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Solouma Solouma
Beni-Suef University

Ibrahim Al-Dayel
Al Imam Mohammad Ibn Saud Islamic University

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On geometry of spherical image in Minkowski space-time with timelike type-2 parallel transport frame

[1,2,*]Emad Solouma and [2]Ibrahim Al-Dayel

[1]Department of Mathematics and Information Science
Faculty of Science
Beni-Suef University
Beni-Suef, Egypt
emadms74@gmail.com

[2]Department of Mathematics and Statistics
College of Science
Al Imam Mohammad Ibn Saud Islamic University
Riyadh, KSA
iaaldayel@imamu.edu.sa

*Corresponding author

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Abstract

In this paper, we investigate a different type of the parallel transport frame in 3-dimensional Minkowski space $\mathfrak{R}_1^3$ by using the binormal vector field of a timelike regular curve as common vector field to introduce, and we recall this frame as " timelike type-2 parallel transport frame". Also, we present new spherical images and call them as timelike type-2 parallel transport spherical images by translating the induced frame vectors to the center of unit Lorentzian sphere in 3-dimensional Minkowski space $\mathfrak{R}_1^3$. Additionally, we obtain the Frenet apparatus of these new spherical images in terms of base curves timelike type-2 parallel transport invariants. Finally, interesting relations are expressed and illustrate an example of the results.

Keywords: Minkowski space-time; Parallel transport frame, slant helix; Spherical images

MSC 2010 No.: 53A04, 53B30, 53C50

1. Introduction

In the Euclidean and Minkowski spaces, the characterization of a regular curve is one of the interesting problems. The curvature κ and the torsion τ of a regular curve have an effective role in the solution of the problem. In fact, the curvature κ and the torsion τ of a regular curve have important rule in the characterization of the shape and size of the curve. One of the mentioned works is spherical image of a regular curve in the Euclidean space. It is a well known concept in the local differential geometry of curves. Such curves are obtained in terms of the Frenet vector fields (Do Carmo (1976); O'Neill (1966); O'Neill (1983)).

In 1802, Lancret proved that a unit speed curve α with $\kappa \neq 0$ is a helix if and only if there is a constant c such that $\tau = c\kappa$. It is known that the helices in the Euclidean 3-space $\mathfrak{R}^3$ is very important curves since they have many general applications in physics and medical science. For example the structure which is studied by Camci et al. (2009); Chouaieb et al. (2006); Cook (1979); Watson and Crick (1953).

Parallel transport frame was introduced by Bishop (1975) by means of parallel vector fields. Recently, many research papers related to this concept have been treated in the Euclidean space (Bükcü and Karacan (2009)), in Minkowski space (Bükcü and Karacan (2008); Bükcü and Karacan (2008); Karacan and Bükcü (2008); Solouma (2017)) and in dual space (Karacan et al. (2008)). Recently, this special frame is extended to study the canal and tubular surfaces, we refer to (Karacan and Bükcü (2007); Karacan and Bükcü (2008)). Yilmaz and Turgut (2010) are introduced a new version of Bishop frame using a common vector field as binormal vector field of a regular curve in Euclidean 3-space E^3 and introduced a new spherical images. In this work, we introduce another type of the parallel transport frame using common vector field as the binormal vector of Frenet frame in 3-dimensional Minkowski space $\mathfrak{R}^3_1$. We call it as "timelike type-2 parallel transport frame" of a timelike regular curves. Thereafter, translating new frame's vector fields to the center of a unit Lorentzian sphere, we obtained new spherical images. We call them as " timelike type-2 parallel transport spherical images" of timelike regular curves. We name them by ζ_1, ζ_2 and binormal parallel transport spherical images. Also, we investigate their Frenet apparatus according to timelike type-2 parallel transport invariants. We establish some relations on general helix and slant helix of spherical images and illustrate an example of main results.

2. Basic concepts

The 3-dimensional Minkowski space $\mathfrak{R}^3_1$ is Euclidean 3-space $\mathfrak{R}^3$ provided with the standard metric given by

$$\langle \, , \rangle = -du_1^2 + du_2^2 + du_3^2,$$

where (u_1, u_2, u_3) is a rectangular coordinate system of $\mathfrak{R}^3_1$. An arbitrary vector $\nu \in \mathfrak{R}^3_1$ can have one of three Lorentzian causal characters; it can be spacelike if $\langle \nu, \nu \rangle > 0$ or $\nu = 0$, timelike if $\langle \nu, \nu \rangle < 0$, and lightlike if $\langle \nu, \nu \rangle = 0$ and $\nu \neq 0$. Similarly, an arbitrary curve $\omega = \omega(s)$ can be locally spacelike, timelike or lightlike if all of its velocity vectors $\omega'(s)$ are spacelike, timelike or lightlike, respectively. Let $\zeta = \zeta(s)$ be a regular curve in $\mathfrak{R}^3_1$. If the tangent vector of this curve forms a constant angle with a fixed constant vector U, then this curve is called a general helix or an inclined curve. The sphere of radius $r > 0$ and with center in the origin in the space $\mathfrak{R}^3_1$is defined by

$$S_1^2 = \{x = (x_1, x_2, x_3) \in \mathfrak{R}^3_1 : \langle x, x \rangle = r^2\}.$$

Let $\psi = \psi(s)$ be a regular curve parametrized by arc-length in $\mathfrak{R}^3_1$ and $\{T, N, B, \kappa, \tau\}$ be its Frenet invariants, where $\{T, N, B\}$, κ and τ are moving Frenet frame, curvature and torsion of $\psi(s)$, respectively. If ψ is a timelike curve, then the Frenet frame has the following properties (Do Carmo (1976); López (2014); O'Neill (1983)).

$$\begin{pmatrix} T'(s) \\ N'(s) \\ B'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ \kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}, \tag{1}$$

where

$$-\langle T, T \rangle = \langle N, N \rangle = \langle B, B \rangle = 1 \text{ and } \langle T, N \rangle = \langle N, B \rangle = \langle T, B \rangle = 0.$$

The parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative (Bishop (1975); Bükcü and Karacan (2008)).

Consider the parallel transport frame $\{T, E_1, E_2\}$ of the timelike curve $\psi(s)$ such that $T(s)$ the unit timelike tangent vector, $E_1(s)$ is unit spacelike principal normal vector, and $E_1(s)$ the unit spacelike binormal vector and

$$\begin{cases} -\langle T, T \rangle = \langle E_1, E_1 \rangle = \langle E_2, E_2 \rangle = 1, \\ \langle T, E_1 \rangle = \langle T, E_2 \rangle = \langle E_1, E_2 \rangle = 0. \end{cases} \tag{2}$$

The parallel transport frame $\{T, E_1, E_2\}$ is expressed ast the following (Bishop (1975); Bükcü and Karacan (2010)).

$$\begin{pmatrix} T'(s) \\ E_1'(s) \\ E_2'(s) \end{pmatrix} = \begin{pmatrix} 0 & k_1(s) & k_2(s) \\ k_1(s) & 0 & 0 \\ k_2(s) & 0 & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ E_1(s) \\ E_2(s) \end{pmatrix}. \tag{3}$$

Here, we shall call the set $\{T, E_1, E_2\}$ as parallel transport trihedra and $k_1(s)$ and $k_2(s)$ as parallel transport curvatures. We can expressed the relation matrix as

$$\begin{pmatrix} T(s) \\ E_1(s) \\ E_2(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta(s) & \sin\vartheta(s) \\ 0 & -\sin\vartheta(s) & \cos\vartheta(s) \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}, \tag{4}$$

where

$$\begin{cases} \vartheta(s) = arc\tan\left(\frac{k_2}{k_1}\right), \quad k_1 \neq 0, \\ \tau(s) = \frac{d\vartheta(s)}{ds}, \\ \kappa(s) = \sqrt{k_1^2(s) + k_{2(s)}^2} \end{cases} . \tag{5}$$

In further researches, spherical images, the tangent and the binormal indicatrix and some characterizations of such curves in Euclidean space and Lorentz-Minkowski spaces are presented (Ali and López (2011); Izumiya and Takeuchi (2004); Kula and Yayli (2005); Kula et al. (2010); Solouma (2017)).

3. Timelike type-2 parallel transport frame of a timelike regular curve

In this section, we define another type of the parallel transport frame using common vector field as the binormal vector of Frenet frame. Let $\mu = \mu(s)$ be a unit speed timelike regular curve in 3-dimensional Minkowski space $\mathfrak{R}_1^3$ with moving Frenet frame given by equation (1). Let us express a relatively parallel adapted frame,

$$\frac{d}{ds}\begin{pmatrix}\zeta_1\\ \zeta_2\\ B\end{pmatrix} = \begin{pmatrix}0 & 0 & \varepsilon_1\\ 0 & 0 & \varepsilon_2\\ \varepsilon_1 & \varepsilon_2 & 0\end{pmatrix}\begin{pmatrix}\zeta_1\\ \zeta_2\\ B\end{pmatrix}, \tag{6}$$

we shall call this frame as "timelike type-2 parallel transport frame". In order to investigate the relation between this frame and with Frenet frame, firstly we write $B' = -\tau N = \varepsilon_1\zeta_1 + \varepsilon_2\zeta_2$.

Then, we have

$$\tau = \sqrt{\varepsilon_1^2 + \varepsilon_2^2}\ . \tag{7}$$

We write the tangent vector according to frame $\{\zeta_1, \zeta_2, B\}$ as

$$T = \sin\vartheta(s)\,\zeta_1 - \cos\vartheta(s)\ \zeta_2,$$

and differentiate with respect to s

$$T' = \kappa\, N = \vartheta'(s)(\cos\vartheta(s)\,\zeta_1 + \sin\vartheta(s)\ \zeta_2) + \sin\vartheta(s)\,\zeta_1' - \cos\vartheta(s)\ \zeta_2'. \tag{8}$$

Substituting $\zeta_1' = \varepsilon_1 B$ and $\zeta_2' = \varepsilon_2 B$ to Eqn. (8), we have

$$\kappa\, N = \vartheta'(s)(\cos\vartheta(s)\,\zeta_1 + \sin\vartheta(s)\ \zeta_2).$$

In the above equation we take $\kappa(s) = \vartheta'(s)$. So, we have

$$N = \cos\vartheta(s)\,\zeta_1 + \sin\vartheta(s)\ \zeta_2.$$

By consequence of the obtained equations, the relation matrix between Frenet and timelike type-2 parallel transport frames can be expressed

$$\begin{pmatrix}T\\ N\\ B\end{pmatrix} = \begin{pmatrix}\sin\vartheta(s) & -\cos\vartheta(s) & 0\\ \cos\vartheta(s) & \sin\vartheta(s) & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\zeta_1\\ \zeta_2\\ B\end{pmatrix}. \tag{9}$$

Also, equation (7) can be written as

$$1 = \sqrt{\frac{\varepsilon_1^2}{\tau^2} + \frac{\varepsilon_2^2}{\tau^2}}\,, \tag{10}$$

and so by equation (10), we may have

$$\begin{cases}\varepsilon_1(s) = \tau\cos\vartheta(s),\\ \varepsilon_2(s) = \tau\sin\vartheta(s)\,.\end{cases}$$

By this way, we conclude $\vartheta(s) = arc\tan\left(\frac{\varepsilon_2}{\varepsilon_1}\right)$. The frame $\{\zeta_1, \zeta_2, B\}$ is properly oriented, and τ and $\vartheta(s) = \int_0^s \kappa(s)ds$ are polar coordinates for the curve $\mu = \mu(s)$. We shall call the set $\{\zeta_1, \zeta_2, B, \varepsilon_1, \varepsilon_2\}$ as timelike type-2 parallel transport invariants of the timelike curve $\mu = \mu(s)$.

4. Spherical images of a timelike regular curve

4.1 ζ_1-timelike parallel transport spherical image

Definition 4.1.

Let $\mu = \mu(s)$ be a timelike regular curve lying fully on the timelike surface M in 3-dimensional Minkowski space $\mathfrak{R}_1^3$ with the moving frame $\{T, N, B\}$. If we translate of the first vector field of timelike type-2 parallel transport frame to the center O of the unit Lorentzian sphere S_1^2, we obtain a spherical image $\Omega = \Omega(\varsigma(s))$. This curve is called ζ_1-timelike parallel transport spherical image or indicatrix of the curve $\mu = \mu(s)$.

Now, we can investigate Frenet invariants of ζ_1-timelike parallel transport spherical image $\Omega = \Omega(\varsigma(s))$. of a timelike regular curve $\mu = \mu(s)$. Differentiating $\Omega = \Omega(\varsigma(s))$. with respect to s, we get

$$\Omega'(\varsigma) = \frac{d\Omega}{d\varsigma}\frac{d\varsigma}{ds} = \varepsilon_1 B \text{ and } T_\Omega\frac{d\varsigma}{ds} = \varepsilon_1 B,$$

where

$$\frac{d\varsigma}{ds} = \varepsilon_1. \tag{11}$$

Then,

$$T_\Omega(\varsigma) = B(s). \tag{12}$$

Differentiating equation (12) with respect to s, we obtain

$$\frac{dT_\Omega}{d\varsigma}\frac{d\varsigma}{ds} = \varepsilon_1\zeta_1 + \varepsilon_2\zeta_2. \tag{13}$$

Substituting equation (11) in equation (13), we get $\frac{dT_\Omega}{d\varsigma} = \zeta_1 + \left(\frac{\varepsilon_2}{\varepsilon_1}\right)\zeta_2$. Then, the curvature and principal normal vector field of curve $\Omega(\varsigma)$ are respectively,

$$\kappa_\Omega(\varsigma) = \left\|\frac{dT_\Omega}{d\varsigma}\right\| = \sqrt{1 - \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2}, \tag{14}$$

and

$$N_\Omega(\varsigma) = \left(\frac{1}{\kappa_\Omega}\right)\zeta_1 + \left(\frac{\varepsilon_2}{\varepsilon_1\kappa_\Omega}\right)\zeta_2.$$

So, the binormal vector of curve $\Omega(\varsigma)$ is

$$B_{\Omega}(\varsigma) = T_{\Omega}(\varsigma) \times N_{\Omega}(\varsigma) = \left(\frac{-\varepsilon_2}{\varepsilon_1 \kappa_{\Omega}}\right)\zeta_1 + \left(\frac{1}{\kappa_{\Omega}}\right)\zeta_2.$$

Using the formula of the torsion, we write a relation

$$\tau_{\Omega} = \frac{\varepsilon_1 \left(\frac{\varepsilon_2}{\varepsilon_1}\right)'}{\varepsilon_1^2 - \varepsilon_2^2}. . \tag{15}$$

Considering equations (14) and (15), we give:

Corollary 4.1.

Let $\Omega = \Omega(\varsigma)$ be ζ_1-timelike parallel transport spherical image of the timelike curve $\mu = \mu(s)$. If the ratio of timelike type-2 parallel transport curvatures of $\mu = \mu(s)$ is constant, i.e., $\left(\frac{\varepsilon_2}{\varepsilon_1}\right) =$ constant, then the ζ_1-timelike parallel transport spherical indicatrix $\Omega(\varsigma)$ is a Lorentzian circle in the osculating plane.

Proof:

Let Let $\Omega = \Omega(\varsigma)$ be ζ_1-timelike parallel transport spherical image of a timelike regular curve $\mu = \mu(s)$. If the ratio of timelike type-2 parallel transport curvatures of $\mu = \mu(s)$ is constant, in terms of equations (14) and (15), we have $\kappa_{\Omega} =$ constant and $\tau_{\Omega} = 0$, respectively. Therefore, Ω is a Lorentzian circle in the osculating plane. ■

Theorem 4.2.

Let $\Omega = \Omega(\varsigma)$ be ζ_1-timelike parallel transport spherical image of the timelike curve $\mu = \mu(s)$. There exists a relation among Frenet invariants of $\Omega = \Omega(\varsigma)$ and timelike type-2 parallel transport curvatures of $\mu = \mu(s)$ as follows

$$\left(\frac{\varepsilon_2}{\varepsilon_1}\right) - \int_{\mathbf{0}}^{\varsigma} \kappa_{\Omega}^2 \, \tau_{\Omega} d\varsigma = 0. \tag{16}$$

Proof:

Let Let $\Omega = \Omega(\varsigma)$ be ζ_1-timelike parallel transport spherical image of a timelike regular curve $\mu = \mu(s)$. Then, equations (11) and (15) hold. Using equation (11) in (15) and by the chain rule, we have

$$\tau_{\Omega} = \frac{\varepsilon_1 \frac{d}{d\varsigma}\left(\frac{\varepsilon_2}{\varepsilon_1}\right)\frac{d\varsigma}{ds}}{\varepsilon_1^2 - \varepsilon_2^2} = \frac{\varepsilon_1^2 \frac{d}{d\varsigma}\left(\frac{\varepsilon_2}{\varepsilon_1}\right)}{\varepsilon_1^2 - \varepsilon_2^2} = \frac{\frac{d}{d\varsigma}\left(\frac{\varepsilon_2}{\varepsilon_1}\right)}{1 - \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2}.$$

From equation (14) and integrating both sides, we have equation (16) as desired. ■

Theorem 4.3.

Let $\Omega = \Omega(\varsigma)$ be ζ_1-timelike parallel transport spherical image of the timelike curve $\mu = \mu(s)$. If Ω is a general helix, then, the timelike type-2 parallel transport curvatures of μ satisfy

$$\frac{\varepsilon_1^2\left(\frac{\varepsilon_2}{\varepsilon_1}\right)'}{\left(\varepsilon_1^2-\varepsilon_2^2\right)^{\frac{3}{2}}} = constant.$$

Proof:

Let Let $\Omega = \Omega(\varsigma)$ be ζ_1-timelike parallel transport spherical image of a timelike regular curve $\mu(s)$. Then Ω is a general helix if and only if its curvature functions satisfying

$$\left(\frac{\tau_\Omega}{\kappa_\Omega}\right) = constant,$$

(see Izumiya and Takeuchi (2004)). Using equations (14) and (15) with applying in the later formula and with simple calculation we will get the result which complete the proof. ■

Theorem 4.4.

Let $\Omega = \Omega(\varsigma)$ be ζ_1-timelike parallel transport spherical image of the timelike curve $\mu = \mu(s)$. If Ω is a slant helix, then, the timelike type-2 parallel transport curvatures of μ satisfy

$$\left[\frac{\varepsilon_1^2\left(\frac{\varepsilon_2}{\varepsilon_1}\right)'}{\left(\varepsilon_1^2-\varepsilon_2^2\right)^{\frac{3}{2}}}\right]' \left\{\frac{\left(\varepsilon_2^2-\varepsilon_1^2\right)^4}{\varepsilon_1^3\left[\left(\frac{\varepsilon_2}{\varepsilon_1}\right)'^2+\left(\varepsilon_1^2-\varepsilon_2^2\right)^3\right]^{\frac{3}{2}}}\right\} = constant.$$

Proof:

Let Let $\Omega = \Omega(\varsigma)$ be ζ_1-timelike parallel transport spherical image of a timelike regular curve $\mu(s)$. Then Ω is a slant helix if and only if its curvature functions satisfying

$$\frac{\kappa_\Omega^2}{\left(\tau_\Omega^2-\kappa_\Omega^2\right)^{\frac{3}{2}}}\left(\frac{\tau_\Omega}{\kappa_\Omega}\right)' = constant,$$

(see Ali and López (2011); Izumiya and Takeuchi (2004)). Substituting from equations (14) and (15) in the later formula one can get the result directly we will get the result which complete the proof. ■

Theorem 4.5.

Let $\Omega = \Omega(\varsigma)$ be ζ_1-timelike parallel transport spherical image of the timelike curve $\mu = \mu(s)$. Then the timelike type-2 parallel transport curvatures of a timelike regular curve μ satisfy the following differential equation

$$\left[\frac{\varepsilon_1^2\left(\frac{\varepsilon_2}{\varepsilon_1}\right)'}{\varepsilon_1^2-\varepsilon_2^2}\right]+\left[\frac{\varepsilon_1\varepsilon_2}{\sqrt{\varepsilon_1^2-\varepsilon_2^2}}\right]'=0.$$

Proof:

Let $\Omega = \Omega(\varsigma)$ be ζ_1-timelike parallel transport spherical image of the timelike curve $\mu = \mu(s)$. Then, Ω lie on the surface of a Lorentzian sphere if and only if

$$\frac{\tau_\Omega}{\kappa_\Omega}+\left[\frac{1}{\tau_\Omega}\left(\frac{1}{\kappa_\Omega}\right)'\right]'=0,$$

(see Izumiya and Takeuchi (2004)). The key of the proof is to applying equations (14) and (15) direcltly in the later formula, which ended the proof. ■

Remark 4.6.

Considering $\vartheta_\Omega = \int_0^\varsigma \kappa_\Omega\, d\varsigma$ and using the transformation matrix, one can obtain the timelike type-2 parallel transport trihedra $\{\zeta_{1\Omega}, \zeta_{2\Omega}, B_\Omega\}$ of the curve $\Omega = \Omega(\varsigma)$.

4.2 ζ_2-timelike parallel transport spherical image

Definition 4.2.

Let $\mu = \mu(s)$ be a timelike regular curve lying fully on the timelike surface M in 3-dimensional Minkowski space $\Re_1^3$ with the moving frame $\{T, N, B\}$. If we translate of the second vector field of timelike type-2 parallel transport frame to the center O of the unit Lorentzian sphere S_1^2, we obtain a spherical image $\mathcal{B} = \mathcal{B}(\varsigma(s))$. This curve is called ζ_2-timelike parallel transport spherical image or indicatrix of the curve $\mu = \mu(s)$.

Let $\mathcal{B} = \mathcal{B}(\varsigma(s))$ be ζ_2-timelike parallel transport spherical of a timelike regular curve $\mu = \mu(s)$. Differentiating $\mathcal{B} = \mathcal{B}(\varsigma(s))$ with respect to s, we get

$$\mathcal{B}'(\varsigma) = \frac{d\mathcal{B}}{d\varsigma}\frac{d\varsigma}{ds} = T_{\mathcal{B}}\frac{d\varsigma}{ds} = \varepsilon_2\, B,$$

where

$$\frac{d\varsigma}{ds} = \varepsilon_2. \tag{17}$$

Similar to ζ_1-timelike parallel transport spherical image, one can have

$$T_{\mathcal{B}}(\varsigma) = B(s). \tag{18}$$

Differentiating equation (18) with respect to s, we obtain

$$\frac{dT_{\mathcal{B}}}{d\varsigma}\frac{d\varsigma}{ds} = \varepsilon_1\zeta_1 + \varepsilon_2\zeta_2. \tag{19}$$

Substituting equation (17) in (19), we get $\frac{dT_{\mathcal{B}}}{d\varsigma} = \left(\frac{\varepsilon_1}{\varepsilon_2}\right)\zeta_1 + \zeta_2$. Then, we can express

$$\kappa_{\mathcal{B}}(\varsigma) = \sqrt{1 - \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2}, \tag{20}$$

and

$$N_{\mathcal{B}}(\varsigma) = \left(\frac{\varepsilon_1}{\varepsilon_2 \kappa_{\mathcal{B}}}\right)\zeta_1 + \left(\frac{1}{\kappa_{\mathcal{B}}}\right)\zeta_2.$$

So, the binormal vector of curve $\mathcal{B}(\varsigma)$ is

$$B_{\mathcal{B}}(\varsigma) = \left(\frac{-1}{\kappa_{\mathcal{B}}}\right)\zeta_1 + \left(\frac{\varepsilon_1}{\varepsilon_2 \kappa_{\mathcal{B}}}\right)\zeta_2.$$

By the formula of the torsion, we write

$$\tau_{\mathcal{B}} = \frac{\varepsilon_2\left(\frac{\varepsilon_1}{\varepsilon_2}\right)'}{\varepsilon_2^2 - \varepsilon_1^2}. \tag{21}$$

Considering equations (20) and (21), we give:

Corollary 4.7.

Let $\mathcal{B} = \mathcal{B}(\varsigma)$ be ζ_2-timelike parallel transport spherical image of the timelike curve $\mu = \mu(s)$. If the ratio of timelike type-2 parallel transport curvatures of $\mu = \mu(s)$ is constant, i.e., $\left(\frac{\varepsilon_1}{\varepsilon_2}\right) =$ constant, then the ζ_2-timelike parallel transport spherical indicatrix $\mathcal{B}(\varsigma)$ is a Lorentzian circle in the osculating plane.

Proof:

Let Let $\mathcal{B} = \mathcal{B}(\varsigma)$ be ζ_2-timelike parallel transport spherical image of a timelike regular curve $\mu = \mu(s)$. If the ratio of timelike type-2 parallel transport curvatures of $\mu = \mu(s)$ is constant, in terms of equations (20) and (21), we have $\kappa_{\mathcal{B}} =$ constant and $\tau_{\mathcal{B}} = 0$, respectively. Therefore, $\mathcal{B}$ is a Lorentzian circle in the osculating plane. ■

Theorem 4.8.

Let $\mathcal{B} = \mathcal{B}(\varsigma)$ be ζ_2-timelike parallel transport spherical image of the timelike curve $\mu = \mu(s)$. Then, there exists a relation among Frenet invariants of $\mathcal{B} = \mathcal{B}(\varsigma)$ and timelike type-2 parallel transport curvatures of $\mu = \mu(s)$ as follows

$$\left(\frac{\varepsilon_1}{\varepsilon_2}\right) + \int_{\mathbf{0}}^{\varsigma} \kappa_{\mathcal{B}}^2\, \tau_{\mathcal{B}} d\varsigma = 0. \tag{22}$$

Proof:

Let Let $\mathcal{B} = \mathcal{B}(\varsigma)$ be ζ_2-timelike parallel transport spherical image of a timelike regular curve $\mu = \mu(s)$. Then, equations (17) and (21) hold. Using equation (17) in (21) and by the chain rule, we have

$$\tau_{\mathcal{B}} = \frac{\varepsilon_2 \frac{d}{d\varsigma}\left(\frac{\varepsilon_1}{\varepsilon_2}\right)\frac{d\varsigma}{ds}}{\varepsilon_2^2 - \varepsilon_1^2} = \frac{\varepsilon_2^2 \frac{d}{d\varsigma}\left(\frac{\varepsilon_1}{\varepsilon_2}\right)}{\varepsilon_2^2 - \varepsilon_1^2} = \frac{-\frac{d}{d\varsigma}\left(\frac{\varepsilon_1}{\varepsilon_2}\right)}{1 - \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2}.$$

From equation (20) and integrating both sides, we have (22) as desired. ■

By uing the preivious method to proof the Theorems 4.3, 4.4 and 4.5, we can use these methods to proof the following theorems.

Theorem 4.9.

Let $\mathcal{B} = \mathcal{B}(\varsigma)$ be ζ_2-timelike parallel transport spherical image of the timelike curve $\mu = \mu(s)$. If $\mathcal{B}$ is a general helix, then, the timelike type-2 parallel transport curvatures of μ satisfy

$$\frac{\varepsilon_2^2\left(\frac{\varepsilon_1}{\varepsilon_2}\right)'}{(\varepsilon_2^2-\varepsilon_1^2)^{\frac{3}{2}}} = constant.$$

Theorem 4.10.

Let $\mathcal{B} = \mathcal{B}(\varsigma)$ be ζ_2-timelike parallel transport spherical image of the timelike curve $\mu = \mu(s)$. If $\mathcal{B}$ is a slant helix, then, the timelike type-2 parallel transport curvatures of μ satisfy

$$\left[\frac{\varepsilon_2^2\left(\frac{\varepsilon_1}{\varepsilon_2}\right)'}{(\varepsilon_2^2-\varepsilon_1^2)^{\frac{3}{2}}}\right]' \left\{\frac{\varepsilon_2(\varepsilon_2^2-\varepsilon_1^2)^4}{\varepsilon_2^3\left[\left(\frac{\varepsilon_1}{\varepsilon_2}\right)'^2+(\varepsilon_2^2-\varepsilon_1^2)^3\right]^{\frac{3}{2}}}\right\} = constant.$$

Theorem 4.11.

Let $\mathcal{B} = \mathcal{B}(\varsigma)$ be ζ_2-timelike parallel transport spherical image of the timelike curve $\mu = \mu(s)$. Then the timelike type-2 parallel transport curvatures of a timelike regular curve μ satisfy the following differential equation

$$\left[\frac{\varepsilon_2^2\left(\frac{\varepsilon_1}{\varepsilon_2}\right)'}{\varepsilon_2^2-\varepsilon_1^2}\right] + \left[\frac{\varepsilon_1\varepsilon_2}{\sqrt{\varepsilon_2^2-\varepsilon_1^2}}\right]' = 0.$$

Remark 4.12.

Considering $\vartheta_{\mathcal{B}} = \int_0^{\varsigma} \kappa_{\mathcal{B}}\, d\varsigma$ and using the transformation matrix, one can obtain the timelike type-2 parallel transport trihedra $\{\zeta_{1\mathcal{B}}, \zeta_{2\mathcal{B}}, B_{\mathcal{B}}\}$ of the curve $\mathcal{B} = \mathcal{B}(\varsigma)$.

4.3 Binormal-timelike parallel transport spherical image

Definition 4.3.

Let $\mu = \mu(s)$ be a timelike regular curve lying fully on the timelike surface M in 3-dimensional Minkowski space $\Re_1^3$ with the moving frame $\{T, N, B\}$. If we translate of the third vector field of timelike type-2 parallel transport frame to the center O of the unit Lorentzian sphere S_1^2, we obtain a spherical image $\phi = \phi(\varsigma(s))$. This curve is called binormal timelike parallel transport spherical image or indicatrix of the curve $\mu = \mu(s)$.

Let $\phi = \phi(\varsigma(s))$ be the binormal timelike parallel transport spherical of a timelike regular curve $\mu = \mu(s)$. One can differentiate ϕ with respect to s

$$\phi'(\varsigma) = \frac{d\phi}{d\varsigma}\frac{d\varsigma}{ds} = \varepsilon_1\zeta_1 + \varepsilon_2\zeta_2,$$

and

$$T_\phi \frac{d\varsigma}{ds} = \varepsilon_1\zeta_1 + \varepsilon_2\zeta_2,$$

where

$$\frac{d\varsigma}{ds} = \sqrt{\varepsilon_2^2 - \varepsilon_1^2}.$$

In terms of timelike type-2 parallel transport frame vector field given in equation (6), we have the tangent vector of the binormal-timelike parallel transport spherical image as follows:

$$T_\phi(\varsigma) = \frac{1}{\sqrt{\varepsilon_2^2-\varepsilon_1^2}}(\varepsilon_1\zeta_1 + \varepsilon_2\zeta_2). \tag{23}$$

In order to determine the first curvature and principal normal vector field of curve ϕ. Differentiating equation (23) with respect to s, we obtain

$$\frac{dT_\phi}{d\varsigma} = \left[\frac{\varepsilon_2^3\left(\frac{\varepsilon_1}{\varepsilon_2}\right)'}{\varepsilon_2^2-\varepsilon_1^2}\right]\zeta_1 - \left[\frac{\varepsilon_1^3\left(\frac{\varepsilon_2}{\varepsilon_1}\right)'}{\varepsilon_2^2-\varepsilon_1^2}\right]\zeta_2 + (\varepsilon_1^2 + \varepsilon_2^2)B. \tag{24}$$

Since, we immediately have

$$\kappa_\phi(\varsigma) = \sqrt{\left[\frac{\varepsilon_1^3\left(\frac{\varepsilon_2}{\varepsilon_1}\right)'}{\varepsilon_2^2-\varepsilon_1^2}\right]^2 - \left[\frac{\varepsilon_2^3\left(\frac{\varepsilon_1}{\varepsilon_2}\right)'}{\varepsilon_2^2-\varepsilon_1^2}\right]^2 + (\varepsilon_1^2 + \varepsilon_2^2)^2}, \tag{25}$$

and

$$N_\phi(\varsigma) = \frac{1}{\kappa_\phi}\left(\left[\frac{\varepsilon_2^3\left(\frac{\varepsilon_1}{\varepsilon_2}\right)'}{\varepsilon_2^2-\varepsilon_1^2}\right]\zeta_1 - \left[\frac{\varepsilon_1^3\left(\frac{\varepsilon_2}{\varepsilon_1}\right)'}{\varepsilon_2^2-\varepsilon_1^2}\right]\zeta_2 + (\varepsilon_1^2 + \varepsilon_2^2)B\right).$$

So, the binormal vector of curve ϕ is

$$B_\phi(\varsigma) = \frac{1}{\kappa_\phi\sqrt{\varepsilon_2^2-\varepsilon_1^2}}\left\{\begin{array}{c}(\varepsilon_1^2+\varepsilon_2^2)(\varepsilon_2\zeta_1 - \varepsilon_1\zeta_2)\\ -\left\{\varepsilon_1\left[\frac{\varepsilon_1^3\left(\frac{\varepsilon_2}{\varepsilon_1}\right)'}{\varepsilon_2^2-\varepsilon_1^2}\right] + \varepsilon_2\left[\frac{\varepsilon_2^3\left(\frac{\varepsilon_1}{\varepsilon_2}\right)'}{\varepsilon_2^2-\varepsilon_1^2}\right]\right\}B\end{array}\right\}.$$

By means of obtained equations, we express the torsion by formula

$$\tau_{\phi}(\varsigma)=\frac{\begin{array}{c}\varepsilon_1'\{3\varepsilon_2'(\varepsilon_1\varepsilon_1'+\varepsilon_2\varepsilon_2')-(\varepsilon_1^2+\varepsilon_2^2)[\varepsilon_2''+\varepsilon_2(\varepsilon_1^2+\varepsilon_2^2)]\}\\+\varepsilon_2\{(\varepsilon_1^2+\varepsilon_2^2)[\varepsilon_1''+\varepsilon_1(\varepsilon_1^2+\varepsilon_2^2)]-3\varepsilon_1(\varepsilon_1\varepsilon_1'+\varepsilon_2\varepsilon_2')\}\end{array}}{\varepsilon_1^4\left(\frac{\varepsilon_2}{\varepsilon_1}\right)'^2-(\varepsilon_2^2-\varepsilon_1^2)(\varepsilon_1^2+\varepsilon_2^2)^2}. \tag{26}$$

Consequently, we determined Frenet invariants of the binormal-timelike parallel transport spherical indicatrix according to timelike type-2 parallel transport invariants. In terms of equations (25) and (26), we give:

Corollary 4.13.

Let $\phi=\phi(\varsigma)$ be the binormal timelike parallel transport spherical image of the timelike curve $\mu=\mu(s)$. If the ratio of timelike type-2 parallel transport curvatures of $\mu=\mu(s)$ is constant i.e., $\left(\frac{\varepsilon_1}{\varepsilon_2}\right)=$ constant, then the binormal timelike parallel transport spherical indicatrix $\phi(\varsigma)$ is a Lorentzian circle in the osculating plane.

Proof:

The proof come directly from equations (25) and (26). ■

Remark 4.14.

Considering $\vartheta_{\phi}=\int_0^{\varsigma}\kappa_{\phi}\,d\varsigma$ and using the transformation matrix, one can obtain the timelike type-2 parallel transport trihedra $\{\zeta_{1\phi},\zeta_{2\phi},B_{\phi}\}$ of the curve $\phi=\phi(\varsigma)$.

5. Examples

Example 5.1.

Let we consider a time-like regular curve μ in $\mathfrak{R}_1^3$ (see Figure 1)

$$\mu=\mu(s)=\left(\sqrt{3}\,s,\sqrt{2}\,\cos s,\sqrt{2}\,\sin s\right). \tag{27}$$

One can calculate its Frenet apparatus as the following

$$\begin{cases}T=\left(\sqrt{3},-\sqrt{2}\,\sin s,\sqrt{2}\,\cos s\right),\\ \kappa=\sqrt{2},\\ \tau=\sqrt{3},\\ N=(0,-\cos s,-\sin s),\\ B=\left(\sqrt{2},\sqrt{3}\,\sin s,-\sqrt{3}\,\cos s\right).\end{cases}$$

Now we focus on the time-like type-2 parallel transport trihedra. Let us express $\vartheta(s)=\int_0^s\sqrt{2}\,dt=\sqrt{2}\,s$. Since, we can write the transformation matrix

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} \sin\sqrt{2}\,s & -\cos\sqrt{2}\,s & 0 \\ \cos\sqrt{2}\,s & \sin\sqrt{2}\,s & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ B \end{pmatrix}.$$

By the method of Cramer, one can obtain timelike type-2 parallel transport trihedra of μ as follows (see Figures 2-4):

$$\zeta_1 = \left(\sqrt{3}\sin(\sqrt{2}\,s), -\sqrt{2}\,\sin s\sin(\sqrt{2}\,s) - \cos s\cos(\sqrt{2}\,s), \sqrt{2}\,\cos s\sin(\sqrt{2}\,s) - \sin s\cos(\sqrt{2}\,s)\right),$$
$$\zeta_2 = \left(-\sqrt{3}\cos(\sqrt{2}\,s), \sqrt{2}\,\sin s\cos(\sqrt{2}\,s) - \cos s\sin(\sqrt{2}\,s), -\sqrt{2}\,\cos s\cos(\sqrt{2}\,s) - \sin s\sin(\sqrt{2}\,s)\right),$$
$$B = \left(\sqrt{2}, \sqrt{3}\,\sin s, -\sqrt{3}\,\cos s\right).$$

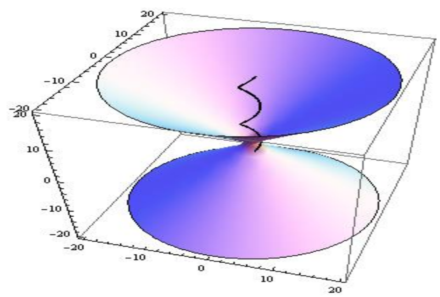

Figure 1. The timelike curve $\mu = \mu(s)$

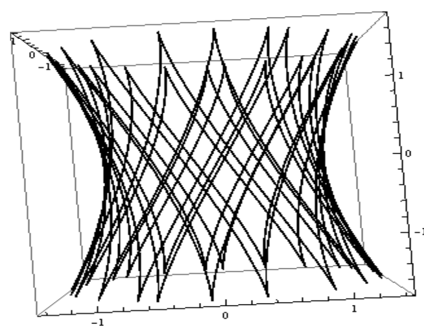

Figure 2. ζ_1-timelike parallel transport spherical image of $\mu = \mu(s)$

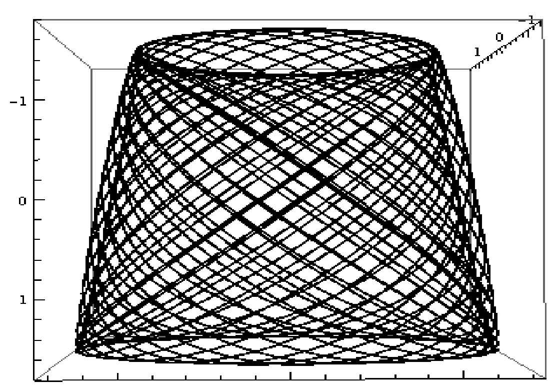

Figure 3. ζ_2-timelike parallel transport spherical image of $\mu = \mu(s)$

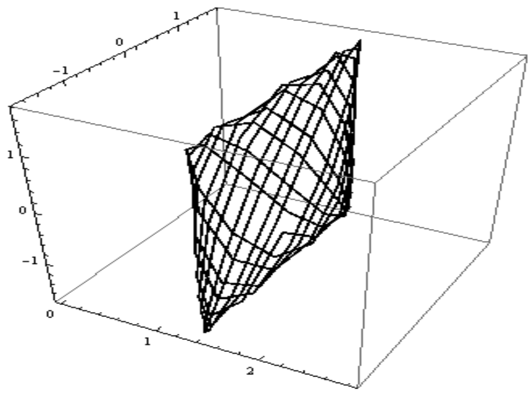

Figure 4. Binormal-timelike parallel transport spherical image of $\mu = \mu(s)$

Example 5.2.

Let we consider a time-like regular curve $\omega(s) = \left(\sqrt{3}\,s, 25\sqrt{2}\,\cos\left(\frac{s}{25}\right), 25\,\sqrt{2}\,\sin\left(\frac{s}{25}\right)\right)$ (see Figure 5). Then it is easy to show that the Frenet frame of the curve $\omega = \omega(s)$ is given as follows:

$$\begin{cases} T(s) = \left(\sqrt{3}, -\sqrt{2}\,\sin\left(\frac{s}{25}\right), \sqrt{2}\,\cos s\left(\frac{s}{25}\right)\right), \\ N(s) = \left(0, -\cos\left(\frac{s}{25}\right), -\sin\left(\frac{s}{25}\right)\right), \\ B(s) = \left(\sqrt{2}, \sqrt{3}\,\sin\left(\frac{s}{25}\right), -\sqrt{3}\,\cos\left(\frac{s}{25}\right)\right). \end{cases}$$

Also the curvature functions are expressed as:

$$\begin{cases} \kappa(s) = \dfrac{\sqrt{2}}{25}, \\ \tau(s) = \dfrac{\sqrt{3}}{25}. \end{cases}$$

In order to form the transformation matrix, we also need $\vartheta(s) = \int_0^s \left(\frac{\sqrt{2}}{25}\right) dt = \left(\frac{\sqrt{2}}{25}\right) s$. The transformation matrix for the curve $\omega = \omega(s)$ has the form

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} \sin\left(\frac{\sqrt{2}\,s}{25}\right) & -\cos\left(\frac{\sqrt{2}\,s}{25}\right) & 0 \\ \cos\left(\frac{\sqrt{2}\,s}{25}\right) & \sin\left(\frac{\sqrt{2}\,s}{25}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ B \end{pmatrix}.$$

By the solution of the system above, we have time-like type-2 parallel transport spherical images of the unit speed curve $\omega = \omega(s)$, see Figures 6, 7 and 8.

$$\zeta_1 = \begin{Bmatrix} \sqrt{3}\sin\left(\frac{\sqrt{2}\,s}{25}\right), \\ -\sqrt{2}\ \sin\left(\frac{s}{25}\right)\sin\left(\frac{\sqrt{2}\,s}{25}\right) - \cos\left(\frac{s}{25}\right)\cos\left(\frac{\sqrt{2}\,s}{25}\right), \\ \sqrt{2}\ \cos\left(\frac{s}{25}\right)\sin\left(\frac{\sqrt{2}\,s}{25}\right) - \sin\left(\frac{s}{25}\right)\cos\left(\frac{\sqrt{2}\,s}{25}\right) \end{Bmatrix},$$

$$\zeta_2 = \begin{Bmatrix} -\sqrt{3}\cos\left(\frac{\sqrt{2}\,s}{25}\right), \\ \sqrt{2}\ \sin\left(\frac{s}{25}\right)\cos\left(\frac{\sqrt{2}\,s}{25}\right) - \cos\left(\frac{s}{25}\right)\sin\left(\frac{\sqrt{2}\,s}{25}\right), \\ -\sqrt{2}\ \cos\left(\frac{s}{25}\right)\cos\left(\frac{\sqrt{2}\,s}{25}\right) - \sin\left(\frac{s}{25}\right)\sin\left(\frac{\sqrt{2}\,s}{25}\right) \end{Bmatrix},$$

$$B = \left(\sqrt{2}, \sqrt{3}\ \sin\left(\frac{s}{25}\right), -\sqrt{3}\ \cos\left(\frac{s}{25}\right)\right).$$

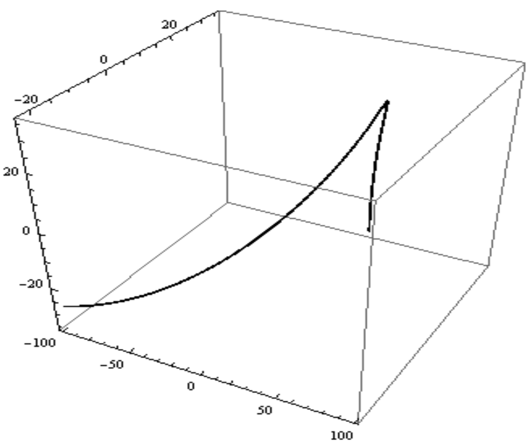

Figure 5. The timelike curve $\omega = \omega(s)$

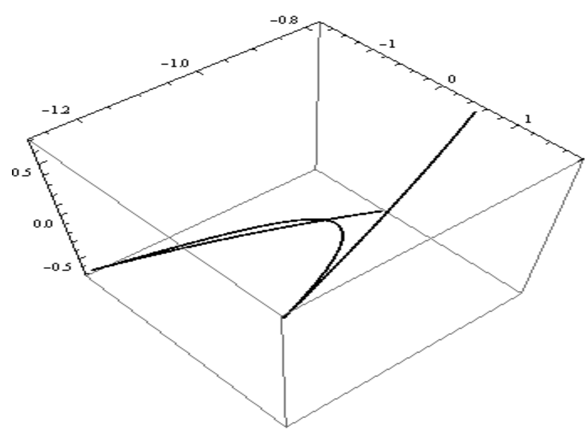

Figure 6. ζ_1-timelike parallel transport spherical image of $\omega = \omega(s)$

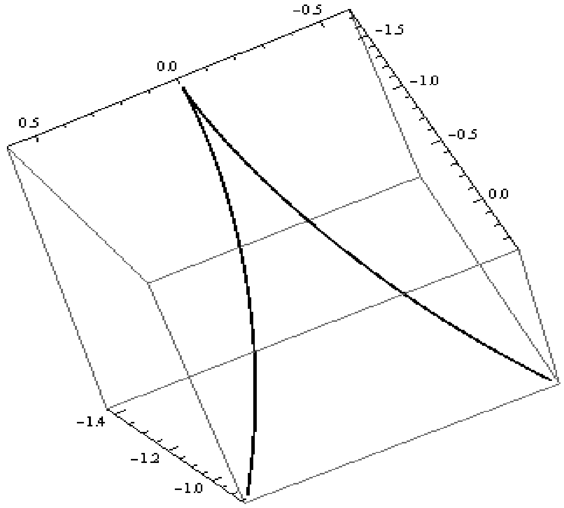

Figure 7. ζ_2-timelike parallel transport spherical image of $\omega = \omega(s)$

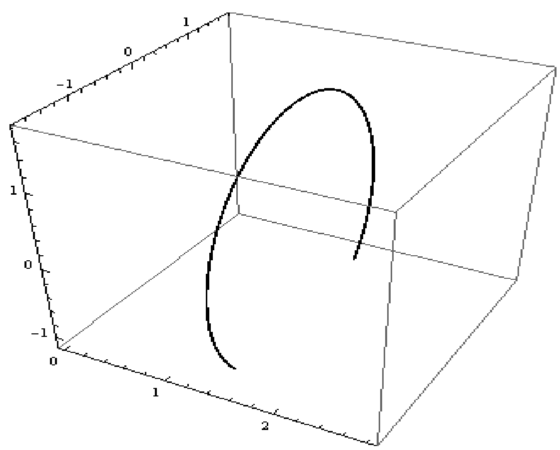

Figure 8. Binormal-timelike parallel transport spherical image of $\omega = \omega(s)$

6. Conclusion

As a conclusion of our results, by using the binormal vector field of a timelike regular curve as common vector field a timelike type-2 parallel transport frame has been. Also, a new timelike type-2 parallel transport spherical images introduced by translating the induced frame vectors to the center of a unit Lorentzian sphere in 3-dimensional Minkowski space $\mathfrak{R}_1^3$. Additionally, the Frenet apparatus of these new spherical images are obtained in terms of base curves timelike type-2 parallel transport invariants.

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REFERENCES

Ali, A.T. and López, R. (2011). Slant helices in Minkowski space E_1^3, J. Korean Math. Soc., Vol. 48, No. 1, pp. 159-167.

Bishop, L.R. (1975). There is more than one way to frame a curve, Amer. Math. Monthly, Vol. 82, No. 11, pp. 246-251.

Bükcü, B. and Karacan, M.K. (2008). Parallel transport frame of the spacelike curve with a spacelike binormal in Minkowski 3-space, Selçuk J. Appl. Math., Vol. 57, No. 1, pp. 13-22.

Bükcü, B. and Karacan, M.K. (2010), Bishop frame of the spacelike curve with a spacelike binormal in Minkowski 3-space, Selçuk J. Appl. Math., Vol.11, No. 1, pp. 15-25.

Bükcü, B. and Karacan, M.K. (2009). The slant helices according to Bishop frame, Int. J. Math. Comput. Sci., Vol. 3, No. 2, pp. 67-70.

Camci, Ç., İlarslan, K., Kula, L. and Hacisalihoğlu, H.H. (2009). Harmonic curvatures and generalized helices in En, Chaos Solitons Fractals, Vol. 4, pp. 2590-2596.

Chouaieb, N., Goriely, A. and Maddocks, J.H. (2006). Helices, Proc. Natl. Acad. Sci. USA, Vol. 103, pp. 398-403.

Cook, T.A. (1979). The Curves of Life. Dover, London.

Do Carmo, M.P. (1976). Differential geometry of curves and surfaces, Prentice Hall, Englewood Cliffs, NJ.

Izumiya, S. and Takeuchi, N. (2004). New special curves and developable surfaces, Turkish J. Math., Vol. 28, No. 2, pp. 531-537.

Karacan, M. K. and Bükcü, B. (2007). An alternative moving frame for tubular surfaces around timelike curves in the Minkowski 3-space, Balkan J. Geom. Appl., Vol. 12, No. 2, pp. 73-80.

Karacan, M. K. and Bükcü, B. (2007). An alternative moving frame for tubular surface around the spacelike curve with a spacelike binormal in Minkowski 3-space, Math. Morav., Vol. 11, pp. 47-54.

Karacan, M. K. and Bükcü, B. (2008). An alternative moving frame for a tubular surface around a spacelike curve with a spacelike normal in Minkowski 3-space, Rend. Circ. Mat. Palermo, Vol. 57, No. 2, pp. 193-201.

Karacan, M. K. and Bükcü, B. (2008). Bishop frame of the timelike curve in Minkowski 3-space, Fen Derg., Vol. 3, No. 1, pp. 80-90.

Karacan, M. K., Bükcü, B. and Yuksel N. (2008). On the dual Bishop Darboux rotation axis of the dual space curve, Appl. Sci., Vol. 10, pp. 115-120.

Kula, L. and Yayli, Y. (2005). On slant helix and its spherical indicatrix, Appl. Math. Comput., Vol. 169, No. 1, pp. 600-607.

Kula, L., Ekmekçi, N., Yayl, Y. and İlarslan, K. (2010). Characterizations of slant helices in Euclidean 3-space, Turkish J. Math., Vol. 34, pp. 261-274.

López, R. (2014). Differential Geometry of Curves and Surfaces in Lorentz-Minkowski Space, International Electronic Journal of Geometry, Vol. 7, No. 1, pp. 44-107.

O'Neill, B. (1966). Elementary differential geometry, Academic Press Inc., New York.

O'Neill, B. (1983). Semi-Riemannian Geometry with Application to Relativity, Academic Press.

Solouma, E.M. (2017). Type-2 spacelike Bishop frame and application to spherical image in Minkowski space time, Int. J. Appl. Comput. Math., Vol. 3, pp. 3575-3591.

Watson, J. D. and Crick, F. H. (1953). Molecular structures of nucleic acids, Nature, Vol. 171, pp. 737-738.

Yilmaz, S. and Turgut, M. (2010). A new version of Bishop frame and an application to spherical image, J. of Math. Analy. And Appl., Vol 371, pp. 764-776.