Theorem 3.3, let us take $\widetilde{\delta} = [1, 1]$. If $\widetilde{\lambda}(x) < \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]$ for all $x \in S$, then $\widetilde{\lambda}$ is an interval valued fuzzy generalized bi-ideal of S. ■

Corollary 3.6.

Let $\widetilde{\lambda}$ be an interval valued $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S. If $\widetilde{\lambda}(x) < \left[\frac{1}{2}, \frac{1}{2}\right]$ for all $x \in S$, then $\widetilde{\lambda}$ is an interval valued fuzzy generalized bi-ideal of S.

Proof:

In Theorem 3.3, let us take $\widetilde{\delta} = [1, 1]$ and $\widetilde{k} = [0, 0]$. If $\widetilde{\lambda}(x) < \left[\frac{1}{2}, \frac{1}{2}\right]$ for all $x \in S$, then $\widetilde{\lambda}$ is an interval valued fuzzy generalized bi-ideal of S. ■

Theorem 3.4.

Let $[0, 0] \leq \widetilde{k} < \widetilde{r} < [1, 1]$ and $[0, 0] < \widetilde{\delta} < \widetilde{s} \leq [1, 1]$. Then, every interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy generalized bi-ideal of S is an interval valued $\left(\in, \in \vee q_{\widetilde{r}}^{\widetilde{s}}\right)$-fuzzy generalized bi-ideal of S.

Proof:

Straightforward. ■

The converse of Theorem 3.4 is not true. For this we have the following example.

Example 3.3.

Consider an ordered semigroup of Example 3.1. Let $\widetilde{\lambda}$ be an interval valued fuzzy subset of S defined by

$$\widetilde{\lambda}(x) = \begin{cases} [0.4, 0.5], & \text{if } x = a, \\ [0.2, 0.3], & \text{if } x \in \{b, c\}, \\ [0.1, 0.2], & \text{if } x = d. \end{cases}$$

Then, $\widetilde{\lambda}$ is an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy generalized bi-ideal of S with $\widetilde{k} \geq [0.3, 0.3]$ and $\widetilde{\delta} \geq [0.5, 0.6]$. Note that $3 < 4$ and

$$\begin{aligned} \widetilde{\lambda}(4) &= [0.1, 0.2] < [0.26, 0.28] \\ &= \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] \\ &= r\min\left\{\widetilde{\lambda}(x), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}, \end{aligned}$$

where $\widetilde{k} = [0.13, -0.4]$ and $\widetilde{\delta} = [0.4, 0.5]$. Thus, $\widetilde{\lambda}$ does not satisfy the first condition of Theorem 3.2, and hence $\widetilde{\lambda}$ is not an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy generalized bi-ideal of S with $\widetilde{k} = [0.13, -0.4]$ and $\widetilde{\delta} = [0.4, 0.5]$. This shows that the converse of Theorem 3.4 is not true.

Theorem 3.5.

For an interval valued fuzzy subset $\widetilde{\lambda}$ in S, the following statements are equivalent:

(i) $\widetilde{\lambda}$ is an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy generalized bi-ideal of S,

(ii) $\left(\forall \widetilde{t} \in D(0, \frac{\delta - k}{2}]\right) \left(U\left(\widetilde{\lambda}; \widetilde{t}\right) \neq \emptyset \Rightarrow U\left(\widetilde{\lambda}; \widetilde{t}\right)\right.$ is a generalized bi-ideal of $\left.S\right)$.

Proof:

Suppose that $\widetilde{\lambda}$ is an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy generalized bi-ideal of S and $\widetilde{t} \in D(0, \frac{\delta - k}{2})$ is such that $U\left(\widetilde{\lambda}; \widetilde{t}\right) \neq \emptyset$. By Theorem 3.2$(i)$, we have $\widetilde{\lambda}(x) \geq r\min\left\{\widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$ for any $x, y \in S$ with $x \leq y$ and $y \in U\left(\widetilde{\lambda}; \widetilde{t}\right)$. It follows that $\widetilde{\lambda}(x) \geq r\min\left\{\widetilde{t}, \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} = \widetilde{t}$ so that $x \in U\left(\widetilde{\lambda}; \widetilde{t}\right)$. Let $x, z \in U\left(\widetilde{\lambda}; \widetilde{t}\right)$. Then, $\widetilde{\lambda}(x) \geq \widetilde{t}$ and $\widetilde{\lambda}(z) \geq \widetilde{t}$. Theorem 3.2$(ii)$ implies that

$$\widetilde{\lambda}(xyz) \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \geq r\min\left\{\widetilde{t}, \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} = \widetilde{t}.$$

Thus, $xyz \in U\left(\widetilde{\lambda}; \widetilde{t}\right)$. Therefore, $U\left(\widetilde{\lambda}; \widetilde{t}\right)$ is a generalized bi-ideal of S where $\widetilde{t} \in D(0, \frac{\delta - k}{2}]$.

Conversely, let $\widetilde{\lambda}$ be an interval valued fuzy subset of S such that $U\left(\widetilde{\lambda}; \widetilde{t}\right) \neq \emptyset$ is a generalized bi-ideal of S for all $\widetilde{t} \in D(0, \frac{\delta - k}{2}]$. If there exist $x, y \in S$ with $x \leq y$ and $y \in \left(\widetilde{\lambda}; \widetilde{t}\right)$ such that $\widetilde{\lambda}(x) < r\min\left\{\widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$, then $\widetilde{\lambda}(y) < \widetilde{t}_x \leq r\min\left\{\widetilde{\lambda}(x), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$ for some $\widetilde{t}_x \in D(0, \frac{\delta - k}{2})$ and so $x \in U\left(\widetilde{\lambda}; \widetilde{t}\right)$ which is a contradiction. Therefore, $\widetilde{\lambda}(x) \geq r\min\left\{\widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$ for all $x, y \in S$ with $x \leq y$. Suppose that for all $x, y, z \in S$, let $\widetilde{\lambda}(xyz) < r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$. Then, $\widetilde{\lambda}(xy) < \widetilde{t}_{xyz} \leq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$ for some $\widetilde{t}_{xyz} \in D(0, \frac{\delta - k}{2})$. It follows that $x \in U\left(\widetilde{\lambda}; \widetilde{t}_{xyz}\right)$ and $z \in U\left(\widetilde{\lambda}; \widetilde{t}_{xyz}\right)$, but $xyz \notin U\left(\widetilde{\lambda}; \widetilde{t}\right)$ which is a contradiction. Hence, $\widetilde{\lambda}(xyz) \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$. Therefore, by Theorem 3.2, we conclude that $\widetilde{\lambda}$ is an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy generalized bi-ideal of S. ■

Corollary 3.7.

For an interval valued fuzzy subset $\widetilde{\lambda}$ in S, the following are equivalent:

(i) $\widetilde{\lambda}$ is an interval valued $\left(\in, \in \vee q_{\widetilde{k}}\right)$-fuzzy generalized bi-ideal of S,

(ii) $\left(\forall \widetilde{t} \in D(0, \frac{1-k}{2}]\right) \left(U\left(\widetilde{\lambda}; \widetilde{t}\right) \neq \emptyset \Rightarrow U\left(\widetilde{\lambda}; \widetilde{t}\right) \text{ is a generalized bi-ideal of } S\right)$.

Proof:

By Theorem 3.5, if we take $\widetilde{\delta} = [1,1]$, then we get the equivalence of (i) and (ii). ■

Corollary 3.8.

For an interval valued fuzzy subset $\widetilde{\lambda}$ in S, the following are equivalent:

(i) $\widetilde{\lambda}$ is an interval valued $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S,

(ii) $\left(\forall \widetilde{t} \in D(0, \frac{1}{2}]\right) \left(U\left(\widetilde{\lambda}; \widetilde{t}\right) \neq \emptyset \Rightarrow U\left(\widetilde{\lambda}; \widetilde{t}\right) \text{ is a generalized bi-ideal of } S\right)$.

Proof:

By Theorem 3.5, if we take $\widetilde{\delta} = [1,1]$, and $\widetilde{k} = [0,0]$, then we get the equivalence of (i) and (ii). ■

For any interval valued fuzzy subset of S and $\widetilde{t} \in D(0, \frac{\delta - k}{2}]$, we consider the following subsets:

$$Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda}; \widetilde{t}\right) := \left\{x \in S \mid \left(x, \widetilde{t}\right) q_{\widetilde{k}}^{\widetilde{\delta}} \widetilde{\lambda}\right\},$$

and

$$\left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}} := \left\{x \in S \mid \left(x, \widetilde{t}\right) \in \vee q_{\widetilde{k}}^{\widetilde{\delta}} \widetilde{\lambda}\right\}.$$

It is clear that $\left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}} = U\left(\widetilde{\lambda}; \widetilde{t}\right) \cup Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda}; \widetilde{t}\right)$.

Theorem 3.6.

Let $\widetilde{\lambda}$ be an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy generalized bi-ideal of S. Then

$$\left(\forall \widetilde{t} \in D(\frac{\delta - k}{2}, 1]\right) \left(Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda}; \widetilde{t}\right) \neq \emptyset \Rightarrow Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda}; \widetilde{t}\right) \text{ is a generalized bi-ideal of } S\right).$$

Proof:

Suppose that $\widetilde{\lambda}$ is an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy generalized bi-ideal of S. Let $\widetilde{t} \in D(\frac{\delta - k}{2}, 1]$ be such that $Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda}; \widetilde{t}\right) \neq \emptyset$. Let $y \in Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda}; \widetilde{t}\right)$ and $x \in S$ be such that $x \leq y$.

Then, $\widetilde{\lambda}(y) + \widetilde{t} > \widetilde{\delta} - \widetilde{k}$. Thus, by Theorem 3.2$(i)$, we have

$$\begin{aligned}\widetilde{\lambda}(x) &\geq r\min\left\{\widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \\ &= \begin{cases} \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right], & \text{if } \widetilde{\lambda}(y) \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] \\ \widetilde{\lambda}(y), & \text{if } \widetilde{\lambda}(y) < \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] \end{cases} \\ &> \widetilde{\delta} - \widetilde{t} - \widetilde{k}.\end{aligned}$$

Thus, $x \in Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda};\widetilde{t}\right)$. Let $x, z \in Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda};\widetilde{t}\right)$. Then, $\widetilde{\lambda}(x) + \widetilde{t} > \widetilde{\delta} - \widetilde{k}$ and $\widetilde{\lambda}(z) + \widetilde{t} > \widetilde{\delta} - \widetilde{k}$. Thus, from Theorem 3.2$(ii)$, it follows that

$$\begin{aligned}\widetilde{\lambda}(xyz) &\geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]\right\} \\ &= \begin{cases} r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\}, & \text{if } r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\} < \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right] \\ \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right], & \text{if } r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\} \geq \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right] \end{cases} \\ &> \widetilde{\delta} - \widetilde{t} - \widetilde{k}.\end{aligned}$$

Thus, $xyz \in Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda};\widetilde{t}\right)$. Therefore, $xy \in Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda};\widetilde{t}\right)$ is a generalized bi-ideal of S. ■

Corollary 3.9.

Let $\widetilde{\lambda}$ be an interval valued $\left(\in, \in \vee q_{\widetilde{k}}\right)$-fuzzy generalized bi-ideal of S. Then

$$\left(\forall \widetilde{t} \in D(\frac{1-k}{2}, 1]\right)\left(Q_{\widetilde{k}}\left(\widetilde{\lambda};\widetilde{t}\right) \neq \emptyset \Rightarrow Q_{\widetilde{k}}\left(\widetilde{\lambda};\widetilde{t}\right) \text{ is a generalized bi-ideal of } S\right).$$

Proof:

In Theorem 3.6, if we take $\widetilde{\delta} = [1, 1]$, then

$$\left(\forall \widetilde{t} \in D(\frac{1-k}{2}, 1]\right)\left(Q_{\widetilde{k}}\left(\widetilde{\lambda};\widetilde{t}\right) \neq \emptyset \Rightarrow Q_{\widetilde{k}}\left(\widetilde{\lambda};\widetilde{t}\right) \text{ is a generalized bi-ideal of } S\right).$$ ■

Corollary 3.10.

Let $\widetilde{\lambda}$ be an interval valued $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S. Then

$$\left(\forall \widetilde{t} \in D(\frac{1}{2}, 1]\right)\left(Q\left(\widetilde{\lambda};\widetilde{t}\right) \neq \emptyset \Rightarrow Q\left(\widetilde{\lambda};\widetilde{t}\right) \text{ is a generalized bi-ideal of } S\right).$$

Proof:

In Theorem 3.6, if we take $\widetilde{\delta}=[1,1]$, and $\widetilde{k}=[0,0]$, then

$$\left(\forall\widetilde{t}\in D(\frac{1}{2},1]\right)\left(Q\left(\widetilde{\lambda};\widetilde{t}\right)\neq\emptyset\Rightarrow Q\left(\widetilde{\lambda};\widetilde{t}\right)\text{ is a generalized bi-ideal of } S\right).$$ ■

Theorem 3.7.

For interval valued fuzzy subset $\widetilde{\lambda}$ of S, the following are equivalent:

(i) $\widetilde{\lambda}$ is an interval valued $\left(\in,\in\vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy generalized bi-ideal of S,

(ii) $\left(\forall\widetilde{t}\in D(0,\delta]\right)\left(\left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}\neq\emptyset\Rightarrow\left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}\text{ is a generalized bi-ideal of } S\right)$.

Proof:

Suppose that $\widetilde{\lambda}$ is an interval valued $\left(\in,\in\vee q_{\widetilde{k}}\right)$-fuzzy generalized bi-ideal of S and $\widetilde{t}\in D(0,\delta]$ is such that $\left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}\neq\emptyset$. Let $x\in\left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}$ and $y\in S$ be such that $x\leq y$. Then, $y\in U\left(\widetilde{\lambda};\widetilde{t}\right)$ or $y\in Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda};\widetilde{t}\right)$, that is, $\widetilde{\lambda}(y)\geq\widetilde{t}$ or $\widetilde{\lambda}(y)+\widetilde{t}>\widetilde{\delta}-\widetilde{k}$. Thus, by Theorem 3.2$(i)$, we get

$$\widetilde{\lambda}(x)\geq r\min\left\{\widetilde{\lambda}(y),\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]\right\}. \qquad (a)$$

So we consider two cases: I) $\widetilde{\lambda}(y)\leq\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]$ and II) $\widetilde{\lambda}(y)>\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]$.
For the first case, if $\widetilde{\lambda}(y)\leq\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]$, then $\widetilde{\lambda}(x)\geq\widetilde{\lambda}(y)$. Thus, if $\widetilde{\lambda}(y)\geq\widetilde{t}$ for some $\widetilde{t}\in D(0,\delta]$. Then, $\widetilde{\lambda}(x)\geq\widetilde{t}$ and so $y\in U\left(\widetilde{\lambda};\widetilde{t}\right)\subseteq\left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}$. If $\widetilde{\lambda}(y)+\widetilde{t}>\widetilde{\delta}-\widetilde{k}$, then $\widetilde{\lambda}(x)+\widetilde{t}\geq\widetilde{\lambda}(y)+\widetilde{t}>\widetilde{\delta}-\widetilde{k}$. This shows that $\left(x,\widetilde{t}\right)q_{\widetilde{k}}^{\widetilde{\delta}}\widetilde{\lambda}$, that is, $x\in Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda};\widetilde{t}\right)\subseteq\left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}$.

From the second case and (a) we have, $\widetilde{\lambda}(x)>\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]$. If $\widetilde{t}\leq\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]$, then $\widetilde{\lambda}(x)\geq\widetilde{t}$ and hence $x\in U\left(\widetilde{\lambda};\widetilde{t}\right)\subseteq\left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}$. If $\widetilde{t}>\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]$, then $\widetilde{\lambda}(x)+\widetilde{t}>\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]+\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]=\widetilde{\delta}-\widetilde{k}$.
It follows that $x\in Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda};\widetilde{t}\right)\subseteq\left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}$. Therefore, $\left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}$ satisfies first condition of generalized bi-ideal in order semigroup S.

Let, $x,z\in\left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}$. Then, $x\in U\left(\widetilde{\lambda};\widetilde{t}\right)$ or $x\in Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda};\widetilde{t}\right)$ and $z\in U\left(\widetilde{\lambda};\widetilde{t}\right)$ or $z\in Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda};\widetilde{t}\right)$. That is, $\widetilde{\lambda}(x)\geq\widetilde{t}$ or $\widetilde{\lambda}(z)+\widetilde{t}>\widetilde{\delta}-\widetilde{k}$ and $\widetilde{\lambda}(z)\geq\widetilde{t}$ or $\widetilde{\lambda}(z)+\widetilde{t}>\widetilde{\delta}-\widetilde{k}$. Thus, we consider the following four cases:

(i) If $\widetilde{\lambda}(x) \geq \widetilde{t}$ and $\widetilde{\lambda}(z) \geq \widetilde{t}$,

(ii) If $\widetilde{\lambda}(x) \geq \widetilde{t}$ and $\widetilde{\lambda}(z) + \widetilde{t} > \widetilde{\delta} - \widetilde{k}$,

(iii) If $\widetilde{\lambda}(x) + \widetilde{t} > \widetilde{\delta} - \widetilde{k}$ and $\widetilde{\lambda}(z) \geq \widetilde{t}$,

(iv) If $\widetilde{\lambda}(x) + \widetilde{t} > \widetilde{\delta} - \widetilde{k}$ and $\widetilde{\lambda}(z) + \widetilde{t} > \widetilde{\delta} - \widetilde{k}$.

Case (i) : From Theorem 3.2(ii) it follows that

$$\begin{aligned}\widetilde{\lambda}(xyz) &\geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \\ &\geq r\min\left\{\widetilde{t}, \widetilde{t}, \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \\ &= r\min\left\{\widetilde{t}, \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \\ &= \begin{cases}\left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right], & \text{if } \widetilde{t} > \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right], \\ \widetilde{t}, & \text{if } \widetilde{t} \leq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right].\end{cases}\end{aligned}$$

This implies that $xyz \in U\left(\widetilde{\lambda}; \widetilde{t}\right)$ or $\widetilde{\lambda}(xyz) + \widetilde{t} > \widetilde{\delta} - \widetilde{k} > \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] + \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] = \widetilde{\delta} - \widetilde{k}$, that is, $xyz \in Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda}; \widetilde{t}\right)$. Hence, $xyz \in \left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}$.

Case (ii) : Assume that $\widetilde{t} > \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. Then, $\widetilde{\delta} - \widetilde{t} - \widetilde{k} \leq \widetilde{\delta} - \widetilde{t} < \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$ and hence,

$$\begin{aligned}&\widetilde{\lambda}(xyz) \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \\ &= \begin{cases} r\min\left\{\widetilde{\lambda}(z), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} > \widetilde{\delta} - \widetilde{k}, & \text{if } r\min\left\{\widetilde{\lambda}(z), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \leq \widetilde{\lambda}(x), \\ \widetilde{\lambda}(x) \geq \widetilde{t}, & \text{if } r\min\left\{\widetilde{\lambda}(z), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} > \widetilde{\lambda}(x).\end{cases}\end{aligned}$$

Thus, $xy \in U\left(\widetilde{\lambda}; \widetilde{t}\right) \cup Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda}; \widetilde{t}\right) = \left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}$.

Now suppose that $\widetilde{t} \leq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. Then,

$$\widetilde{\lambda}(xyz) \geq r \min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$$
$$= \begin{cases} r\min\left\{\widetilde{\lambda}(x), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} > \widetilde{\delta} - \widetilde{k}, & \text{if } r\min\left\{\widetilde{\lambda}(x), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \leq \widetilde{\lambda}(z), \\ \widetilde{\lambda}(z) \geq \widetilde{\delta} - \widetilde{t} - \widetilde{k}, & \text{if } r\min\left\{\widetilde{\lambda}(x), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} > \widetilde{\lambda}(z). \end{cases}$$

Thus, $xyz \in U\left(\widetilde{\lambda}; \widetilde{t}\right) \cup Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda}; \widetilde{t}\right) = \left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}$.

Case (iii) : Proof of case (iii) is similar to case (ii).

Case (iv) : Suppose that $\widetilde{t} > \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. Then, $\widetilde{\delta} - \widetilde{t} - \widetilde{k} < \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. Hence,

$$\begin{aligned} \widetilde{\lambda}(xyz) &\geq r \min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \\ &= \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] \\ &> \widetilde{\delta} - \widetilde{t} - \widetilde{k}, \end{aligned}$$

whenever $r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\} < \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. Thus, $xyz \in Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda}; \widetilde{t}\right) \subseteq \left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}$. If $\widetilde{t} \leq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$, then

$$\widetilde{\lambda}(xyz) \geq r \min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$$
$$= \begin{cases} \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right], & \text{if } r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\} \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right], \\ r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\} > \widetilde{\delta} - \widetilde{t} - \widetilde{k}, & \text{if } r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\} < \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]. \end{cases}$$

This implies that $xyz \in U\left(\widetilde{\lambda}; \widetilde{t}\right) \cup Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda}; \widetilde{t}\right) = \left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}$.

Conversely, assume that (ii) holds. If there exist $x, y \in S$ such that $x \leq y$ and $\widetilde{\lambda}(x) < r\min\left\{\widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$, then $\widetilde{\lambda}(x) < \widetilde{t}_x \leq r\min\left\{\widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$ for some $\widetilde{t}_x \in D(0, \frac{\delta - k}{2}]$. It follows that $x \in U\left(\widetilde{\lambda}; \widetilde{t}_x\right) \subseteq \left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}$ but $x \notin Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda}; \widetilde{t}\right)$. Also, we have $\widetilde{\lambda}(x) + \widetilde{t}_x < 2\widetilde{t}_x \leq \widetilde{\delta} - \widetilde{k}$, and hence $\left(x, \widetilde{t}_x\right) \overline{q_{\widetilde{k}}^{\widetilde{\delta}}} \widetilde{\lambda}$, that is, $x \notin Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda}; \widetilde{t}_x\right)$. Therefore, $x \notin \left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_x}$, which is a contradiction. Hence, $\widetilde{\lambda}(x) \geq$

$r\min\left\{\widetilde{\lambda}\left(y\right),\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]\right\}$ for all $x,y\in S$ with $x\leq y$. Assume that there exist $x,y,z\in S$ such that $\widetilde{\lambda}\left(xyz\right)<r\min\left\{\widetilde{\lambda}\left(x\right),\widetilde{\lambda}\left(z\right),\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]\right\}$, then $\widetilde{\lambda}\left(xyz\right)<\widetilde{t}\leq r\min\left\{\widetilde{\lambda}\left(x\right),\widetilde{\lambda}\left(z\right),\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]\right\}$ for some $\widetilde{t}\in D(0,\frac{\delta-k}{2}]$. It follows that $x\in U\left(\widetilde{\lambda};\widetilde{t}\right)\subseteq\left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}$ and $z\in U\left(\widetilde{\lambda};\widetilde{t}\right)\subseteq\left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}$, thus, by second condition of generalized bi-ideal in order semigroup S we have $xyz\subseteq\left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}$. Thus, $\widetilde{\lambda}\left(xyz\right)+\widetilde{t}$ or $\widetilde{\lambda}\left(xyz\right)+\widetilde{t}\geq\widetilde{\delta}-\widetilde{k}$, which is a contradiction. Hence, $\widetilde{\lambda}\left(xyz\right)\geq r\min\left\{\widetilde{\lambda}\left(x\right),\widetilde{\lambda}\left(y\right),\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]\right\}$. Therefore, by Theorem 3.2, we conclude that $\widetilde{\lambda}$ is an interval valued $\left(\in,\in\vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy generalized bi-ideal of S. ■

Corollary 3.11.

For interval valued fuzzy subset $\widetilde{\lambda}$ of S, the following are equivalent:

(i) $\widetilde{\lambda}$ is an interval valued $\left(\in,\in\vee q_{\widetilde{k}}\right)$-fuzzy generalized bi-ideal of S,

(ii) $\left(\forall\widetilde{t}\in D(0,1]\right)\left(\left[\widetilde{\lambda}_{\widetilde{k}}\right]_{\widetilde{t}}\neq\emptyset\Rightarrow\left[\widetilde{\lambda}_{\widetilde{k}}\right]_{\widetilde{t}}\right.$ is a generalized bi-ideal of $\left.S\right)$.

Proof:

By Theorem 3.7, if we take $\widetilde{\delta}=[1,1]$, then we get the equivalence of (i) and (ii). ■

Corollary 3.12.

For interval valued fuzzy subset $\widetilde{\lambda}$ of S, the following are equivalent:

(i) $\widetilde{\lambda}$ is an interval valued $(\in,\in\vee q)$-fuzzy generalized bi-ideal of S,

(ii) $\left(\forall\widetilde{t}\in D(0,1]\right)\left(\left[\widetilde{\lambda}\right]_{\widetilde{t}}\neq\emptyset\Rightarrow\left[\widetilde{\lambda}\right]_{\widetilde{t}}\right.$ is a generalized bi-ideal of $\left.S\right)$.

Proof:

By Theorem 3.7, if we take $\widetilde{\delta}=[1,1]$ and $\widetilde{k}=[0,0]$, then we get the equivalence of (i) and (ii). ■

An interval valued fuzzy subset $\widetilde{\lambda}$ of S is called proper if $Im(\widetilde{\lambda})$ has at least two elements. Two interval valued fuzzy subsets are called equivalent if they have same family of level subsets. Otherwise, they are called non-equivalent.

Theorem 3.8.

Let $\widetilde{\lambda}$ is an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy generalized bi-ideal of S such that

$$\#\left\{\widetilde{\lambda}(x) \,|\widetilde{\lambda}(x) < \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \geq 2.$$

Then, there exist two proper non-equivalent interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy generalized bi-ideals of S such that $\widetilde{\lambda}$ can be expressed as the union of them.

Proof:

Let $\left\{\widetilde{\lambda}(x) \,|\widetilde{\lambda}(x) < \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} = \left\{\widetilde{t}_1, \widetilde{t}_2, ..., \widetilde{t}_n\right\}$, where, $\widetilde{t}_1 > \widetilde{t}_2 > ... > \widetilde{t}_n$ and $n \geq 2$. Then, the chain of $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-level generalized bi-ideal of S is $\left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]} \subseteq \left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_1} \subseteq \left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_2} \subseteq ... \subseteq \left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_n} = S$. Let $\widetilde{\Theta}$ and $\widetilde{\Xi}$ be interval valued fuzzy subsets of S defined by

$$\widetilde{\Theta}(x) = \begin{cases} \widetilde{t}_1, & \text{if } x \in \left[\widetilde{\Theta}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_1}, \\ \widetilde{t}_2, & \text{if } x \in \left[\widetilde{\Theta}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_2} \setminus \left[\widetilde{\Theta}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_1}, \\ ... \\ \widetilde{t}_n, & \text{if } x \in \left[\widetilde{\Theta}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_n} \setminus \left[\widetilde{\Theta}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_{n-1}}, \end{cases}$$

and

$$\widetilde{\Xi}(x) = \begin{cases} \widetilde{\Xi}(x), & \text{if } x \in \left[\widetilde{\Xi}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]}, \\ \widetilde{k}, & \text{if } x \in \left[\widetilde{\Xi}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_2} \setminus \left[\widetilde{\Xi}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]}, \\ \widetilde{t}_3, & \text{if } x \in \left[\widetilde{\Xi}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_2} \setminus \left[\widetilde{\Xi}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_1}, \\ ... \\ \widetilde{t}_n, & \text{if } x \in \left[\widetilde{\Xi}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_n} \setminus \left[\widetilde{\Xi}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_{n-1}}, \end{cases}$$

respectively, where $\widetilde{t}_3 < \widetilde{k} < \widetilde{t}_2$. Then, $\widetilde{\Theta}$ and $\widetilde{\Xi}$ are interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy generalized bi-ideal of S, and $\widetilde{\Theta}$,$\widetilde{\Xi} \in \widetilde{\lambda}$. The chain of $\left(\in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-level generalized bi-ideals of $\widetilde{\Theta}$ and $\widetilde{\Xi}$ are given by $\left[\widetilde{\Theta}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_1} \subseteq \left[\widetilde{\Theta}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_2} \subseteq ... \subseteq \left[\widetilde{\Theta}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_n}$ and $\left[\widetilde{\Xi}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_1} \subseteq \left[\widetilde{\Xi}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_2} \subseteq ... \subseteq \left[\widetilde{\Xi}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_n}$, respectively. Therefore, $\widetilde{\Theta}$ and $\widetilde{\Xi}$ are non equivalent and clearly, $\widetilde{\lambda} = \widetilde{\Theta} \cup \widetilde{\Xi}$. Hence proved. ■

4. Implication-based Interval Valued Fuzzy Generalized Bi-ideal

Fuzzy logic is an extension of set theoretic multivalued logic in which the truth values are linguistic variables or terms of the linguistic variable truth. Some operators, for example $\wedge, \vee, \neg, \rightarrow$ in fuzzy logic are also defined by using truth tables and the extension principle can be applied to derive definitions of the operators. In fuzzy logic, the truth value of fuzzy proposition Φ is denoted by $[\Phi]$. For a universe X of discourse, we display the fuzzy logical and corresponding set-theoretical notations used in this paper:

$$x \in [\widetilde{\lambda}] = \widetilde{\lambda}\,(x)\,, \tag{2}$$

$$[\Phi \vee \Psi] = \min\{[\Phi], [\Psi]\}, \tag{3}$$

$$[\Phi \to \Psi] = \min\{1, 1 - [\Phi] + [\Psi]\}, \tag{4}$$

$$[\forall \Phi\,(x)] = \inf_{x \in X} [\Phi\,(x)], \tag{5}$$

$$\Phi \text{ if and only if } [\Phi] = 1 \text{ for all valuation.} \tag{6}$$

The truth valuation rules given in (4) are those in the Łukasiewicz system of continuous-valued logic. Of course, various implication operators have been defined. We show only a selection of them in the following.

(a) Gaines-Rescher implication operator (I_{GR}) :

$$(I_{GR})\,(x, y) = \begin{cases} 1, & \text{if } x \leq y, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Gödel implication operator (I_G) :

$$(I_G)\,(x, y) = \begin{cases} 1, & \text{if } x \leq y, \\ y, & \text{otherwise.} \end{cases}$$

(c) The contraposition of Gödel implication operator (I_{cG}) :

$$(I_{cG})\,(x, y) = \begin{cases} 1, & \text{if } x \leq y, \\ 1 - x, & \text{otherwise.} \end{cases}$$

Ying (1991) introduced the concept of fuzzifying topology. We can expand his/her idea to ordered semigroups, and we define an interval valued fuzzifying generalized bi-ideal as follows.

Definition 4.1.

An interval valued fuzzy subset $\widetilde{\lambda}$ of S is called an interval valued fuzzifying generalized bi-ideal of S if it satisfies the following conditions:

(1) $(\forall x, y \in S) \left(\left(\models \left[x \in \widetilde{\lambda} \right] \to \left[y \in \widetilde{\lambda} \right] \right), x \leq y \right)$,

(2) $(\forall x, y, z \in S) \left(\models r\min \left\{ \left[x \in \widetilde{\lambda} \right] \to \left[y \in \widetilde{\lambda} \right] \right\} \to \left[xyz \in \widetilde{\lambda} \right] \right)$.

Obviously, the conditions (1) and (2) are equivalent to (i) and (ii), respectively, of the definition of interval valued $(\in, \in \vee q_k^\delta)$-fuzzy generalized bi-ideals of an ordered semigroup S. Therefore, an interval valued fuzzifying generalized bi-ideal is an ordinary interval valued fuzzy generalized bi-ideal. Kuroki (1991) introduced the concept of $\widetilde{t}$-tautology, i.e., for all valuations,

$$\models_{\widetilde{t}} \widetilde{\Phi} \text{ if and only if } \left[\widetilde{\Phi}\right] \geq \widetilde{t}. \tag{7}$$

Definition 4.2.

An interval valued fuzzy subset $\widetilde{\lambda}$ of S and $\widetilde{t} \in D(0,1]$ is called a $\widetilde{t}$-implication-based interval valued fuzzy generalized bi-ideal of S if it satisfies:

(3) $(\forall x, y \in S)\left(\left(\models_{\widetilde{t}} \left[x \in \widetilde{\lambda}\right] \to \left[y \in \widetilde{\lambda}\right]\right), x \leq y\right)$,

(4) $(\forall x, y, z \in S)\left(\models_{\widetilde{t}} r\min\left\{\left[x \in \widetilde{\lambda}\right] \to \left[y \in \widetilde{\lambda}\right]\right\} \to \left[xyz \in \widetilde{\lambda}\right]\right)$.

Example 4.1.

Consider an ordered semigroup S defined in Example 3.5. We define an interval valued fuzzy subset $\widetilde{\lambda}$ by $\widetilde{\lambda}(x) = \begin{cases} [0.4, 0.5], & \text{if } x \in \{a, c\}, \\ [0.3, 0.4], & \text{if } x \in \{b, d\}. \end{cases}$

Let $\widetilde{t} = [0.1, 0.2]$. Then, $\widetilde{\lambda}$ is a $\widetilde{t}$-implication-based interval valued fuzzy generalized bi-ideal of S.

Theorem 4.1.

For any interval valued fuzzy subset $\widetilde{\lambda}$ of S, we have,

(i) If $I = I_{GR}$, then $\widetilde{\lambda}$ is a $\frac{\delta - k}{2}$-implication-based interval fuzzy generalized bi-ideal of S if and only if $\widetilde{\lambda}$ is an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy generalized bi-ideal of S,

(ii) If $I = I_G$, then $\widetilde{\lambda}$ is a $\frac{\delta - k}{2}$-implication-based interval valued fuzzy generalized bi-ideal of S if and only if $\widetilde{\lambda}$ is an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy generalized bi-ideal of S,

(iii) If $I = I_{cG}$, then $\widetilde{\lambda}$ is a $\frac{\delta - k}{2}$-implication-based interval valued fuzzy generalized bi-ideal of S if and only if $\widetilde{\lambda}$ satisfies the following conditions:

$(iiia)$ $(\forall x, y \in S)\left(x \leq y \Rightarrow r\max - \left\{\widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \geq r\min\left\{\widetilde{\lambda}(y), \widetilde{\delta}\right\}\right)$,

$(iiib)$ $r\max\left\{\widetilde{\lambda}(xyz), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \widetilde{\delta}\right\}$, for all, $x, y, z \in S$.

Proof:

(i) : Proof of (i) is straightforward.

(ii) : Assume that $\widetilde{\lambda}$ is a $\dfrac{\delta-k}{2}$- implication based interval valued fuzzy generalized bi-ideal of S. Then,

$$(e) : (\forall x, y \in S)\left(x \leq y \Rightarrow I_G\left(\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right) \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right).$$

$$(f) : (\forall x, y, z \in S)\left(I_G\left(r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\}, \widetilde{\lambda}(xyz),\right) \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right).$$

Let $x, y \in S$ be such that $x \leq y$. Using (e) we have, $\widetilde{\lambda}(y) \geq \widetilde{\lambda}(x)$ or $\widetilde{\lambda}(x) > \widetilde{\lambda}(y) \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$, Hence, $\widetilde{\lambda}(x) \geq r\min\left\{\widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$. From (f) we have $\widetilde{\lambda}(xyz) \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\}$ or $r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\} > \widetilde{\lambda}(xyz) \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. Hence, $\widetilde{\lambda}(xyz) \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$. Hence,

$$\widetilde{\lambda}(xyz) > r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}.$$

Therefore, by Theorem 3.2, it follows that $\widetilde{\lambda}$ is an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy generalized bi-ideal of S.

Conversely, assume that $\widetilde{\lambda}$ is an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy generalized bi-ideal of S. Let, $x, y \in S$ be such that $x \leq y$. Thus, by Theorem 3.2(i), it follows that

$$I_G\left(\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right) = \begin{cases} \widetilde{\delta} \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right], \\ \text{if } r\min\left\{\widetilde{\lambda}(x), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} = \widetilde{\lambda}(x), \\ \widetilde{\lambda}(y) \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right], \\ \text{if } r\min\left\{\widetilde{\lambda}(x), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} = \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]. \end{cases}$$

From Theorem 3.2(ii), if $r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} = r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\}$, then $\widetilde{\lambda}(xyz) \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\}$. Also, $I_G\left(r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\}, \widetilde{\lambda}(xyz)\right) = \widetilde{\delta} \geq$

$\left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. If $r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} = \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$, then $\widetilde{\lambda}(xyz) \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$ and hence $I_G\left(r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\}, \widetilde{\lambda}(xyz)\right) \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. Therefore, $\widetilde{\lambda}$ is a $\frac{\widetilde{\delta} - \widetilde{k}}{2}$-implication-based interval valued fuzzy generalized bi-ideal of S.

(iii) : Assume that $\widetilde{\lambda}$ satisfies $(iiia)$ and $(iiib)$. Let $x, y \in S$ be such that $x \leq y$. In $(iiia)$, if $r\min\left\{\widetilde{\lambda}(x), \widetilde{\delta}\right\} = \widetilde{\delta}$, then $r\max\left\{\widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} = \widetilde{\delta}$ and so $\widetilde{\lambda}(x) = \widetilde{\delta} \geq \widetilde{\lambda}(y)$. Consequently, $I_{cG}\left(\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right) = \widetilde{\delta} \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. If $\widetilde{\lambda}(y) < \widetilde{\delta}$, then

$$r\max\left\{\widetilde{\lambda}(x), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \geq \widetilde{\lambda}(y). \qquad (b)$$

Now, if $\widetilde{\lambda}(x) > \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$, in (b), then $\widetilde{\lambda}(x) \geq \widetilde{\lambda}(y)$ and so $I_{cG}\left(\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right) = \widetilde{\delta} \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. If $\widetilde{\lambda}(x) \leq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$, in (b), then $\widetilde{\lambda}(y) \leq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. Hence,

$$I_{cG}\left(\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right) = \begin{cases} \widetilde{\delta} \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right], & \text{if } \widetilde{\lambda}(x) \geq \widetilde{\lambda}(y), \\ \widetilde{\delta} - \widetilde{\lambda}(y) \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right], & \text{otherwise.} \end{cases}$$

In $(ii2)$, if $r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \widetilde{\delta}\right\} = \widetilde{\delta}$, then $r\max\left\{\widetilde{\lambda}(xyz), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} = \widetilde{\delta}$ and hence, $\widetilde{\lambda}(xyz) = \widetilde{\delta} \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right\}$. Therefore, $I_{cG}\left(r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\}, \widetilde{\lambda}(xyz)\right) = \widetilde{\delta} \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. If $r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \widetilde{\delta}\right\} = r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\}$, then

$$r\max\left\{\widetilde{\lambda}(xyz), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\}. \qquad (c)$$

Hence, if $r\max\left\{\widetilde{\lambda}(xyz), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} = \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$, in (c) then $\widetilde{\lambda}(xyz) \leq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$ and $r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\} \leq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. Therefore,

$$I_{cG}\left(r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\}, \widetilde{\lambda}(xyz)\right) = \widetilde{\delta} \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right],$$

whenever $\widetilde{\lambda}(xyz) \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\}$ and $I_{cG}\left(r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\}, \widetilde{\lambda}(xyz)\right) = \widetilde{\delta} - r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\} \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$, whenever $\widetilde{\lambda}(xyz) < r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\}$. Now, if $r\max\left\{\widetilde{\lambda}(xyz), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} = \widetilde{\lambda}(xyz)$, in (c) then $\widetilde{\lambda}(xyz) \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\}$ and so $I_{cG}\left(r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\}, \widetilde{\lambda}(xyz)\right) = \widetilde{\delta} \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. Therefore, $\widetilde{\lambda}$ is a $\frac{\widetilde{\delta} - \widetilde{k}}{2}$-implication-based interval valued fuzzy generalized bi-ideal of S.

Conversely, suppose that $\widetilde{\lambda}$ is a $\frac{\widetilde{\delta} - \widetilde{k}}{2}$-implication-based interval valued fuzzy generalized bi-ideal of S. Then

(a) $(\forall x, y \in S)\left(x \leq y \Rightarrow I_G\left(\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right) \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right)$,

(b) $(\forall x, y, z \in S)\left(I_G\left(r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\}, \widetilde{\lambda}(xyz)\right) \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right)$.

Let $x, y \in S$ be such that $x \leq y$. Then, from (a), it follows that $I_G\left(\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right) = \widetilde{\delta}$ or $\widetilde{\delta} - \widetilde{\lambda}(y) \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$ so that $\widetilde{\lambda}(y) \leq \widetilde{\lambda}(x)$ or $\widetilde{\lambda}(y) \leq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. Therefore, $r\max\left\{\widetilde{\lambda}(x), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \geq \widetilde{\lambda}(y) = r\min\left\{\widetilde{\lambda}(y), \widetilde{\delta}\right\}$. From (b), for all $x, y, z \in S$, it follows that $I_G\left(r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\}, \widetilde{\lambda}(xyz)\right) = \widetilde{\delta}$, that is, $r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\} \leq \widetilde{\lambda}(xyz)$, or $\widetilde{\delta} - r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\} \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. Hence, $r\max\left\{\widetilde{\lambda}(z), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\} = r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \widetilde{\delta}\right\}$. Hence, $\widetilde{\lambda}(xy) \leq \widetilde{\lambda}(x)$ or $\widetilde{\lambda}(xy) \leq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. Therefore, $r\max\left\{\widetilde{\lambda}(x), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \geq \widetilde{\lambda}(xy) = r\min\left\{\widetilde{\lambda}(xy), \widetilde{\delta}\right\}$. ■

Corollary 4.1.

For any interval valued fuzzy subset $\widetilde{\lambda}$ of S, we have:

(i) If $I = I_{GR}$, then $\widetilde{\lambda}$ is a $\frac{1-k}{2}$-implication-based interval fuzzy generalized bi-ideal of S if and only if $\widetilde{\lambda}$ is an interval valued $\left(\in, \in \vee q_{\widetilde{k}}\right)$-fuzzy generalized bi-ideal of S,

(ii) If $I = I_G$, then $\widetilde{\lambda}$ is a $\frac{1-k}{2}$-implication-based interval valued fuzzy generalized bi-ideal of S

if and only if $\widetilde{\lambda}$ is an interval valued $\left(\in, \in \vee q_{\widetilde{k}}\right)$-fuzzy generalized bi-ideal of S,

(iii) If $I = I_{cG}$, then $\widetilde{\lambda}$ is a $\dfrac{1-k}{2}$-implication-based interval valued fuzzy generalized bi-ideal of S if and only if $\widetilde{\lambda}$ satisfies the following conditions:

$(iiia)$ $(\forall x, y \in S)\left(x \leq y \Rightarrow r\max - \left\{\widetilde{\lambda}(y), \left[\dfrac{1-k^-}{2}, \dfrac{1-k^+}{2}\right]\right\} \geq r\min\left\{\widetilde{\lambda}(y), [1,1]\right\}\right)$,

$(iiib)$ $r\max\left\{\widetilde{\lambda}(xyz), \left[\dfrac{1-k^-}{2}, \dfrac{1-k^+}{2}\right]\right\} \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), [1,1]\right\}$,

for all, $x, y, z \in S$.

Proof:

By Theorem 4.1, if we take $\widetilde{\delta} = [1,1]$, then we get $(i), (iii), (iii)$. ∎

Corollary 4.2.

For any interval valued fuzzy subset $\widetilde{\lambda}$ of S, we have:

(i) If $I = I_{GR}$, then $\widetilde{\lambda}$ is a $\dfrac{1}{2}$-implication-based interval fuzzy generalized bi-ideal of S if and only if $\widetilde{\lambda}$ is an interval valued $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S,

(ii) If $I = I_G$, then $\widetilde{\lambda}$ is a $\dfrac{1}{2}$-implication-based interval valued fuzzy generalized bi-ideal of S if and only if $\widetilde{\lambda}$ is an interval valued $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S,

(iii) If $I = I_{cG}$, then $\widetilde{\lambda}$ is a $\dfrac{1}{2}$-implication-based interval valued fuzzy generalized bi-ideal of S if and only if $\widetilde{\lambda}$ satisfies the following conditions:

$(iiia)$ $(\forall x, y \in S)\left(x \leq y \Rightarrow r\max - \left\{\widetilde{\lambda}(y), \left[\dfrac{1}{2}, \dfrac{1}{2}\right]\right\} \geq r\min\left\{\widetilde{\lambda}(y), [1,1]\right\}\right)$,

$(iiib)$ $r\max\left\{\widetilde{\lambda}(xyz), \left[\dfrac{1}{2}, \dfrac{1}{2}\right]\right\} \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), [1,1]\right\}$,

for all, $x, y, z \in S$.

Proof:

By Theorem 4.1, if we take $\widetilde{\delta} = [1,1]$ and $\widetilde{k} = [0,0]$, then we get $(i), (iii), (iii)$. ∎

5. Conclusion and Discussion

The concept of an interval valued $(\in, \in \vee q_k^\delta)$-fuzzy generalized bi-ideal of an ordered semigroup is a generalization of the concepts studied by Khan et al. (2013). If we take $\delta = 1$, then we get an interval valued $(\in, \in \vee q_k^\delta)$-fuzzy generalized bi-ideal of an ordered semigroup. If we take $\delta = 1$ and $k = 0$, then we get interval valued $(\in, \in \vee q_k^\delta)$-fuzzy generalized bi-ideal of an ordered semigroup, which means that these fuzzy ideals become a special case of an interval valued $(\in, \in \vee q_k^\delta)$-fuzzy generalized bi-ideal of an ordered semigroup. Due to the motivation and inspiration of the concept, we study the concept of an interval valued $(\in, \in \vee q_k^\delta)$-fuzzy generalized bi-ideal of an ordered semigroup. In the domain of modern mathematics, the use of algebraic structures in computer science, control theory and fuzzy automata theory constantly increase the attention of researchers. Algebraic structures mostly semigroups play a vital role in such applied branches. Further, the fuzzification of several subsystems of semigroups is used in numerous models including uncertainties. In this paper, we introduced new types of substructure of semigroup called $(\in, \in \vee q_k^\delta)$-fuzzy substructure and characterized ordered semigroups in terms of an interval valued $(\in, \in \vee q_k^\delta)$-fuzzy generalized bi-ideal which is the generalization of interval valued $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal and interval valued $(\in, \in \vee q)$-fuzzy generalized bi-ideal.

$1)$ We can further apply the present concept to other algebraic structures, such as Ring, Hemiring, Nearring, etc.

$2)$ We will define interval valued $(\in, \in \vee q_k^\delta)$-fuzzy soft generalized bi-ideals.

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