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# Indexed Absolute Summability Factor of Improper Integrals

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## Abstract

In this paper, we have defined the summability for improper integrals and established a theorem on indexed absolute Cesáro summability factors of improper integral under sufficient conditions. Some auxiliary results (well known) have also been deduced from the main result under suitable conditions.

**Keywords:** Absolute summability; Nörlund summability; Improper Integrals; Inequalities for Integrals

**MSC 2010 No.:** 40C10, 40G05

## 1. Introduction

Considering the $(N, p_n)$ and $(K, 1, \alpha)$ summability methods, Parashar (1981) obtained the minimum set of conditions for an infinite series to be $(K, 1, \alpha)$ summable. Bor (1986a) found a relationship between two summability techniques $(C, 1)_k$ and $|N, p_n|_k$, and Bor (1986b) used the $|N, p_n|_k$ for generalization of a theorem based on minimal set of sufficient conditions for infinite series. Sonker and Munjal (2016a, 2016b) determined a theorem on generalized absolute Cesáro summability method with the sufficient conditions for infinite series and they used the concept of triangular matrices for obtaining the minimal set of sufficient conditions of infinite series to be bounded. Sonker and Munjal (2017b, 2017c) found the approximation of the function $f \in Lip(\alpha, p)$ using infinite matrices of Cesáro summability method and they obtained boundedness conditions of absolute summability factors. In this way by using the advanced summability method, we can improve the quality of the filters. Borwein and Thorpe (1986) extend many results on ordinary and absolute summability methods of integral. Çanak and Totur (2011, 2013) worked on the concept of Cesáro summability method with a very interesting result for integrals. In the same direction, we extend the results of Mazhar (1972) with the help of some new generalized conditions and absolute Nörlund summability method $|N, p_n|_k$ factor for integrals.

**Definition 1.1.**

Let $\sum a_n$ be an infinite series with sequence of partial sums $s_n$ and

$$\sigma_n = \frac{1}{n}\sum_{k=1}^{n} s_k. \tag{1}$$

The series $\sum a_n$ is said to be $(C, 1)$ summable if

$$\lim_{n\to\infty} = s, \tag{2}$$

where $s$ is a finite number. The series $\sum a_n$ is said to be $|C, 1|_k, k \geq 1$ summable (Flett (1957)), if

$$\sum_{n=1}^{\infty} n^{k-1}|\sigma_n - \sigma_{n-1}|^k < \infty. \tag{3}$$

**Definition 1.2.**

Let $f$ a real valued continuous function defined in the interval $[0, \infty)$ and $s(x) = \int_0^x f(t)dt$. The Cesáro mean of $s(x)$, usually denoted by $\sigma(x)$, is defined (Borwein and Thorpe (1986)) as

$$\sigma(x) = \frac{1}{x}\int_0^x (x-t)f(t)dt, \tag{4}$$

and

$$\sigma(x) = \frac{1}{x}\int_0^x s(t)dt. \tag{5}$$

The integral $\int_0^\infty f(t)dt$ is said to be summable $|C, 1|$ if

$$\int_0^\infty |\sigma^{'}(x)|dx < \infty, \tag{6}$$

and is said to be summable $|C,1|_k,\ k \geq 1$ if

$$\int_0^\infty x^{k-1}|\sigma^{'}(x)|dx < \infty. \tag{7}$$

Clearly, we have

$$s(x) - \sigma(x) = \frac{1}{x}\int_0^x tf(t)dt.$$

Let

$$s(x) - \sigma(x) = \nu(x). \tag{8}$$

Then (7) can be written as

$$\int_0^\infty x^{k-1}|\nu(x)|^k dx < \infty. \tag{9}$$

The improper integral $\int_0^\infty f(t)dt$ is said to be summable $|C,1,\delta|_k,\ \delta \geq 0,\ \delta k \leq 1$ if

$$\int_0^\infty x^{\delta k+k-1}|\nu(x)|^k dx < \infty. \tag{10}$$

## 2. Known Results

Concerning with $|C,1|_k$, Ozgen (2016) obtained the following results for integrals.

**Theorem 2.1.**

Let $\gamma(x)$ be a positive monotonic non-decreasing function such that

$$\lambda(x)\gamma(x) = O(1),\ \text{as } x \to \infty, \tag{11}$$

$$\int_0^x u|\lambda^{''}(u)|\gamma(u)du = O(1), \tag{12}$$

$$\int_0^x \frac{|\nu(u)|^k}{u} = O(\gamma(x)),\ \text{as } x \to \infty. \tag{13}$$

Then the integral $\int_0^\infty f(t)dt$ is summable $|C,1|_k,\ k \geq 1$.

Recently, Sonker and Munjal (2017a) extended Theorem 2.1 to $|N,p_n|_k$ summability method by establishing the following theorem.

**Theorem 2.2.**

Let $p(0) > 0,\ p(x) \geq 0$ and $p(x)$ be a non-increasing function. Let $\chi(x)$ be a positive non-decreasing function and there be two functions $\beta(x)$ and $\epsilon(x)$ such that

$$|\epsilon^{'}(x)| \leq \beta(x), \tag{14}$$

$$\beta(x) \to 0, \text{as } x \to \infty, \tag{15}$$

$$\int_0^\infty u|\beta^{'}(u)|\chi(u)du < \infty, \tag{16}$$

$$|\epsilon(x)|\chi(x) = O(1), \tag{17}$$

and

$$\int_0^x u^{-1}|\nu(u)|^k du = O(\chi(x)), \text{as } x \to \infty. \tag{18}$$

Then the integral $\int_0^\infty \epsilon(t)f(t)dt$ is summable $|N, p_n|_k$ for $k \geq 1$.

## 3. Main Results

In the present research article, we have extended the result of Ozgen (2016) by using the $|C, 1, \delta|_k$ summability method and we prove the following theorem.

**Theorem 3.1.**

Let $\chi(x)$ be a positive non-decreasing function and there be two functions $\beta(x)$ and $\epsilon(x)$ such that

$$|\epsilon^{'}(x)| \leq \beta(x), \tag{19}$$

$$\beta(x) \to 0, \text{as } x \to \infty, \tag{20}$$

$$\int_0^\infty u|\beta^{'}(u)|\chi(u)du < \infty, \tag{21}$$

$$|\epsilon(x)|\chi(x) = O(1), \tag{22}$$

and

$$\int_0^x u^{\delta k-1}|\nu(u)|^k du = O(\chi(x)), \text{as } x \to \infty. \tag{23}$$

The integral $\int_0^\infty \epsilon(t)f(t)dt < \infty$ is summable $|C, 1, \delta|_k$, for $k \geq 1,\ \delta k \leq 1$.

**Note:** The above theorem can be proved by using the concept of example that $\int_0^\infty x|\beta^{'}(x)|\chi(x)dx < \infty$ is weaker $\int_0^\infty x|\epsilon^{''}(x)|\chi(x)dx < \infty$, and hence the introduction of the function $\{\beta(x)\}$ is justified.

***Proof:***

It may be possible to choose the function $\beta(x)$ such that $|\epsilon^{'}(x)| \leq \beta(x)$ when $\epsilon^{'}(x)$ oscillates, $\beta(x)$ may be chosen such that $|\beta(x)| < |\epsilon^{''}(x)|$. Hence $\beta^{'}(x) < |\epsilon^{''}(x)|$, so that $\int_0^\infty x|\beta^{'}(x)|\chi(x)dx < \infty$ is a weaker requirement than $\int_0^\infty x|\epsilon^{''}(x)|\chi(x)dx < \infty$.

***Proof of the Theorem:***

Let $T(x)$ be the function $(C, 1)$ mean of the integral $\int_0^\infty \epsilon(t)dt$. The integral $|C, 1, \delta|_k$ summable if

$$\int_0^x x^{\delta k+k-1}|T^{'}(x)|dx = O(1), \text{as } x \to \infty, \tag{24}$$

where $T(x)$ is given by

$$\begin{aligned}
T(x) &= \frac{1}{x}\int_0^x \int_0^t \epsilon(u)f(u)dudt \\
&= \frac{1}{x}\int_0^x \epsilon(u)f(u)du \int_0^t dt \\
&= \frac{1}{x}\int_0^x (x-u)\epsilon(u)f(u)du \\
&= \int_0^x \left(1-\frac{u}{x}\right)\epsilon(u)f(u)du. 
\end{aligned} \tag{25}$$

On differentiating both sides with respect to $x$, we get

$$\begin{aligned}
T^{'}(x) &= \frac{1}{x^2}\int_0^x u\epsilon(u)f(u)du \\
&= \frac{\epsilon(x)}{x^2}\int_0^x uf(u)du - \frac{1}{x^2}\int_0^x \epsilon^{'}(u)\int_0^u tf(t)dtdu \\
&= \frac{\epsilon(x)\nu(x)}{x} - \frac{1}{x^2}\int_0^x u\epsilon^{'}(u)\left(\frac{1}{u}\int_0^u tf(t)dt\right)du \\
&= \frac{\epsilon(x)\nu(x)}{x} - \frac{1}{x^2}\int_0^x u\epsilon^{'}(u)\nu(u)du \\
&= T_1(x)+T_2(x).
\end{aligned} \tag{26}$$

Applying Minkowski's inequality (Royden (2015)), we have

$$|T^{'}(x)|^k = |T_1+T_2|^k < 2^k\left(|T_1|^k+|T_2|^k\right). \tag{27}$$

Further, by Hölder's inequality (Royden (2015)), we have

$$\begin{aligned}
\int_0^x t^{\delta k+k-1}|T_1(t)|^k dt &= \int_0^x t^{\delta k+k-1}\frac{|\nu(t)|^k|\epsilon(t)|^k}{|t|^k}dt \\
&= \int_0^x t^{\delta k-1}|\nu(t)|^k|\epsilon(t)|^{k-1}|\epsilon(t)|dt \\
&\leq \int_0^x t^{\delta k-1}|\nu(t)|^k|\epsilon(t)|dt \\
&= |\epsilon(x)|\int_0^x t^{\delta k-1}|\nu(t)|^k dt - \int_0^x |\epsilon^{'}(t)|\left(\int_0^t y^{\delta k-1}|\nu(y)|^k dy\right)dt \\
&= O(1)|\epsilon(x)|\chi(x) - \int_0^x \beta(t)\chi(t)dt \\
&= O(1) - \int_0^x |\beta^{'}(x)|dx\int_0^x \chi(u)du \\
&\leq O(1) - \int_0^\infty x|\beta^{'}(x)|\chi(x)dx \\
&= O(1), \text{as } x \to \infty.
\end{aligned} \tag{28}$$

By virtue of the hypothesis of Theorem 3.1,

$$\begin{aligned}
\int_0^x t^{\delta k+k-1}|T_2(t)|^k dt &= \int_0^x t^{\delta k+k-1}\frac{1}{t^{2k}}\left|\int_0^t u\epsilon^{'}(u)\nu(u)du\right|^k dt \\
&\leq \int_0^x t^{\delta k-1}\left(\int_0^t u^k|\epsilon^{'}(u)|^k|\nu(u)|^k du\right)\left(\frac{1}{t}\int_0^t du\right)^{k-1} dt \\
&= \int_0^x |u\epsilon^{'}(u)|^{k-1}|u\epsilon^{'}(u)||\nu(u)|^k du \int_u^x \frac{dt}{t^{1-\delta k}} \\
&= O(1)\int_0^x |u\epsilon^{'}(u)||\nu(u)|^k(u^{\delta k}-x^{\delta k})du \\
&\leq \int_0^x |u\epsilon^{'}(u)||\nu(u)|^k u^{\delta k}du \\
&= x|\epsilon^{'}(x)|\int_0^x |\nu(u)|^k u^{\delta k}du - \int_0^x (u|\epsilon^{'}(u)|)^{'}\int_0^u |\nu(t)|^k t^{\delta k}dt du \\
&= x|\beta(x)|\chi(x) - O(1)\int_0^x |\beta(u)|\chi(u)du - O(1)\int_0^x u|\beta^{'}(u)|\chi(u)du \\
&= O(1), \text{as } x \to \infty. \qquad (29)
\end{aligned}$$

On collecting (25) - (29), we have

$$\int_0^x t^{\delta k+k-1}|T^{'}(t)|^k dt = O(1),$$

and this completes the proof of the theorem. ∎

## 4. Conclusion

The main result of this research article is an attempt to formulate the problem of absolute summability factor of integrals which make a more modified filter. Through the investigation, we concluded that the improper integral is absolute Nörlund summable under the minimal sufficient conditions. Further, this study has a number of direct applications in rectification of signals in FIR filter (finite impulse response filter) and IIR filter (infinite impulse response filter). In a nutshell absolute summability method is a motivation for the researchers interested in studies of improper integrals.

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