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On Digital Metric Space Satisfying Certain Rational Inequalities

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Abstract

In this paper, we have established some new results by extending some existing theorems in the setting of Digital Metric Space. We also proved some results in Digital Metric Space which were established earlier in the context of Complete Metric Space by different authors.

Keywords: Digital-Metric space; Fixed point; Cauchy sequence; Self-Mapping; Digital image; Digital contraction

MSC 2010 No.: 47H10, 54H25, 46J10, 46J15

1. Introduction

Fixed point theory (FPT) is one the most interesting fields in the context of modern analysis. It has shown applications of its results in a variety of fields such as mathematical analysis, mathematical economics, physics, chemistry, computer science, digital image processing and many more. The branch of analysis that is related to metric FPT is functional analysis. It started as a part of classical analysis. It gives deep insights into finding solutions of problems related particularly to pure mathematics. It is evident that functional analysis has a wide spectrum of applications in almost all branches of science. The notions of convergence, metric spaces, linear and nonlinear operators are fundamental to the study of functional analysis. The importance of metric space to functional analysis is somewhat analogous to real line with its importance in calculus. In fact metric spaces have been incorporated to form the basis of unification in the theory of functional analysis. In 1922, Banach gave the contraction principle which ensures the existence and uniqueness of the fixed point of certain self-maps.

In the latter half of the 20th century FPT has established itself as a capable tool for nonlinear analysis. In the past 20 years FPT has seen enormous development. Much of the work presented in this paper relates to the work carried out in this time. If we talk about the applications of FPT, there are obviously numerous. For instance, investigating solutions (if exist) of integral equations, system of linear equations, differential equations etc. But that is not it; in fact, Fixed Point schemes have been applied to multifarious fields such as biology, chemistry, engineering, physics and economics. FPT has constructive applications in category theory, functional equations, control theory, game theory and multitude of other areas. Fixed Point theorems are exercised in existence theory of random differential equations, numerical methods like Newton-Rapshon method and Picard's Existence Theorem, etc.

Digital image processing is one of the most flourishing area of engineering research. It has tremendous potential of utilizing available computing power with help of optimized algorithms and produce wonderful results of much importance. It was Rosenfeld in 1979 who coined the term Digital Topology. He discussed it in a paper titled "Digital Topology," which then became the founding literature of Digital Topology. It discusses the topological aspects of the images, i.e., connectedness, adjacency, etc. Thus, digital topology, by dealing with topological and geometrical properties, opens up possibilities for analysis (recognition, detection, etc.) of images. Along with applications related to image processing, digital topology is also useful in areas of artificial intelligence dealing with spatial structures. The geometry and topology of images helped a lot in the field of image processing and pattern recognition. Kong and Rosenfeld and Rosenfeld in their work published in 1989 reviewed concepts and surveyed established results in digital topology. The general topology is not appropriate for studying digital images as it considers infinitely many points in smallest possible neighbourhood, while digital topology consider finite number of points in the neighbourhood of interest.

The objective of this paper is to prove the completeness of the digital metric space by establishing Kannan Contraction Fixed Point Theorem. Similarly, Chatterjea and Reich Contraction Fixed Point theorem are also established for the said purpose. A couple of Fixed Point Theorems using relational inequalities are also established in the paper.

2. Preliminaries

The fundamental object in digital topology is a Lattice, which is used to represent a digital image in n-dimensions. The lattice has lattice points (with integer coordinates) which are called **pixels(2d)** or **voxels(3d)**. A **Digital Plane** $\mathbb{Z}^{\mathbf{2}}$ is a set of all the points in $\mathbb{R}^2$ and 3-D digital space $\mathbb{Z}^{\mathbf{3}}$ is a set of all the points in $\mathbb{R}^3$ having integer coordinates.

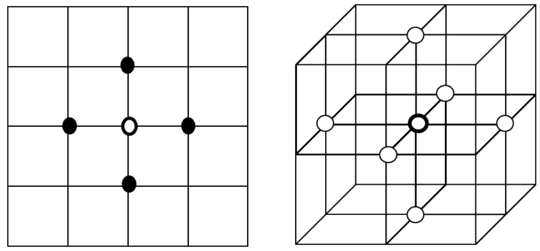

Figure 1. 2D and 3D digital planes

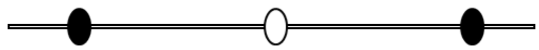

Figure 2. 2-adjacency

Definition 2.1.

Let $l,\ n$ be positive integers with $1 \leq l \leq n$. Consider two distinct points $p = \{p_1,\ p_2,\ \ldots,\ p_n\},\ q = \{q_1, q_2,\ \ldots,\ q_n\}\ \epsilon\ \mathbb{Z}^n$. The points p and q are $k_l - adjacent$ if there are at most l indices i such that $|p_i - q_i| = 1$ and for all other indices $j,\ |p_j - q_j| \neq 1,\ p_j = q_j$.

Definition 2.2.

(2-adjacency): Two points on digital plane are said to be 2-adjacent if $|p - q| = 1$.

Definition 2.3.

(4-neighbours): The $4 - neighbours$ of a point p_{ij} are its four horizontal and vertical neighbours $(i \pm 1, j)$ and $(i, j \pm 1)$. $4 - neighbours$ of a point p_{ij} are denoted by $N_4(p_{ij})$.

Definition 2.4.

(6-neighbours): The $6 - neighbours$ of a point p_{ij} are its four horizontal and vertical neighbours $(i \pm 1, j)$ and $(i, j \pm 1)$ along with 2 neighbours $(i + 1, j + 1)$ and $(i - 1, j - 1)$ or $(i - 1, j + 1)$ and $(i + 1, j - 1)$, i.e., from either of the diagonals. $6 - neighbours$ of a point p_{ij} are denoted by $N_6(p_{ij})$.

Definition 2.5.

(8-neighbours): The $8 - neighbours$ of a point p_{ij} consist of its $4 - neighbours$ together with its four diagonal neighbours $(i + 1, j \pm 1)\ and\ (i - 1, j \pm 1)$ and are denoted by $N_8(p_{ij})$. The diagonal neighbours of a point p_{ij} are denoted by $N_D\left(p_{ij}\right)$.

The $4 - neighbours,\ N_4\left(p_{ij}\right)$ and $4\ diagonal\ neighbours,\ N_D\left(p_{ij}\right)$ are together called as $8 - neighbours$ of the point p_{ij} and are denoted by $N_8(p_{ij})$.

Definition 2.6.

(4-adjacency): Two points $p\ and\ q$ on a digital plane $(\mathbb{Z}^2)$ are said to be 4-adjacent if $q \in N_4(p_{ij})$.

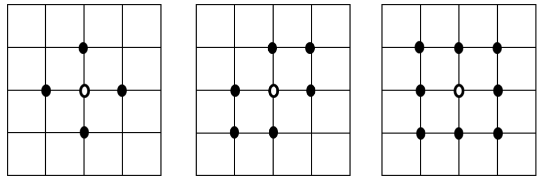

Figure 3. 4-adjacency (left), 6-adjacency (center) 8-adjacency (right)

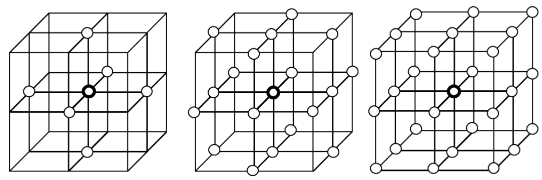

Figure 4. 6 (left),18 (center)& 26-adjacency (right)

Definition 2.7.

(6-adjacency): Two points $p\ and\ q$ on a digital plane $(\mathbb{Z}^2)$ are said to be 6-adjacent if $q \in N_6(p_{ij})$.

Definition 2.8.

(8-adjacency): Two points $p\ and\ q$ on a digital plane $(\mathbb{Z}^2)$ are said to be 8-adjacent if $q \in N_8(p_{ij})$.

Definition 2.9.

(6-adjacency in $\mathbb{Z}^3$): Two points $p\ and\ q$ are $6-adjacent$ in 3-D digital space $(\mathbb{Z}^3)$ if $point\ q$ is located at coordinates $(i \pm 1, j, k)$, $(i, j \pm 1, k)\ or\ (i,\ j,\ k \pm 1)$ from the point p_{ijk}.

Definition 2.10.

(18-adjacency in $\mathbb{Z}^3$): Two points $p\ and\ q$ are $18-adjacent$ in a 3-D digital space $(\mathbb{Z}^3)$ if point q is located at coordinates $(i \pm 1, j \pm 1,\ kt), (i \pm 1,\ j \mp 1,\ k), (i \pm 1,\ j, k \pm 1), (i \pm 1,\ j,\ k \mp 1), (i,\ j \pm 1,\ k \pm 1) or\ (i, j \pm 1,\ k \mp 1)$ from the point p_{ijk}.

Definition 2.11.

(26-adjacency in $\mathbb{Z}^3$): Two points $p\ and\ q$ are $26-adjacent$ in a 3-D digital space $(\mathbb{Z}^3)$ if point q is located at coordinates $(i \pm 1,\ j \pm 1,\ k \pm 1)$, $(i \pm 1,\ j \pm 1,\ k \mp 1)$, $(i \pm 1,\ j \mp 1,\ k \pm 1)\ or\ (i \mp 1,\ j \pm 1,\ k \pm 1)$ from the point p_{ijk}.

Definition 2.12.

A digital image is a pair (X, κ), where $\Phi\ \neq X\ \subset \mathbb{Z}^n$ for some positive integer n and κ is an adjacency relation on X. Technically, a digital image (X, κ) is an undirected graph whose vertex set is the set of members of X and whose edge set is the set of unordered pairs $\{x_0,\ x_1\}\ \subset X$ such that $x_0\ \neq x_1$ and $x_0\ and\ x_1$are $\kappa - adjacent.$

3. Main Results

The following theorem is a variation of Kannan Contraction Fixed Point Theorem. Using similar rational inequality and a self-map, Ege et al. proved similar results. The following theorem uses two self-maps $S \; and \; T$ on X.

Theorem 3.1.

Let (X, κ) be a digital image where $X \subset \mathbb{Z}^n$ and κ is an adjacency relation in X. Let $(X, \; d, \; \kappa)$ be a digital metric space and S, T be self maps on X satisfying the following,

$$d(Sx, Ty) \leq \alpha d(x, Sx) + d(y, Ty),$$

for all $x, y \; \in X \; and \; 0 < \alpha < \frac{1}{2}$. Then, S and T have a unique common fixed point in X.

Proof:

Let $x_0 \in X$ be any arbitrary in X. Define a sequence $< x_n >$ in X such that, $x_1 = S(x_0)$, $x_2 = T(x_1)$, in general $S\left(x_{2n}\right) = x_{2n+1}$ and T$(x_{2n+1}) = x_{2n+2}$.

Consider

$$d\left(x_1, x_2\right) = d\left(Sx_0, Tx_1\right).$$

Now

$$\begin{aligned} d\left(x_1, x_2\right) &\leq \alpha\{d\left(x_0, \; Sx_0\right) + d\left(x_1, \; Tx_1\right)\}, \\ d\left(x_1, x_2\right) &\leq \alpha\left\{d\left(x_0, \; x_1\right) + d\left(x_1, \; x_2\right)\right\}, \\ d\left(x_1, x_2\right) - \alpha d\left(x_1, \; x_2\right) &\leq \alpha d\left(x_0, \; x_1\right), \\ (1-\alpha) d\left(x_1, \; x_2\right) &\leq \alpha d\left(x_0, \; x_1\right), \\ d\left(x_1, \; x_2\right) &\leq \left[\frac{\alpha}{1-\alpha}\right] d\left(x_0, \; x_1\right). \end{aligned}$$

In a similar way, we have

$$\begin{aligned} d\left(x_2, \; x_3\right) &\leq \left[\frac{\alpha}{1-\alpha}\right] d\left(x_1, \; x_2\right), \\ d\left(x_2, \; x_3\right) &\leq \left[\frac{\alpha}{1-\alpha}\right]^2 d\left(x_0, \; x_1\right), \end{aligned}$$

and continuing like this, we have

$$d(x_n,\ x_{n+1}) \leq \left[\frac{\alpha}{1-\alpha}\right]^n d(x_0,\ x_1),$$
$$d(x_{n+1},\ x_{n+2}) \leq \left[\frac{\alpha}{1-\alpha}\right]^{n+1} d(x_0,\ x_1).$$

Now, as $n \to \infty$, $\left(\frac{\alpha}{1-\alpha}\right)^{n+1} d(x_0,\ x_1) \to 0$. This suggests that the sequence $< x_n >$ is a $Cauchy\ Sequence$ in digital metric space (X, d, κ). Thus, there is a point $u \in X$ such that $x_n \to u$. Therefore, sub sequence $< Sx_{2n} > \to u$ and $< Tx_{2n+1} > \to u$ since S and T are $(\kappa, \kappa) - \mathrm{continuous}$ functions so we have, $Su = u,\ Tu = u$. If we put $Ty = Sy$, we obtain the results of Ege et al. Let u be the common fixed point of S and T. Then, by the condition of the theorem,

$$\begin{aligned} d(u, u) &= d(Su, Tu) \\ &= \alpha d(u, Su) + d(u, Tu) \\ &= \alpha d(u, u) + d(u, u) \\ &= \alpha d(u, u), \\ d(u, u) &= 0. \end{aligned}$$

Let u, v be common fixed points of S and T. Then,

$$\begin{aligned} d(u, v) &= d(Su, Tv), \\ &= \alpha d(u, Su) + d(v, Tv), \\ &= \alpha d(u, u) + d(v, v) = 0. \end{aligned}$$

Thus, $d(u, v) = 0$, hence, $u = v.$ This proves the uniqueness.

Now the following theorem is a variation of Chatterjea Fixed Point Theorem. Using similar rational inequality and a self-map, Ege et al. proved similar results. The following theorem uses two self-maps S and T on X. ■

Theorem 3.2.

Let (X, κ) be a digital image and (X, d, κ) be a digital metric space where $X \subset \mathbb{Z}^n$, and κ is an adjacency relation in X, and S, T be self maps on X satisfying the following,

$$d(Sx, Ty) \leq \alpha d(x, Ty) + d(y, Sx),$$

for all $x, y \in X$ and $0 < \alpha < \frac{1}{2}$. Then, S and T have a unique common fixed point in X.

Proof:

Let $x_0 \in X$ be any arbitrary in X. Define a sequence $< x_n >$ in X such that $x_1 = S(x_0)$,

$x_2 = T(x_1)$ in general $S(x_{2n}) = x_{2n+1}$ and $T(x_{2n+1}) = x_{2n+2}$.

Consider

$$d(x_1, x_2) = d(Sx_0, Tx_1).$$

Now

$$\begin{aligned} d(x_1, x_2) &\leq \alpha d(x_0,\ Tx_1) + d(x_1, Sx_0), \\ d(x_1, x_2) &\leq \alpha d(x_0,\ x_2) + d(x_1,\ x_1), \\ d(x_1, x_2) &\leq \alpha d(x_0, x_1) + d(x_1, x_2), \\ d(x_1, x_2) - \alpha d(x_1, x_2) &\leq \alpha d(x_0,\ x_1), \\ (1-\alpha) d(x_1,\ x_2) &\leq \alpha d(x_0,\ x_1), \end{aligned}$$

$$d(x_1,\ x_2) \leq \left[\frac{\alpha}{1-\alpha}\right] d(x_0,\ x_1).$$

In a similar way, we have

$$\begin{aligned} d(x_2, x_3) &\leq \left[\frac{\alpha}{1-\alpha}\right] d(x_1, x_2), \\ d(x_2, x_3) &\leq \left[\frac{\alpha}{1-\alpha}\right]^2 d(x_0, x_1), \end{aligned}$$

and continuing like this we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \left[\frac{\alpha}{1-\alpha}\right]^n d(x_0, x_1), \\ d(x_{n+1}, x_{n+2}) &\leq \left[\frac{\alpha}{1-\alpha}\right]^{n+1} d(x_0,\ x_1), \end{aligned}$$

Now, as $n \to \infty$, $\left(\frac{\alpha}{1-\alpha}\right)^{n+1} d\left(x_0,\ x_1\right) \to 0$. This suggests that the sequence $< x_n >$ is a Cauchy Sequence in digital metric space (X, d, κ). Thus, there is a point $u \in X$ such that $x_n \to u$. Therefore, sub-sequence $< Sx_{2n} > \to u$ and $< Tx_{2n+1} > \to u$ since S and T are $(\kappa\ ,\ \kappa)$ $-$continuous functions so, we have, $Su = u$, $Tu = u$.

If we put $Ty = Sy$, we obtain the results of Ege et al. Let u be the common fixed point of S and

T. Then, by the condition of the theorem,

$$\begin{aligned} d(u,u) &= d(Su,Tu) \\ &= \alpha d(u,Tu) + d(u,Su) \\ &= \alpha d(u,u) + d(u,u) \\ &= \alpha d(u,u), \\ d(u,u) &= 0. \end{aligned}$$

Let u,v be common fixed points of S and T. Then,

$$\begin{aligned} d(u,v) &= d(Su,Tv) \\ &= \alpha d(u,Tv) + d(v,Su), \\ d(u,v) &= \alpha d(u,v) + d(v,u). \end{aligned} \tag{1}$$

Put $u = v$ and $v = u$, we get

$$d(v,u) = \alpha d(v,u) + d(u,v). \tag{2}$$

Subtracting Equation 2 from 1, we get

$$d(u,v) - d(v,u) \leq 0.$$

Thus, $d(u,v) = 0$. Hence $u = v$. This proves the uniqueness. If we put $Ty = Sy$, we obtain the results of Ege et al. ■

The next theorem is a variation of Reich Fixed Point Theorem. Using similar rational inequality and a self-map Ege et al. proved similar results. The following theorem uses two self-maps S and T on X.

Theorem 3.3.

Let (X,κ) be a digital image and (X,d,κ) be a digital metric space where, $X \subset \mathbb{Z}^n$, and κ is an adjacency relation in X, and S, T be self-maps on X satisfying the following,

$$d(Sx,Ty) \leq a.d(x,Sx) + b.d(y,Ty) + c.d(x,y),$$

for all $x, y \in X$. Here a, b and c are nonnegative real numbers such that $(a+b+c) < 1$. Then, S and T have a unique common fixed point in X.

Proof:

Let $x_0 \in X$ be any arbitrary in X. Define a sequence $< x_n >$ in X such that $x_1 = S(x_0)$, $x_2 = T(x_1)$ in general $Sx_{2n} = x_{2n+1}$ and $T\,(x_{2n+1}) = x_{2n+2}$.

Consider

$$d(x_1, x_2) = d(Sx_0, Tx_1).$$

Now

$$\begin{aligned} d(x_1, x_2) &\leq a.d(x_0, Sx_0) + b.d(x_1, Tx_1) + c.d(x_0, x_1), \\ d(x_1, x_2) &\leq a.d(x_0, x_1) + b.d(x_1, x_2) + c.d(x_0, x_1), \\ d(x_1, x_2) &\leq (a + c)d(x_0, x_1) + b.d(x_1, x_2), \\ d(x_1, x_2) - b.d(x_1, x_2) &\leq (a + c)d(x_0, x_1), \\ (1 - b)d(x_1, x_2) &\leq (a + c)d(x_0, x_1), \\ d(x_1, x_2) &\leq \left[\frac{a + c}{1 - b}\right] d(x_0, x_1). \end{aligned}$$

In a similar way, we have

$$\begin{aligned} d(x_2, x_3) &\leq \left[\frac{a + c}{1 - b}\right] d(x_1, x_2), \\ d(x_2, x_3) &\leq \left[\frac{a + c}{1 - b}\right]^2 d(x_0,\ x_1), \end{aligned}$$

and continuing like this we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \left[\frac{a + c}{1 - b}\right]^n d(x_0, x_1), \\ d(x_{n+1}, x_{n+2}) &\leq \left[\frac{a + c}{1 - b}\right]^{n+1} d(x_0, x_1), \end{aligned}$$

$n \to \infty, (\frac{a + c}{1 - b})^{n+1} d(x_0, x_1) \to 0$. This suggests that the sequence $< x_n >$ is a Cauchy Sequence in digital metric space $(X,\ d,\ \kappa)$. Thus, there is a point $u \in X$ such that $x_n \to u$. Therefore, sub sequence $< Sx_{2n} > \to u$ and $< Tx_{2n+1} > \to u$ since S and T are $(\kappa, \kappa) - \text{continuous}$ functions so we have $Su = u$ and $Tu = u$.

If we put $Ty = Sy$, we obtain the results of Ege et al.

Let u be the common fixed point of S and T. Then, by the condition of the theorem,

$$\begin{aligned} d(u, u) &= d(Su, Tu) \\ &= a.d(u, Su) + b.d(v, Tu) + c.d(u, u) \\ &= a.d(u, u) + b.d(u, u) + c.d(u, u) \\ &= (a + b + c)d(u, u), \end{aligned}$$

which gives

$$d(u, u) = 0.$$

Let u,v be common fixed points of S and T. Then,

$$\begin{aligned} d(u, v) &= d(Su, Tv) \\ &= a.d(u, Su) + b.d(v, Tv) + c.d(u, v) \\ &= a.d(u, u) + b.d(v, v) + c.d(u, v) \\ d(u, v) &= c.d(u, v). \end{aligned}$$

This is only possible when $d(u, v) = 0$, therefore, $u = v$. This proves the uniqueness. If we put $Ty = Sy$, we obtain the results of Ege et al. ■

Theorem 3.4.

Let (X, κ) be a digital image and (X, d, κ) be a digital metric space where $X \subset \mathbb{Z}^n$, and κ is an adjacency relation in X, and S be a self-map on X satisfying the following,

$$d(Sx, Sy) \leq \frac{[a.d\,(y, Sy)\,[1 + d\,(x, Sx)]]}{[1 + d\,(x, y)]} + b.d\,(x, y)\,,$$

for all $x, y \in X$. Here a and b are non-negative real numbers such that, $(a + b) < 1$. Then, S has a unique fixed point in X. Dass and Gupta proved this result in the context of complete metric space. Now, we prove the result in the setting of digital metric space.

Proof:

Let $x_0 \in X$ be any arbitrary in X. Define a sequence $< x_n >$ in X such that, $x_1 = S(x_0)$, $x_2 = S(x_1)$ in general $S(x_{2n}) = x_{2n+1}$ and S $(x_{2n+1}) = x_{2n+2}$.

Consider

$$\begin{aligned} d(x_1, x_2) &= d(Sx_0, Sx_1), \\ d(Sx_0, Sx_1) &\leq \frac{a.d(x_1, Sx_1)[1 + d(x_0, Sx_0)]}{[1 + d(x_0, x_1)]} + b.d(x_0, x_1), \\ d(x_1, x_2) &\leq \frac{a.\ d(x_1, x_2)[1 + d(x_0, x_1)]}{[1 + d(x_0, x_1)]} + b.\ d(x_0, x_1), \\ d(x_1, x_2) &\leq a.\ d(x_1, x_2) + b.\ d(x_0, x_1), \\ d(x_1, x_2) &\leq \left(\frac{b}{1 - a}\right).\ d(x_0, x_1). \end{aligned}$$

Similarly,

$$d(x_2,x_3) \leq \left(\frac{b}{1-a}\right).\, d(x_1,x_2),$$
$$d(x_2,x_3) \leq \left(\frac{b}{1-a}\right)^2.\, d(x_0,x_1).$$

Now let

$$c = \left(\frac{b}{1-a}\right).$$

We get

$$d(x_2,x_3) \leq c^2.\, d(x_0,x_1).$$

In general, we can easily get

$$d(x_n,x_{n+1}) \leq c^n.\, d(x_0,x_1),$$
$$d(x_{n+1},x_{n+2}) \leq c^{n+1}.\, d(x_0,x_1).$$

From triangular inequality,

$$d(x_n,x_{n+k}) \leq (c^n + c^{n+1} + \cdots + c^{n+k-1}).d(x_0,x_1),$$
$$d(x_n,x_{n+k}) \leq \frac{c^n}{1-c}.d(x_0,x_1).$$

Now, as $n \to \infty, \dfrac{c^n}{1-c} d(x_0,x_1) \to 0$. This suggests that the sequence $< x_n >$ is a Cauchy Sequence in digital metric space (X,d,κ). Thus, there is a point $u \in X$ such that $x_n \to u$. Therefore, sub sequence $< Sx_{2n} >\to u$ and $< Sx_{2n+1} >\to u$ since S is $(\kappa,\kappa) -$ continuous functions so we have $Su = u$.

Let u be the common fixed point of S. Then, by the condition of the theorem,

$$d(u,u) = d(Su,Su),$$
$$d(u,u) \leq \frac{a.\, d\,(u,Su)\,[1 + d\,(u,Su)]}{1 + d\,(u,u)} + b.\, d\,(u,u)\,,$$
$$d\,(u,u) \leq \frac{a.\, d\,(u,u)\,[1 + d\,(u,u)]}{1 + d\,(u,u)} + b.\, d\,(u,u)],$$
$$d\,(u,u) \leq b.\, d\,(u,u)\,,$$
$$d(u,u) = 0 \text{ as } 0 < b < 1.$$

This proves the uniqueness. ∎

Theorem 3.5.

Let (X,κ) be a digital image and (X,d,κ) be a digital metric space where $X \subset \mathbb{Z}^n$, and κ is an adjacency relation in X, and S be a self-map on X satisfying the following,

$$d\left(Sx,Sy\right)\leq\frac{\left[a.\,d\left(y,Sy\right)\left[1+d\left(x,Sx\right)\right]\right]}{\left[1+d\left(x,y\right)\right]}+b.\,d\left(x,y\right)+c.\,\left[\frac{d\left(y,Sy\right)+d\left(y,Sx\right)}{\left[1+d\left(y,Sy\right)d\left(y,Sx\right)\right]}\right],$$

for all $x,y\in X$. Here a and b are non-negative real numbers such that $(a+b+c)<1$. Then, S has a unique fixed point in X. Qureshi et al. proved this result in the context of complete dq-metric space. Now, we prove the result in the setting of digital metric space.

Proof:

Let $x_0\in X$ be any arbitrary in X. Define a sequence $<x_n>$ in X such that $x_1=S(x_0)$, $x_2=S(x_1)$ in general $S(x_{2n})=x_{2n+1}$ and $S(x_{2n+1})=x_{2n+2}$.

Consider

$$\begin{aligned}
d(x_1,x_2)&=d\left(Sx_0,Sx_1\right),\\
d\left(Sx_0,Sx_1\right)&\leq\frac{a.\,d\left(x_1,Sx_1\right)\left[1+d\left(x_0,Sx_0\right)\right]}{1+d\left(x_0,x_1\right)}+b.d\left(x_0,x_1\right)\\
&\quad+c.\,\left[\frac{d\left(x_1,Sx_1\right)+d\left(x_1,Sx_0\right)}{1+d\left(x_1,Sx_1\right)d\left(x_1,Sx_0\right)}\right],\\
d\left(x_1,x_2\right)&\leq\frac{a.\,d\left(x_1,x_2\right)\left[1+d\left(x_0,x_1\right)\right]}{1+d\left(x_0,x_1\right)}+b.\,d\left(x_0,x_1\right)\\
&\quad+c.\,\left[\frac{d\left(x_1,x_2\right)+d\left(x_1,x_1\right)}{1+d\left(x_1,x_2\right)d\left(x_1,x_1\right)}\right],\\
d\left(x_1,x_2\right)&\leq a.\,d\left(x_1,x_2\right)+b.\,d\left(x_0,x_1\right)+c.\,d\left(x_1,x_2\right)],\\
d\left(x_1,x_2\right)&\leq\left[\frac{b}{1-(a+c)}\right].\,d\left(x_0,x_1\right)].
\end{aligned}$$

Similarly,

$$\begin{aligned}
d\left(x_2,x_3\right)&\leq\left[\frac{b}{1-(a+c)}\right].d\left(x_1,x_2\right),\\
d\left(x_2,x_3\right)&\leq\left[\frac{b}{1-(a+c)}\right]^2.d\left(x_0,x_1\right)].
\end{aligned}$$

Now let

$$h=\left[\frac{b}{1-(a+c)}\right].$$

We get

$$d\left(x_2,x_3\right)\leq h^2.\,d\left(x_0,x_1\right).$$

In general,

$$d\left(x_n, x_{n+1}\right) \leq h^n . d\left(x_0, x_1\right),$$
$$d\left(x_{n+1}, x_{n+2}\right) \leq h^{n+1} .\, d\left(x_0, x_1\right).$$

From triangular inequality,

$$d\left(x_n, x_{n+k}\right) \leq \left(h^n + h^{n+1} + \cdots + h^{n+k-1}\right).\, d\left(x_0, x_1\right),$$
$$d\left(x_n, x_{n+k}\right) \leq \frac{h^n}{1-h} \,.\, d\left(x_0, x_1\right).$$

Now, as $n \rightarrow \infty$, $\frac{h^n}{1-h} d\left(x_0, x_1\right) \rightarrow 0$. This suggests that the sequence $< x_n >$ is a Cauchy Sequence in digital metric space (X, d, κ). Thus, there is a point $u \in X$ such that $x_n \rightarrow u$. Therefore, sub sequence $< Sx_{2n} > \rightarrow u$ and $< Sx_{2n+1} > \rightarrow u$ since S is $(\kappa, \kappa) -$ continuous functions so we have $Su = u$.

Let u be the common fixed point of S. Then, by the condition of the theorem,

$$d(u, u) = d(Su, Su),$$

$$d\left(u, u\right) \leq \frac{a.\, d\left(u, Su\right)\left[1 + d\left(u, Su\right)\right]}{1 + d\left(u, u\right)} + b.\, d\left(u, u\right) + c.\, \left[\frac{d\left(u, Su\right) + d\left(u, Su\right)}{1 + d\left(u, Su\right) d\left(u, Su\right)}\right],$$

$$d(u, u) \leq \frac{a.\, d\left(u, u\right)\left[1 + d\left(u, u\right)\right]}{1 + d\left(u, u\right)} + b.\, d\left(u, u\right) + c.\, \left[\frac{d\left(u, u\right) + d\left(u, u\right)}{1 + d\left(u, u\right) d\left(u, u\right)}\right],$$

$$d(u, u) \leq b.d(u, u),$$

$$d(u, u) = 0 \text{ as } 0 < b < 1.$$

This proves the uniqueness. ∎

4. Conclusion

Digital topology concepts are widely used in growing, thinning and region filling of digital images. Due to such applications these concepts are also used in digital image compression. Computer graphics and image processing algorithms with enhanced hardware capabilities are now capable of performing extensive computations. In this case, fixed point theorems in digital metric spaces can provide better acceleration to such algorithms, leading to improved performance. The work addressed in this paper proved Kannan, Chatterjea and Reich Contraction theorems in the setting of digital metric space. There is a scope of taking these mappings further by considering cyclic contractions in digital metric spaces.

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