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Generalized Fractional Calculus Operators Involving the Product Of the Jacobi Type Orthogonal Polynomials And Multivariable Polynomials

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Abstract

The objective of this paper is to establish four theorems for the Marichev-Saigo-Maeda fractional integration and differentiation formula to the product of the finite classes of the classical orthogonal polynomials with the general class of multivariable polynomials. The results are presented in terms of the Wright generalized hypergeometric function. Corresponding assertions in terms of Saigo, Erdélyi-Kober and Riemann-Liouville type of fractional integrals are also presented. Further, we point out also their relevance.

Keywords: Fractional integral operators; Classical orthogonal polynomials; Generalized Wright function' general class of multivariable polynomials

MSC 2010 No.: 26A33, 33C45, 33C05, 33C20, 33C60, 26A09, 33E20

1. Introduction

Fractional calculus is a very fastest growing subject of mathematics which deals with derivatives and integrals of arbitrary orders. The fractional calculus operators involving various special functions has found in various sub-fields of applicable mathematical analysis such as fluid dynamics, nonlinear biological systems, astrophysics, stochastic dynamical system, image processing, statistical distribution theory, and in quantum mechanics. Recently, many studies are found relating to fractional calculus in the papers of Kalla (1969a), Kalla (1969b), Kalla and Saxena (1969), Kalla and Saxena (1974), Saigo (1978), Saigo (1979), Saigo (1980), Saigo and Maeda (1996), etc. A detailed explanation of such operators with their properties and applications can be found in the research monographs by Kiryakova (1994), Miller and Ross (1993), etc.

2. Preliminaries

For our present study, we recall the generalization of the hypergeometric fractional integrals and derivative, including the Saigo operators has been introduced by Marichev (1974) (for details, see Samko et al. (1993), p. 194) and later extended and studied by Saigo and Maeda (1996)(p. 393) in term of any complex order with Appell function F_3 (.) in the kernel, as follows.

Let $\varepsilon, \varepsilon', \delta, \delta', \gamma \in \mathbb{C}$ and $x > 0$, then for $\Re(\gamma) > 0$,

$$\left(I_{0,x}^{\varepsilon,\varepsilon',\delta,\delta',\gamma} f\right)(x) = \frac{x^{-\varepsilon}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} t^{-\varepsilon'} F_3\left(\varepsilon, \varepsilon', \delta, \delta'; \gamma; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)\, dt, \quad (1)$$

and

$$\left(I_{x,\infty}^{\varepsilon,\varepsilon',\delta,\delta',\gamma} f\right)(x) = \frac{x^{-\varepsilon'}}{\Gamma(\gamma)} \int_x^\infty (t-x)^{\gamma-1} t^{-\varepsilon} F_3\left(\varepsilon, \varepsilon', \delta, \delta'; \gamma; 1-\frac{x}{t}, 1-\frac{t}{x}\right) f(t)\, dt. \quad (2)$$

Similarly, for $\Re(\gamma) \geq 0,\ k = [\Re(\gamma)] + 1$,

$$\begin{aligned}
\left(D_{0,x}^{\varepsilon,\varepsilon',\delta,\delta',\gamma} f\right)(x) &= \left(I_{0,x}^{-\varepsilon',-\varepsilon,-\delta',-\delta,-\gamma} f\right)(x) \\
&= \left(\frac{d}{dx}\right)^k \left(I_{0,x}^{-\varepsilon',-\varepsilon,-\delta'+k,-\delta,-\gamma+k} f\right)(x) \\
&= \frac{1}{\Gamma(k-\gamma)} \left(\frac{d}{dx}\right)^k (x)^{\varepsilon'} \int_0^x (x-t)^{k-\gamma-1} t^{\varepsilon} \\
&\times F_3\left(-\varepsilon', -\varepsilon, k-\delta', -\delta; k-\gamma; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)\, dt, \quad (3)
\end{aligned}$$

and

$$\begin{aligned}\left(D_{x,\infty}^{\varepsilon,\varepsilon',\delta,\delta',\gamma} f\right)(x) &= \left(I_{x,\infty}^{-\varepsilon',-\varepsilon,-\delta',-\delta,-\gamma} f\right)(x) \\ &= \left(-\frac{d}{dx}\right)^k \left(I_{x,\infty}^{-\varepsilon',-\varepsilon,-\delta',-\delta+k,-\gamma+k} f\right)(x) \\ &= \frac{1}{\Gamma(k-\gamma)}\left(\frac{d}{dx}\right)^k (x)^{\varepsilon} \int_x^{\infty} (t-x)^{k-\gamma-1} t^{\varepsilon'} \\ &\times F_3\left(-\varepsilon', -\varepsilon, \delta', k-\delta; k-\gamma; 1-\frac{x}{t}, 1-\frac{t}{x}\right) f(t)\, dt, \end{aligned} \tag{4}$$

where $F_3(.)$ denotes Appell function in two variables defined as

$$F_3\left(\varepsilon, \varepsilon', \delta, \delta', \gamma; 1-\frac{x}{t}, 1-\frac{t}{x}\right) = \sum_{m,n=0}^{\infty} \frac{(\varepsilon)_m (\varepsilon')_n (\delta)_m (\delta')_n}{(\gamma)_{m+n}} \frac{x^m}{m!}\frac{x^n}{n!}, \tag{5}$$

$$(\max\{|x|, |y|\} < 1).$$

Using the definition of the Appell function $F_3(.)$, the reader may refer to the monograph by Srivastava and Karlsson (1985). In recent times, many researchers (see Agarwal and Choi (2016), Baleanu et al. (2016), Kumar et al. (2016), Purohit et al. (2011), Purohit et al. (2012), Saxena et al. (2009), Suthar et al. (2018), Suthar and Ayene (2018)) have studied the image formulas for MSM fractional integral operators involving various special functions.

Remark 2.1.

The Appell function defined in above equation reduces to Gauss hypergeometric function ${}_2F_1$ as given in the following relations:

$$F_3(\varepsilon, \gamma-\varepsilon, \delta, \gamma-\delta, \gamma; x, y) = {}_2F_1(\varepsilon, \delta; \gamma; x+y-xy), \tag{6}$$

and

$$F_3(\varepsilon, 0, \delta, \delta', \gamma; x, y) = {}_2F_1(\varepsilon, \delta; \gamma; x), \tag{7}$$

and

$$F_3(0, \varepsilon', \delta, \delta', \gamma; x, y) = {}_2F_1(\varepsilon', \delta', \gamma; y). \tag{8}$$

Power function formulas of the above discussed fractional integral and derivative operators are required for our present study as given in the following forms:

$$\left(I_{0,x}^{\varepsilon,\varepsilon',\delta,\delta',\gamma} t^{\lambda-1}\right)(x) = \Gamma\left[\begin{matrix}\lambda, \lambda+\gamma-\varepsilon-\varepsilon'-\delta, \lambda+\delta'-\varepsilon' \\ \lambda+\delta', \lambda+\gamma-\varepsilon-\varepsilon', \lambda+\gamma-\varepsilon'-\delta\end{matrix}\right] x^{\lambda-\varepsilon-\varepsilon'+\gamma-1}, \tag{9}$$

$$\Re(\gamma) > 0, (\Re(\lambda) > \max\{0, \Re(\varepsilon+\varepsilon'+\delta-\gamma), \Re(\varepsilon'-\delta')\}),$$

$$\left(I_{x,\infty}^{\varepsilon,\varepsilon',\delta,\delta',\gamma} t^{\lambda-1}\right)(x) = \Gamma\left[\begin{matrix}1-\lambda-\gamma+\varepsilon+\varepsilon', 1-\lambda+\varepsilon+\delta'-\gamma, 1-\lambda-\delta \\ 1-\lambda, 1-\lambda+\varepsilon+\varepsilon'+\delta'-\gamma, 1-\lambda+\varepsilon-\delta\end{matrix}\right]$$

$$\times x^{\lambda-\varepsilon-\varepsilon'+\gamma-1}, \tag{10}$$

$$\left(\Re(\gamma)>0, \Re(\lambda)<\min\left\{\Re(-\delta), \Re(\varepsilon+\varepsilon'-\gamma), \Re(\varepsilon+\delta'-\gamma)\right\}\right),$$

$$\left(D_{0,x}^{\varepsilon,\varepsilon',\delta,\delta',\gamma}t^{\lambda-1}\right)(x)=\Gamma\left[\begin{matrix}\lambda, \lambda-\gamma+\varepsilon+\varepsilon'+\delta', \lambda-\delta+\varepsilon\\ \lambda-\delta, \lambda-\gamma+\varepsilon+\varepsilon', \lambda-\gamma+\varepsilon+\delta'\end{matrix}\right]x^{\lambda-\gamma+\varepsilon+\varepsilon'-1}, \tag{11}$$

$$\left(\Re(\lambda)>\max\left\{0, \Re(\gamma-\varepsilon-\varepsilon'-\delta'), \Re(\delta-\varepsilon)\right\}\right),$$

$$\left(D_{x,\infty}^{\varepsilon,\varepsilon',\delta,\delta',\gamma}t^{\lambda-1}\right)(x)=\Gamma\left[\begin{matrix}1-\lambda+\delta', 1-\lambda+\gamma-\varepsilon-\varepsilon', 1-\lambda+\gamma-\varepsilon'-\delta\\ 1-\lambda, 1-\lambda+\gamma-\varepsilon-\varepsilon'-\delta, 1-\lambda-\varepsilon'+\delta'\end{matrix}\right]$$

$$\times x^{\lambda-\gamma+\varepsilon+\varepsilon'-1}, \tag{12}$$

$$\left(\Re(\lambda)<1+\min\left\{\Re(\delta'), \Re(\gamma-\varepsilon-\varepsilon'), \Re(\gamma-\varepsilon'-\delta)\right\}\right).$$

The symbol occurring in (9-12) is given by

$$\Gamma\left[\begin{matrix}a, b, c\\ d, e, f\end{matrix}\right]=\frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(d)\Gamma(e)\Gamma(f)}. \tag{13}$$

An interesting further generalization of the generalized hyper-geometric function is due to Wright (1935) in a series representation of the form

$${}_p\psi_q={}_p\psi_q\left[\left.\begin{matrix}(a_i,\varepsilon_i)_{1,p}\\ (b_j,\delta_j)_{1,q}\end{matrix}\right| x\right]=\sum_{k=0}^{\infty}\frac{\Pi_{i=1}^{p}\Gamma(a_i+\varepsilon_i k)\,x^k}{\Pi_{j=1}^{q}\Gamma(b_j+\delta_j k)\,k!}, \tag{14}$$

where $a_i, b_j\in\mathbb{C}$, and real $\varepsilon_i, \delta_j\in\Re=(-\infty,\infty)$, $(\varepsilon_i, \delta_j\neq 0; i=1,2,\cdots p;\ \ j=1,2,\cdots q)$ and the asymptotic expansion of ${}_p\psi_q(x)$ for all values of the argument x, under the condition:

$$1+\sum_{j=1}^{q}\delta_j-\sum_{i=1}^{p}\varepsilon_i>0. \tag{15}$$

Also, we recall the following multivariable generalization of the polynomials $S_L^k(x)$ which was considered by Srivastava and Garg (1987) (pp. 686, Eq. (1.4)):

$$S_L^{k_1,\ldots,k_r}(x_1,\ldots,x_r)=\sum_{k_1,\ldots,k_r=0}^{h_1k_1+\ldots+h_rk_r\leq L}(-L)_{h_1k_1+\ldots+h_rk_r}\ A(L;k_1,\ldots,k_r)\frac{x_1^{k_1}}{k_1!}\cdots\frac{x_r^{k_r}}{k_r!}, \tag{16}$$

whereas the coefficient $\mathrm{A}(\mathrm{L};\mathrm{k}_1,\ldots,k_r)$, $(L, k_i\in N_0\backslash\{0\}, i=1,\ldots,r)$ are arbitrary chosen constants, real or complex. Clearly by setting $r=1$ of the polynomials defined by (16) would correspond to the polynomials defined by Srivastava (1972).

The solution to the differential equation

$$x(x+1)y_n''(x)+((2-p)x+(1+q))y_n'(x)-n(n-1+p)y_n(x)=0,$$

is given by the polynomial

$$M_n^{(p,q)}(x) = (-1)^n n! \sum_{k=0}^{\infty} \binom{p-n-1}{k} \binom{q+n}{n-k} (-x)^k, \tag{17}$$

with respect to the weight function $W_{p,q}(x) = x^q (1+x)^{-(p+q)}$. The polynomials given in (17) are orthogonal on [0,∞) if and only if $p > 2n+1$ and$q > -1$. The polynomials $M_n^{(p,q)}$ can be related with hypergeometric functions as

$$M_n^{(p,q)}(x) = (-1)^n n! \binom{q+n}{n} {}_2F_1(-n, n+1-p; q+1; -x), \tag{18}$$

Also the Jacobi polynomials $p_n^{(\varepsilon,\delta)}$ and $M_n^{(p,q)}$ can be related as

$$M_n^{(p,q)}(x) = (-1)^n n! P_n^{(q,-p-q)}(2x+1) \Leftrightarrow P_n^{(p,q)}(x) = \frac{(-1)^n}{n!} M_n^{(-p-q,p)} \left(\frac{x-1}{2}\right). \tag{19}$$

Details related to this finite class of classical orthogonal polynomials can be found from Malik and Swaminathan (2012), Malik and Swaminathan (2011), and Masjedjamei (2002).

3. Main Results

Thorough this paper, we suppose that $\varepsilon, \varepsilon', \delta, \delta', \gamma, \tau \in \mathbb{C}$, such that $\Re(\gamma) > 0$, Further, let the constants satisfy the condition $a_i, b_j \in \mathbb{C}$, and $\varepsilon_i, \delta_j \in \Re = (-\infty, \infty)$, $\varepsilon_i, \delta_j \neq 0$; $(i = 1, 2, \cdots p; j = 1, 2, \cdots q)$, such that condition (15) is also satisfied.

3.1. Composition with generalized fractional integration

In this section, we establish image formulas for the product of Srivastava polynomial and finite classes of the classical orthogonal polynomials involving left and right-sided operators of Marichev-Saigo-Meada fractional integral operators, in terms of the generalized Wright function. These formulas are given by the following theorems.

Theorem 3.1.

Let $\Re(\gamma) > 0$ and $\Re(\tau) > \max\{0, \Re(\varepsilon + \varepsilon' + \delta - \gamma), \Re(\varepsilon' - \delta')\}$, then the generalized fractional integration $I_{0,x}^{\varepsilon,\varepsilon',\delta,\delta',\gamma}$ of the product finite classes of the classical orthogonal polynomials $M_n^{(p,q)}(.)$ and multivariable polynomials $S_L^{k_1,\ldots,k_r}(.)$ is given by

$$\left(I_{0,x}^{\varepsilon,\varepsilon',\delta,\delta',\gamma} t^{\tau-1} S_L^{k_1,\ldots,k_r}\left(a_1 t^{\lambda_1}, \ldots, a_r t^{\lambda_r}\right) M_n^{(p,q)}\left(t^{\xi}\right)\right)(x)$$

$$= (-1)^n \sum_{k_1,\ldots,k_r=0}^{h_1k_1+\ldots+h_rk_r \leq L} (-L)_{h_1k_1+\ldots+h_rk_r} \; A(L; k_1, \ldots, k_r) \frac{a_1^{k_1}}{k_1!} \cdots \frac{a_r^{k_r}}{k_r!}$$

$$\times x^{\tau-\varepsilon-\varepsilon'+\gamma+\sum_{i=1}^{r} \lambda_i k_i - 1} \frac{\Gamma(q+n+1)}{\Gamma(-n)\Gamma(1+n-p)}$$

$$\times {}_5\psi_4 \left[\begin{array}{c} (-n,1), (1+n-p,1), (\tau+\sum_{i=1}^{r}\lambda_i k_i, \xi), \\ (q+1,1), (\tau+\delta'+\sum_{i=1}^{r}\lambda_i k_i, \xi), \end{array} \right.$$

$$\left. \begin{array}{c} (\tau+\gamma-\varepsilon-\varepsilon'-\delta+\sum_{i=1}^{r}\lambda_i k_i, \xi), (\tau+\delta'-\varepsilon'+\sum_{i=1}^{r}\lambda_i k_i, \xi) \\ (\tau+\gamma-\varepsilon-\varepsilon'+\sum_{i=1}^{r}\lambda_i k_i, \xi), (\tau+\gamma-\varepsilon'-\delta+\sum_{i=1}^{r}\lambda_i k_i, \xi) \end{array} \right| -x^{\xi} \right]. \tag{20}$$

Proof:

Let Ω be the left side of (20). Using (16) and (17), changing the order of the integration, and summation yields:

$$\Omega = \sum_{k_1,...,k_r=0}^{h_1k_1+...+h_rk_r \le L} (-L)_{h_1k_1+...+h_rk_r} \; A(L;k_1,...,k_r) \frac{a_1^{k_1}}{k_1!}...\frac{a_r^{k_r}}{k_r!}$$

$$\times (-1)^n n! \sum_{k=0}^{\infty} \binom{p-n-1}{k} \binom{q+n}{n-k} \left(I_{0,x}^{\varepsilon,\varepsilon',\delta,\delta',\gamma} t^{\tau+\xi k+\sum_{i=1}^{r}\lambda_i k_i - 1} \right)(x), \tag{21}$$

for any $k = 0,1,2,...,n$. Since $\Re(\tau+k) > \Re(\tau) > \max\{0, \Re(\varepsilon+\varepsilon'+\delta-\gamma), \; \Re(\varepsilon'-\delta')\}$, and by applying (9), we obtain

$$\Omega = \sum_{k_1,...,k_r=0}^{h_1k_1+...+h_rk_r \le L} (-L)_{h_1k_1+...+h_rk_r} \; A(L;k_1,...,k_r) \frac{a_1^{k_1}}{k_1!}...\frac{a_r^{k_r}}{k_r!}$$

$$\times (-1)^n \frac{\Gamma(q+n+1)}{\Gamma(-n)\Gamma(1+n-p)} x^{\tau-\varepsilon-\varepsilon'+\gamma+\sum_{i=1}^{r}\lambda_i k_i - 1}$$

$$\times \sum_{k=0}^{\infty} \frac{\Gamma(1+n-p+k)\Gamma(-n+k)\Gamma(\tau+\sum_{i=1}^{r}\lambda_i k_i+\xi k)}{\Gamma(q+k+1)\Gamma(\tau+\delta'+\sum_{i=1}^{r}\lambda_i k_i+\xi k)}$$

$$\times \frac{\Gamma(\tau-\gamma-\varepsilon-\varepsilon'-\delta+\sum_{i=1}^{r}\lambda_i k_i+\xi k)\Gamma(\tau+\delta'-\varepsilon'+\sum_{i=1}^{r}\lambda_i k_i+\xi k)}{\Gamma(\tau+\gamma-\varepsilon-\varepsilon'+\sum_{i=1}^{r}\lambda_i k_i+\xi k)\Gamma(\tau+\gamma-\varepsilon'-\delta+\sum_{i=1}^{r}\lambda_i k_i+\xi k)} \frac{(-x^{\xi})^k}{k!}. \tag{22}$$

Interpreting the right-hand side of the above equation (22), in view of the definition (14), we arrive at the result (20). ■

Theorem 3.2.

Let $\Re(\gamma) > 0$ and $\Re(1-\gamma-\tau) < 1 + \min\{\Re(-\delta), \Re(\varepsilon+\varepsilon'-\gamma), \Re(\varepsilon+\delta'-\gamma)\}$. Then, the generalized fractional integration $I_{x,\infty}^{\varepsilon,\varepsilon',\delta,\delta',\gamma}$ of the product finite classes of the classical orthogonal polynomials $M_n^{(p,q)}(.)$ and multivariable polynomials $S_L^{k_1,...,k_r}(.)$ is given by

$$\left(I_{x,\infty}^{\varepsilon,\varepsilon',\delta,\delta',\gamma} t^{-\gamma-\tau} S_L^{k_1,...,k_r}\left(a_1 t^{\lambda_1},...,a_r t^{\lambda_r}\right) M_n^{(p,q)}\left(1/t^{\xi}\right) \right)(x)$$

$$= (-1)^n \sum_{k_1,...,k_r=0}^{h_1k_1+...+h_rk_r \le L} (-L)_{h_1k_1+...+h_rk_r} \; A(L;k_1,...,k_r) \frac{a_1^{k_1}}{k_1!}...\frac{a_r^{k_r}}{k_r!}$$

$$\times x^{-\tau-\varepsilon-\varepsilon'+\sum_{i=1}^{r}\lambda_i k_i}$$

$$\times \frac{\Gamma(q+n+1)}{\Gamma(-n)\Gamma(1+n-p)} {}_5\psi_4 \left[\begin{matrix} (-n,1),(1+n-p,1),(\varepsilon+\varepsilon'+\tau-\sum_{i=1}^{r}\lambda_i k_i,\xi), \\ (q+1,1),(\gamma+\tau-\sum_{i=1}^{r}\lambda_i k_i,\xi), \end{matrix} \right.$$

$$\left. \begin{matrix} (\varepsilon+\delta'+\tau-\sum_{i=1}^{r}\lambda_i k_i,\xi),(\gamma-\delta+\tau-\sum_{i=1}^{r}\lambda_i k_i,\xi) \\ (\varepsilon+\varepsilon'+\delta'+\tau-\sum_{i=1}^{r}\lambda_i k_i,\xi),(\gamma+\varepsilon-\delta+\tau-\sum_{i=1}^{r}\lambda_i k_i,\xi) \end{matrix} \middle| -\frac{1}{x^{\xi}} \right]. \tag{23}$$

Proof:

On using (16) and (17), the left-hand side of (23) and changing the order of the integration and summation can be written as:

$$\Omega = \sum_{k_1,\ldots,k_r=0}^{h_1k_1+\ldots+h_rk_r\leq L} (-L)_{h_1k_1+\ldots+h_rk_r}\; A(L;k_1,\ldots,k_r)\frac{a_1^{k_1}}{k_1!}\cdots\frac{a_r^{k_r}}{k_r!}$$

$$\times(-1)^n n!\sum_{k=0}^{\infty}\binom{p-n-1}{k}\binom{q+n}{n-k}\left(I_{x,\infty}^{\varepsilon,\varepsilon',\delta,\delta',\gamma}t^{-\gamma-\tau-\xi k+\sum_{i=1}^{r}\lambda_i k_i}\right)(x), \tag{24}$$

which on using the image formula (10), arrive at

$$\Omega = \sum_{k_1,\ldots,k_r=0}^{h_1k_1+\ldots+h_rk_r\leq L} (-L)_{h_1k_1+\ldots+h_rk_r}\; A(L;k_1,\ldots,k_r)\frac{a_1^{k_1}}{k_1!}\cdots\frac{a_r^{k_r}}{k_r!}x^{-\tau-\varepsilon-\varepsilon'}$$

$$\times(-1)^n\frac{\Gamma(q+n+1)}{\Gamma(-n)\Gamma(1+n-p)}x^{\sum_{i=1}^{r}\lambda_i k_i}$$

$$\times\sum_{k=0}^{\infty}\frac{\Gamma(1+n-p+k)\Gamma(-n+k)\Gamma(\varepsilon+\varepsilon'+\tau-\sum_{i=1}^{r}\lambda_i k_i+\xi k)}{\Gamma(q+k+1)\Gamma(\gamma+\tau-\sum_{i=1}^{r}\lambda_i k_i+\xi k)}$$

$$\times\frac{\Gamma(\varepsilon+\delta'+\tau-\sum_{i=1}^{r}\lambda_i k_i+\xi k)\Gamma(\gamma-\delta+\tau-\sum_{i=1}^{r}\lambda_i k_i+\xi k)}{\Gamma(\varepsilon+\varepsilon'+\delta'+\tau-\sum_{i=1}^{r}\lambda_i k_i+\xi k)\Gamma(\gamma+\varepsilon-\delta+\tau-\sum_{i=1}^{r}\lambda_i k_i+\xi k)}\frac{\left(-x^{-\xi}\right)^k}{k!}. \tag{25}$$

Interpreting the right-hand side of (25), in view of the definition (14), we arrive at the result (23).■

If we set $S_L^{k_j}$ reduce to unity. i.e. $S_L^{k_j}(x)\to 1$ in (20) and (23), then the Theorem 3.1 and Theorem 3.2 takes the following form.

Corollary 3.3.

Let $\Re(\gamma)>0$ and $\Re(1-\gamma-\tau)<1+\min\{\Re(-\delta),\Re(\varepsilon+\varepsilon'-\gamma),\ \Re(\varepsilon+\delta'-\gamma)\}$, then the generalized fractional integration $I_{x,\infty}^{\varepsilon,\varepsilon',\delta,\delta',\gamma}$ of the finite classes of the classical orthogonal polynomials $M_n^{(p,q)}(.)$ exits, and the following integral holds true:

$$\left(I_{0,x}^{\varepsilon,\varepsilon',\delta,\delta',\gamma}t^{\tau-1}M_n^{(p,q)}\left(t^{\xi}\right)\right)(x)=(-1)^n x^{\tau-\varepsilon-\varepsilon'+\gamma-1}\frac{\Gamma(q+n+1)}{\Gamma(-n)\Gamma(1+n-p)}$$

$$\times {}_5\psi_4 \left[\begin{array}{c} (-n,1), (1+n-p,1), (\tau,\xi), (\tau+\gamma-\varepsilon-\varepsilon'-\delta,\xi), (\tau+\delta'-\varepsilon',\xi) \\ (q+1,1), (\tau+\delta',\xi), (\tau+\gamma-\varepsilon-\varepsilon',\xi), (\tau+\gamma-\varepsilon'-\delta,\xi) \end{array} \middle| -x^{\xi} \right]. \tag{26}$$

Corollary 3.4.

Let $\Re(\gamma) > 0$ and $\Re(\tau) > \max\{0, \Re(\varepsilon+\varepsilon'+\delta-\gamma), \Re(\varepsilon'-\delta')\}$, then the generalized fractional integration $I_{0,x}^{\varepsilon,\varepsilon',\delta,\delta',\gamma}$ of the finite classes of the classical orthogonal polynomials $M_n^{(p,q)}(.)$ exits, and the following integral holds true:

$$\left(I_{x,\infty}^{\varepsilon,\varepsilon',\delta,\delta',\gamma} t^{-\gamma-\tau} M_n^{(p,q)}\left(1/t^{\xi}\right)\right)(x) = (-1)^n \frac{\Gamma(q+n+1)}{\Gamma(-n)\Gamma(1+n-p)} x^{-\tau-\varepsilon-\varepsilon'}$$

$$\times {}_5\psi_4 \left[\begin{array}{c} (-n,1), (1+n-p,1), (\varepsilon+\varepsilon'+\tau,\xi), (\varepsilon+\delta'+\tau,\xi), (\gamma-\delta+\tau,\xi) \\ (q+1,1), (\gamma+\tau,\xi), (\varepsilon+\varepsilon'+\delta'+\tau,\xi), (\gamma+\varepsilon-\delta+\tau,\xi) \end{array} \middle| -\frac{1}{x^{\xi}} \right]. \tag{27}$$

3.2. Composition with generalized fractional derivative

In this section, we establish image formulas for the product of Srivastava polynomial and finite classes of the classical orthogonal polynomials involving left and right-sided operators of Marichev-Saigo-Meada fractional derivative operators, in terms of the generalized Wright function. These formulas are given by the following theorems.

Theorem 3.5.

The generalized fractional derivative $D_{0,x}^{\varepsilon,\varepsilon',\delta,\delta',\gamma}$ of the product finite classes of the classical orthogonal polynomials $M_n^{(p,q)}(.)$ and multivariable polynomials $S_L^{k_1,\dots,k_r}(.)$ is given by

$$\left(D_{0,x}^{\varepsilon,\varepsilon',\delta,\delta',\gamma} t^{\tau-1} S_L^{k_1,\dots,k_r}\left(a_1 t^{\lambda_1},\dots,a_r t^{\lambda_r}\right) M_n^{(p,q)}\left(t^{\xi}\right)\right)(x)$$

$$= (-1)^n \sum_{k_1,\dots,k_r=0}^{h_1k_1+\dots+h_rk_r \le L} (-L)_{h_1k_1+\dots+h_rk_r} \; A(L;k_1,\dots,k_r) \frac{a_1^{k_1}}{k_1!} \cdots \frac{a_r^{k_r}}{k_r!}$$

$$\times x^{\tau+\varepsilon+\varepsilon'-\gamma+\sum_{i=1}^r \lambda_i k_i - 1} \frac{\Gamma(q+n+1)}{\Gamma(-n)\Gamma(1+n-p)}$$

$$\times {}_5\psi_4 \left[\begin{array}{c} (-n,1), (1+n-p,1), (\tau+\sum_{i=1}^r \lambda_i k_i,\xi), \\ (q+1,1), (\tau-\delta+\sum_{i=1}^r \lambda_i k_i,\xi), \end{array} \right.$$

$$\left. \begin{array}{c} (\tau-\gamma+\varepsilon+\varepsilon'+\delta'+\sum_{i=1}^r \lambda_i k_i,\xi), (\tau-\delta+\varepsilon+\sum_{i=1}^r \lambda_i k_i,\xi) \\ (\tau-\gamma+\varepsilon+\varepsilon'+\sum_{i=1}^r \lambda_i k_i,\xi), (\tau-\gamma+\varepsilon+\delta'+\sum_{i=1}^r \lambda_i k_i,\xi) \end{array} \middle| -x^{\xi} \right], \tag{28}$$

where $\Re(\gamma) > 0$, $\Re(\tau) > \max\{0, \Re(-\varepsilon+\delta), \Re(-\varepsilon-\varepsilon'-\delta'+\gamma)\}$.

Proof:

On using (16) and (17), writing the function in the series form, the left-hand side of (28), leads to

$$\Omega = \sum_{k_1,\ldots,k_r=0}^{h_1k_1+\ldots+h_rk_r\le L} (-L)_{h_1k_1+\ldots+h_rk_r}\ A(L;k_1,\ldots,k_r)\,\frac{a_1^{k_1}}{k_1!}\cdots\frac{a_r^{k_r}}{k_r!}$$

$$\times(-1)^n\,n!\sum_{k=0}^{\infty}\binom{p-n-1}{k}\binom{q+n}{n-k}\left(I_{0,x}^{-\varepsilon',-\varepsilon,-\delta',-\delta,-\gamma}t^{\tau+\xi k+\sum_{i=1}^{r}\lambda_ik_i-1}\right)(x)\,. \tag{29}$$

Now upon using the image formulas (11), which is valid under the condition stated with Theorem 3.5, we obtain

$$\Omega = (-1)^n \sum_{k_1,\ldots,k_r=0}^{h_1k_1+\ldots+h_rk_r\le L} (-L)_{h_1k_1+\ldots+h_rk_r}\ A(L;k_1,\ldots,k_r)\,\frac{a_1^{k_1}}{k_1!}\cdots\frac{a_r^{k_r}}{k_r!}$$

$$\times x^{\tau-\gamma+\varepsilon+\varepsilon'+\sum_{i=1}^{r}\lambda_ik_i-1}\,\frac{\Gamma(q+n+1)}{\Gamma(-n)\,\Gamma(1+n-p)}$$

$$\times\sum_{k=0}^{\infty}\frac{\Gamma(1+n-p+k)\,\Gamma(-n+k)\,\Gamma\left(\tau+\sum_{i=1}^{r}\lambda_ik_i+\xi k\right)}{\Gamma(q+k+1)\,\Gamma\left(\tau-\delta+\sum_{i=1}^{r}\lambda_ik_i+\xi k\right)}$$

$$\times\frac{\Gamma\left(\tau-\gamma+\varepsilon+\varepsilon'+\delta'+\sum_{i=1}^{r}\lambda_ik_i+\xi k\right)\Gamma\left(\tau-\delta+\varepsilon+\sum_{i=1}^{r}\lambda_ik_i+\xi k\right)}{\Gamma\left(\tau-\gamma+\varepsilon+\varepsilon'+\sum_{i=1}^{r}\lambda_ik_i+\xi k\right)\Gamma\left(\tau-\gamma+\varepsilon+\delta'+\sum_{i=1}^{r}\lambda_ik_i+\xi k\right)}\frac{\left(-x^{\xi}\right)^k}{k!}. \tag{30}$$

Interpreting the right-hand side of the above equation (30), in view of the definition (14), we arrive at the result (28). ∎

Theorem 3.6.

The generalized fractional derivative $D_{x,\infty}^{\varepsilon,\varepsilon',\delta,\delta',\gamma}$ of the product finite classes of the classical orthogonal polynomials $M_n^{(p,q)}(.)$ and multivariable polynomials $S_L^{k_1,\ldots,k_r}(.)$ is given by

$$\left(D_{x,\infty}^{\varepsilon,\varepsilon',\delta,\delta',\gamma}t^{\gamma-\tau}S_L^{k_1,\ldots,k_r}\left(a_1t^{\lambda_1},\ldots,a_rt^{\lambda_r}\right)M_n^{(p,q)}\left(1/t^{\xi}\right)\right)(x)$$

$$= (-1)^n \sum_{k_1,\ldots,k_r=0}^{h_1k_1+\ldots+h_rk_r\le L} (-L)_{h_1k_1+\ldots+h_rk_r}\ A(L;k_1,\ldots,k_r)\,\frac{a_1^{k_1}}{k_1!}\cdots\frac{a_r^{k_r}}{k_r!}$$

$$\times x^{-\tau+\varepsilon+\varepsilon'+\sum_{i=1}^{r}\lambda_ik_i}\,\frac{\Gamma(q+n+1)}{\Gamma(-n)\,\Gamma(1+n-p)}$$

$$\times{}_5\psi_4\left[\begin{matrix}(-n,1),(1+n-p,1),\left(\tau-\varepsilon-\varepsilon'-\sum_{i=1}^{r}\lambda_ik_i,\xi\right),\\ (q+1,1),\left(\tau-\gamma-\sum_{i=1}^{r}\lambda_ik_i,\xi\right),\end{matrix}\right.$$

$$\left.\begin{matrix}\left(\tau-\varepsilon'-\delta-\sum_{i=1}^{r}\lambda_ik_i,\xi\right),\left(\tau-\gamma+\delta'-\sum_{i=1}^{r}\lambda_ik_i,\xi\right)\\ \left(\tau-\varepsilon-\varepsilon'-\delta-\sum_{i=1}^{r}\lambda_ik_i,\xi\right),\left(\tau-\gamma-\varepsilon'+\delta'-\sum_{i=1}^{r}\lambda_ik_i,\xi\right)\end{matrix}\right|-\frac{1}{x^{\xi}}\right], \tag{31}$$

where $\Re(\gamma)>0$, $\Re(1-\gamma-\tau)<1+\min\{\Re(-\delta),\Re(\varepsilon+\varepsilon'-\gamma),\Re(\varepsilon+\delta'-\gamma)\}$.

Proof:

On using (16) and (17), the left-hand side of (31) can be written as:

$$\Omega = \sum_{k_1,\ldots,k_r=0}^{h_1k_1+\ldots+h_rk_r\leq L} (-L)_{h_1k_1+\ldots+h_rk_r} \; A(L;k_1,\ldots,k_r) \frac{a_1^{k_1}}{k_1!}\cdots\frac{a_r^{k_r}}{k_r!}$$

$$\times (-1)^n n! \sum_{k=0}^{\infty} \binom{p-n-1}{k} \binom{q+n}{n-k} \left(I_{x,\infty}^{-\varepsilon',-\varepsilon,-\delta',-\delta,-\gamma} t^{\gamma-\tau-\xi k+\sum_{i=1}^{r}\lambda_i k_i} \right)(x), \quad (32)$$

which on using the image formula (12), we arrive at

$$\Omega = \sum_{k_1,\ldots,k_r=0}^{h_1k_1+\ldots+h_rk_r\leq L} (-L)_{h_1k_1+\ldots+h_rk_r} \; A(L;k_1,\ldots,k_r) \frac{a_1^{k_1}}{k_1!}\cdots\frac{a_r^{k_r}}{k_r!} x^{\tau+\varepsilon+\varepsilon'}$$

$$\times (-1)^n \frac{\Gamma(q+n+1)}{\Gamma(-n)\Gamma(1+n-p)} x^{\sum_{i=1}^{r}\lambda_i k_i}$$

$$\times \sum_{k=0}^{\infty} \frac{\Gamma(1+n-p+k)\Gamma(-n+k)\Gamma(\tau-\varepsilon-\varepsilon'-\sum_{i=1}^{r}\lambda_i k_i+\xi k)}{\Gamma(q+k+1)\Gamma(\tau-\gamma-\sum_{i=1}^{r}\lambda_i k_i+\xi k)}$$

$$\times \frac{\Gamma(\tau-\varepsilon'-\delta-\sum_{i=1}^{r}\lambda_i k_i+\xi k)\Gamma(\tau-\gamma+\delta'-\sum_{i=1}^{r}\lambda_i k_i+\xi k)}{\Gamma(\tau-\varepsilon-\varepsilon'-\delta-\sum_{i=1}^{r}\lambda_i k_i+\xi k)\Gamma(\tau-\gamma-\varepsilon'+\delta'-\sum_{i=1}^{r}\lambda_i k_i+\xi k)} \frac{(-x^{-\xi})^k}{k!}. \quad (33)$$

Interpreting the right-hand side of (33), in view of the definition (14), we arrive at the result (31).■

If we set $S_L^{k_j}$ reduce to unity, i.e., $S_L^{k_j}(x) \to 1$ in (28) and (31), then the Theorem 3.5 and Theorem 3.6 takes the following form:

Corollary 3.7.

The generalized fractional derivative $D_{0,x}^{\varepsilon,\varepsilon',\delta,\delta',\gamma}$ of the product finite classes of the classical orthogonal polynomials $M_n^{(p,q)}(.)$ is given by

$$\left(D_{0,x}^{\varepsilon,\varepsilon',\delta,\delta',\gamma} t^{\tau-1} M_n^{(p,q)}(t^{\xi}) \right)(x) = (-1)^n \frac{\Gamma(q+n+1)}{\Gamma(-n)\Gamma(1+n-p)} x^{\tau+\varepsilon+\varepsilon'-\gamma-1}$$

$$\times {}_5\psi_4 \left[\begin{matrix} (-n,1),(1+n-p,1),(\tau,\xi),(\tau-\gamma+\varepsilon+\varepsilon'+\delta',\xi),(\tau-\delta+\varepsilon,\xi) \\ (q+1,1),(\tau-\delta,\xi),(\tau-\gamma+\varepsilon+\varepsilon',\xi),(\tau-\gamma+\varepsilon+\delta',\xi) \end{matrix} \middle| -x^{\xi} \right], \quad (34)$$

where $\Re(\gamma) > 0$, $\Re(\tau) > \max\{0, \Re(-\varepsilon+\delta), \Re(-\varepsilon-\varepsilon'-\delta'+\gamma)\}$.

Corollary 3.8.

The generalized fractional derivative $D_{x,\infty}^{\varepsilon,\varepsilon',\delta,\delta',\gamma}$ of the product finite classes of the classical orthogonal polynomials $M_n^{(p,q)}(.)$ is given by

$$\left(D_{x,\infty}^{\varepsilon,\varepsilon',\delta,\delta',\gamma} t^{\gamma-\tau} M_n^{(p,q)}(1/t^{\xi}) \right)(x) = (-1)^n \frac{\Gamma(q+n+1)}{\Gamma(-n)\Gamma(1+n-p)} x^{-\tau+\varepsilon+\varepsilon'}$$

$$\times {}_5\psi_4 \left[\begin{array}{c} (-n,1),(1+n-p,1),(\tau-\varepsilon-\varepsilon',\xi),(\tau-\varepsilon'-\delta,\xi),(\tau-\gamma+\delta',\xi) \\ (q+1,1),(\tau-\gamma,\xi),(\tau-\varepsilon-\varepsilon'-\delta,\xi),(\tau-\gamma-\varepsilon'+\delta',\xi) \end{array} \middle| -\frac{1}{x^{\xi}} \right], \quad (35)$$

where $\Re(\gamma)>0$, $\Re(1-\gamma-\tau)<1+\min\{\Re(-\delta),\Re(\varepsilon+\varepsilon'-\gamma),\Re(\varepsilon+\delta'-\gamma)\}$.

4. Special Cases

In this section, we consider some special cases of the main results derived in the preceding section. If we set $\varepsilon=\varepsilon+\delta,\varepsilon'=0,\delta=-\eta$ and $\gamma=\varepsilon$ in the above defined operators (1) and (2) reduces to Saigo fractional integral operators (Saigo (1978)) for $\varepsilon,\delta,\eta\in\mathbb{C}$, and $\Re(\varepsilon)>0$ are defined in the following manner:

$$\left(I_{0,x}^{\varepsilon,\delta,\eta}f\right)(x)=\frac{x^{-\varepsilon-\delta}}{\Gamma(\varepsilon)}\int_0^x (x-t)^{\varepsilon-1}\,{}_2F_1\left(\varepsilon+\delta,-\eta;\varepsilon;1-\frac{t}{x}\right)f(t)\,dt, \quad (36)$$

and

$$\left(I_{x,\infty}^{\varepsilon,\delta,\eta}f\right)(x)=\frac{1}{\Gamma(\varepsilon)}\int_x^\infty (t-x)^{\varepsilon-1}\,t_2^{-\varepsilon-\delta}F_1\left(\varepsilon+\delta,-\eta;\varepsilon;1-\frac{x}{t}\right)f(t)\,dt. \quad (37)$$

Corollary 4.1.

Let $\varepsilon,,\delta,\eta,\tau\in\mathbb{C}$, $\Re(\varepsilon)>0$, $\Re(\tau)>\max[0,\Re(\delta-\eta)]$. Then for $x>0$,

$$\left(I_{0,x}^{\varepsilon,\delta,\eta}t^{\tau-1}S_L^{k_1,\ldots,k_r}\left(a_1t^{\lambda_1},\ldots,a_rt^{\lambda_r}\right)M_n^{(p,q)}\left(t^{\xi}\right)\right)(x)$$

$$=(-1)^n\sum_{k_1,\ldots,k_r=0}^{h_1k_1+\ldots+h_rk_r\le L}(-L)_{h_1k_1+\ldots+h_rk_r}\;A(L;k_1,\ldots,k_r)\frac{a_1^{k_1}}{k_1!}\cdots\frac{a_r^{k_r}}{k_r!}$$

$$\times x^{\tau-\delta+\sum_{i=1}^r\lambda_ik_i-1}\frac{\Gamma(q+n+1)}{\Gamma(-n)\Gamma(1+n-p)}{}_4\psi_3\left[\begin{array}{c}(-n,1),(1+n-p,1),\\(q+1,1),\end{array}\right.$$

$$\left.\begin{array}{c}(\tau-\delta+\eta+\sum_{i=1}^r\lambda_ik_i,\xi),(\tau+\sum_{i=1}^r\lambda_ik_i,\xi)\\(\tau-\delta+\sum_{i=1}^r\lambda_ik_i,\xi),(\tau+\varepsilon+\eta+\sum_{i=1}^r\lambda_ik_i,\xi)\end{array}\middle|-x^{\xi}\right]. \quad (38)$$

Corollary 4.2.

Let $\varepsilon,\delta,\eta,\tau\in\mathbb{C}$, $\Re(\varepsilon)>0$, $\Re(\tau)<1+\min[\Re(\delta),\Re(\eta)]$. Then for $x>0$, the following formula holds true:

$$\left(I_{x,\infty}^{\varepsilon,\delta,\eta}t^{-\gamma-\tau}S_L^{k_1,\ldots,k_r}\left(a_1t^{\lambda_1},\ldots,a_rt^{\lambda_r}\right)M_n^{(p,q)}\left(1/t^{\xi}\right)\right)(x)$$

$$=(-1)^n\sum_{k_1,\ldots,k_r=0}^{h_1k_1+\ldots+h_rk_r\le L}(-L)_{h_1k_1+\ldots+h_rk_r}\;A(L;k_1,\ldots,k_r)\frac{a_1^{k_1}}{k_1!}\cdots\frac{a_r^{k_r}}{k_r!}$$

$$\times x^{\tau-\delta+\sum_{i=1}^{r}\lambda_i k_i}\frac{\Gamma(q+n+1)}{\Gamma(-n)\Gamma(1+n-p)}{}_4\psi_3\left[\begin{array}{c}(-n,1),(1+n-p,1),\\(q+1,1),\end{array}\right.$$

$$\left.\begin{array}{c}(\varepsilon+\delta+\tau-\sum_{i=1}^{r}\lambda_i k_i,\xi),(\varepsilon+\eta+\tau-\sum_{i=1}^{r}\lambda_i k_i,\xi)\\(\varepsilon+\tau-\sum_{i=1}^{r}\lambda_i k_i,\xi),(2\varepsilon+\delta+\eta+\tau-\sum_{i=1}^{r}\lambda_i k_i,\xi)\end{array}\right|-x^{\xi}\Bigg]. \tag{39}$$

For $\delta=-\varepsilon$ with $\varepsilon\in\mathbb{C}$ and $\Re(\varepsilon)>0$, (36) and (37) respectively reduces to the Riemann-Liouville fractional integral operators defined for $x>0$ as follows:

$$\left(I_{0,x}^{\varepsilon,-\varepsilon,\eta}f\right)(x)=\left(I_{0,x}^{\varepsilon}f\right)(x)=\frac{1}{\Gamma(\varepsilon)}\int_0^x(x-t)^{\varepsilon-1}f(t)\,dt, \tag{40}$$

$$\left(I_{x.\infty}^{\varepsilon,-\varepsilon,\eta}f\right)(x)=\left(I_{x,\infty}^{\varepsilon}f\right)(x)=\frac{1}{\Gamma(\varepsilon)}\int_x^\infty(t-x)^{\varepsilon-1}f(t)\,dt. \tag{41}$$

Corollary 4.3.

Let $\varepsilon,\tau\in\mathbb{C}$, $\Re(\varepsilon)>0$ and $\Re(\tau)>0$. Then for $x>0$, the following formula holds true:

$$\left(I_{0,x}^{\varepsilon}t^{\tau-1}S_L^{k_1,\ldots,k_r}\left(a_1t^{\lambda_1},\ldots,a_rt^{\lambda_r}\right)M_n^{(p,q)}\left(t^{\xi}\right)\right)(x)$$

$$=(-1)^n\sum_{k_1,\ldots,k_r=0}^{h_1k_1+\ldots+h_rk_r\le L}(-L)_{h_1k_1+\ldots+h_rk_r}\;A(L;k_1,\ldots,k_r)\frac{a_1^{k_1}}{k_1!}\cdots\frac{a_r^{k_r}}{k_r!}$$

$$\times x^{\tau+\varepsilon+\sum_{i=1}^{r}\lambda_i k_i-1}\frac{\Gamma(q+n+1)}{\Gamma(-n)\Gamma(1+n-p)}$$

$$\times{}_3\psi_2\left[\begin{array}{c}(-n,1),(1+n-p,1),(\tau+\sum_{i=1}^{r}\lambda_i k_i,\xi)\\(q+1,1),(\tau+\varepsilon+\sum_{i=1}^{r}\lambda_i k_i,\xi)\end{array}\right|-x^{\xi}\Bigg]. \tag{42}$$

Corollary 4.4.

Let $\varepsilon,\tau\in\mathbb{C}$, $\Re(\varepsilon)>0$, $\Re(\tau)<1+\Re(-\varepsilon)$ and let $\tau+\varepsilon\neq1,2,\cdots$. Then the following formula holds true:

$$\left(I_{x,\infty}^{\varepsilon}t^{-\gamma-\tau}S_L^{k_1,\ldots,k_r}\left(a_1t^{\lambda_1},\ldots,a_rt^{\lambda_r}\right)M_n^{(p,q)}\left(1/t^{\xi}\right)\right)(x)$$

$$=(-1)^n\sum_{k_1,\ldots,k_r=0}^{h_1k_1+\ldots+h_rk_r\le L}(-L)_{h_1k_1+\ldots+h_rk_r}\;A(L;k_1,\ldots,k_r)\frac{a_1^{k_1}}{k_1!}\cdots\frac{a_r^{k_r}}{k_r!}$$

$$\times x^{\tau+\varepsilon+\sum_{i=1}^{r}\lambda_i k_i}\frac{\Gamma(q+n+1)}{\Gamma(-n)\Gamma(1+n-p)}$$

$$\times{}_3\psi_2\left[\begin{array}{c}(-n,1),(1+n-p,1),(\tau-\sum_{i=1}^{r}\lambda_i k_i,\xi)\\(q+1,1),(\varepsilon+\tau-\sum_{i=1}^{r}\lambda_i k_i,\xi)\end{array}\right|-x^{\xi}\Bigg]. \tag{43}$$

Also, if $\delta = 0$ in (36) and (37) takes the respective forms for $\varepsilon, \eta \in \mathbb{C}$, and $\Re(\varepsilon) > 0$ are the so called Erdélyi-Kober fractional integral operators defined for $x > 0$ as follows:

$$\left(I_{0,x}^{\varepsilon,0,\eta} f\right)(x) = \left(I_{\eta,\varepsilon}^{+} f\right)(x) = \frac{x^{-\varepsilon-\eta}}{\Gamma(\varepsilon)} \int_0^x (x-t)^{\varepsilon-1} t^{\eta} f(t)\, dt, \tag{44}$$

$$\left(I_{x,\infty}^{\varepsilon,0,\eta} f\right)(x) = \left(K_{\eta,\varepsilon}^{-} f\right)(x) = \frac{x^{\eta}}{\Gamma(\varepsilon)} \int_x^\infty (t-x)^{\varepsilon-1} t^{-\varepsilon-\eta} f(t)\, dt. \tag{45}$$

Corollary 4.5.

Let $\varepsilon, \tau \in \mathbb{C}$, $\Re(\varepsilon) > 0$ and $\Re(\tau) > 0$ for $x > 0$, then the following formula holds true:

$$\left(I_{\eta,\varepsilon}^{+} t^{\tau-1} S_L^{k_1,\ldots,k_r}\left(a_1 t^{\lambda_1}, \ldots, a_r t^{\lambda_r}\right) M_n^{(p,q)}\left(t^{\xi}\right)\right)(x)$$

$$= (-1)^n \sum_{k_1,\ldots,k_r=0}^{h_1k_1+\ldots+h_rk_r \le L} (-L)_{h_1k_1+\ldots+h_rk_r}\ A(L;k_1,\ldots,k_r) \frac{a_1^{k_1}}{k_1!} \cdots \frac{a_r^{k_r}}{k_r!}$$

$$\times x^{\tau+\sum_{i=1}^r \lambda_i k_i - 1} \frac{\Gamma(q+n+1)}{\Gamma(-n)\Gamma(1+n-p)}$$

$$\times {}_3\psi_2 \left[\begin{matrix} (-n,1),(1+n-p,1),(\tau+\eta+\sum_{i=1}^r \lambda_i k_i, \xi) \\ (q+1,1),(\tau+\varepsilon+\eta+\sum_{i=1}^r \lambda_i k_i, \xi) \end{matrix} \middle| -x^{\xi}\right]. \tag{46}$$

Corollary 4.6.

Let $\varepsilon, \tau \in \mathbb{C}$, $\Re(\varepsilon) > 0$, $\Re(\tau) < 1 + \min[0, \Re(\eta)]$ and let $\tau - \eta \neq 1, 2, \cdots$. Then for $x > 0$, the following formula holds true:

$$\left(K_{\eta,\varepsilon}^{-} t^{-\gamma-\tau} S_L^{k_1,\ldots,k_r}\left(a_1 t^{\lambda_1}, \ldots, a_r t^{\lambda_r}\right) M_n^{(p,q)}\left(1/t^{\xi}\right)\right)(x)$$

$$= (-1)^n \sum_{k_1,\ldots,k_r=0}^{h_1k_1+\ldots+h_rk_r \le L} (-L)_{h_1k_1+\ldots+h_rk_r}\ A(L;k_1,\ldots,k_r) \frac{a_1^{k_1}}{k_1!} \cdots \frac{a_r^{k_r}}{k_r!}$$

$$\times x^{\tau+\sum_{i=1}^r \lambda_i k_i} \frac{\Gamma(q+n+1)}{\Gamma(-n)\Gamma(1+n-p)}$$

$$\times {}_3\psi_2 \left[\begin{matrix} (-n,1),(1+n-p,1),(\varepsilon+\eta+\tau-\sum_{i=1}^r \lambda_i k_i, \xi) \\ (q+1,1),(2\varepsilon+\eta+\tau-\sum_{i=1}^r \lambda_i k_i, \xi) \end{matrix} \middle| -x^{\xi}\right]. \tag{47}$$

If we put $S_L^{k_j}$ reduce to unity, i.e., $S_L^{k_j}(x) \to 1$ in (28) and (31), then the Corollaries (3.1) to (3.6) takes the following form.

Corollary 4.7.

Let $\varepsilon, \delta, \eta, \tau \in \mathbb{C}$, $\Re(\varepsilon) > 0$, $\Re(\tau) > \max[0, \Re(\delta - \eta)]$. Then for $x > 0$

$$\left(I_{0,x}^{\varepsilon,\delta,\eta} t^{\tau-1} M_n^{(p,q)}\left(t^{\xi}\right)\right)(x) = (-1)^n \frac{\Gamma(q+n+1)}{\Gamma(-n)\Gamma(1+n-p)} x^{\tau-\delta-1}$$

$$\times {}_4\psi_3 \left[\begin{matrix} (-n,1), (1+n-p,1), (\tau-\delta+\eta,\xi), (\tau,\xi) \\ (q+1,1), (\tau-\delta,\xi), (\tau+\varepsilon+\eta,\xi) \end{matrix} \middle| -x^{\xi}\right]. \tag{48}$$

Corollary 4.8.

Let $\varepsilon, \delta, \eta, \tau \in \mathbb{C}$, $\Re(\varepsilon) > 0$, $\Re(\tau) < 1 + \min[\Re(\delta), \Re(\eta)]$. Then, for $x > 0$ the following formula holds true:

$$\left(I_{x,\infty}^{\varepsilon,\delta,\eta} t^{-\gamma-\tau} M_n^{(p,q)}\left(1/t^{\xi}\right)\right)(x) = (-1)^n \frac{\Gamma(q+n+1)}{\Gamma(-n)\Gamma(1+n-p)} x^{\tau-\delta}$$

$$\times {}_4\psi_3 \left[\begin{matrix} (-n,1), (1+n-p,1), (\varepsilon+\delta+\tau,\xi), (\varepsilon+\eta+\tau,\xi) \\ (q+1,1), (\varepsilon+\tau,\xi), (2\varepsilon+\delta+\eta+\tau,\xi) \end{matrix} \middle| -x^{\xi}\right]. \tag{49}$$

Corollary 4.9.

Let $\varepsilon, \tau \in \mathbb{C}$, $\Re(\varepsilon) > 0$ and $\Re(\tau) > 0$. Then, for $x > 0$ the following formula holds true:

$$\left(I_{0,x}^{\varepsilon} t^{\tau-1} M_n^{(p,q)}\left(t^{\xi}\right)\right)(x) = (-1)^n \frac{\Gamma(q+n+1)}{\Gamma(-n)\Gamma(1+n-p)} x^{\tau+\varepsilon-1}$$

$$\times {}_3\psi_2 \left[\begin{matrix} (-n,1), (1+n-p,1), (\tau,\xi) \\ (q+1,1), (\tau+\varepsilon,\xi) \end{matrix} \middle| -x^{\xi}\right]. \tag{50}$$

Corollary 4.10.

Let $\varepsilon, \tau \in \mathbb{C}$, $\Re(\varepsilon) > 0$, $\Re(\tau) < 1 + \Re(-\varepsilon)$ and let $\tau + \varepsilon \neq 1, 2, \cdots$. Then the following formula holds true:

$$\left(I_{x,\infty}^{\varepsilon} t^{-\gamma-\tau} M_n^{(p,q)}\left(1/t^{\xi}\right)\right)(x) = (-1)^n \frac{\Gamma(q+n+1)}{\Gamma(-n)\Gamma(1+n-p)} x^{\tau+\varepsilon}$$

$$\times {}_3\psi_2 \left[\begin{matrix} (-n,1), (1+n-p,1), (\tau,\xi) \\ (q+1,1), (\varepsilon+\tau,\xi) \end{matrix} \middle| -x^{\xi}\right]. \tag{51}$$

Corollary 4.11.

Let $\varepsilon, \tau \in \mathbb{C}$, $\Re(\varepsilon) > 0$ and $\Re(\tau) > 0$ for $x > 0$, then the following formula holds true:

$$\left(I_{\eta,\varepsilon}^{+} t^{\tau-1} M_n^{(p,q)}\left(t^{\xi}\right)\right)(x) = (-1)^n \frac{\Gamma(q+n+1)}{\Gamma(-n)\Gamma(1+n-p)} x^{\tau-1}$$

$$\times {}_3\psi_2 \left[\begin{matrix} (-n,1), (1+n-p,1), (\tau+\eta,\xi) \\ (q+1,1), (\tau+\varepsilon+\eta,\xi) \end{matrix} \middle| -x^{\xi}\right]. \tag{52}$$

Corollary 4.12.

Let $\varepsilon, \tau \in \mathbb{C}$, $\Re(\varepsilon) > 0$, $\Re(\tau) < 1 + \min[0, \Re(\eta)]$ and let $\tau - \eta \neq 1, 2, \cdots$. Then, for $x > 0$ the following formula holds true:

$$\left(K_{\eta,\varepsilon}^{-} t^{-\gamma-\tau} M_n^{(p,q)}\left(1/t^{\xi}\right)\right)(x) = (-1)^n \frac{\Gamma(q+n+1)}{\Gamma(-n)\,\Gamma(1+n-p)} x^{\tau}$$

$$\times {}_3\psi_2 \left[\begin{matrix} (-n,1), (1+n-p,1), (\varepsilon+\eta+\tau,\xi) \\ (q+1,1), (2\varepsilon+\eta+\tau,\xi) \end{matrix} \middle| -x^{\xi} \right]. \tag{53}$$

5. Conclusion

In the remarkably enormous literature on special functions, diverse polynomial systems in more than one variable have been studied from the number of point of view (see Erdélyi et al. (1953), Rainville (1971), and Srivastava and Manocha (1984)). Thus, the generalized fractional integral and derivative operators derived in systematically in this paper are capable of being applied to many of these polynomials in one, two, and more variables. Indeed, by assigning appropriate special values to the coefficient occurring in the definition (16), the polynomials can be reduced not only to the different classical orthogonal polynomials such as the Jacobi polynomial, Hermite polynomials and Leguerre polynomials but also to the Bessel polynomials, the generalized heat polynomials, the Konhauser biorthogonal polynomials, the generalized hypergeometric polynomials of Bateman, Brafmen, Fesenmyer, Gould-Hopper, Rice, Sylvester, and so on.

We can conclude that all above derived results in our work generalize numerous well-known results (see, e.g., Kilbas and Sebastian (2008) and Malik and Mondal (2017)) and are capable of yielding a number of applications in the theory of special functions.

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