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Some Summation Theorems for Clausen's Hypergeometric Functions with Unit Argument

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Abstract

Motivated by the work on hypergeometric summation theorems, we establish new summation formula for Clausen's hypergeometric function with unit argument in terms of pi and natural logarithms of some rational and irrational numbers. For the application purpose, we derive some new and modified summation theorems for Clausen's hypergeometric functions using our new formula.

Keywords: Clausen's hypergeometric function; Digamma (Psi) function; Generalized hypergeometric series; Euler's constant.

MSC 2010 No.: 33B15, 33C20, 33C05, 33C90

1. Introduction

The aim of this research article is to establish an elegant and manifestly relevant summation formula for the Clausen's series (e.g., see Bailey (1935), Rainville (1960), Slater (1966), Srivastava and Manocha (1984)) with the positive argument and to support interest in generalized hypergeometric functions.

Indeed, the classical field of hypergeometric functions ${}_pF_q(z)$ has been recently achieved a substantial progress by investigating, generalizing and producing various relationships between them. Often, many functions contain the integers and fractions in their numerator and denominator parameters in different ways (see mentioned papers of Gradshteyn and Ryzhik (2014), Krattenhalter and Rivoal (2006), Lavoi et al. (1996), Milgram (2007), Miller and Srivastava (2010), Miller and Paris (2011, 2012), Prudnikov et al. (1990), Rainville (1960), Rao et al. (2005), Shpot (2007, and see also references therein)).

With reference to the importance of applications in mathematics, statistics and mathematical physics, the hypergeometric functions readily reduce to a gamma function. A large number of authors notably C.F. Gauss, M.M. Kummer, S. Ramanujan and many others have explored the field of hypergeometric function and its applications.

The papers by Karl (1974), Kim et al. (2014), Lewanowicz (1997), Rathie and Rakha (2008), Rathie and Paris (2013) mentioned at the beginning, as well as the present one, discuss the summation formulas for the functions ${}_3F_2(1)$ that belong to the category of Clausen's hypergeometric function.

We are now motivated enough by the work in the directions of summation theorems for Clausen's hypergeometric functions indicated above. Term by term integration of special polynomials and Appell functions appear in the expansions in Chu (2012), Gauss (1813), Khan et al. (2019), Mubeen et al. (2007), Nisar et al. (2018), Nisar et al. (2017), and Srivastava et al. (1984).

2. Preliminaries

The *generalized hypergeometric function* ${}_pF_q$ is defined by

$${}_pF_q\left(\begin{matrix}(a_p);\\(b_q);\end{matrix}\, z\right) = \sum_{m=0}^{\infty} \frac{[(a_p)]_m}{[(b_q)]_m} \frac{z^m}{m!}, \tag{1}$$

where

- p and q are positive integers or zero,
- z is a complex variable,
- (a_p) designates the set $a_1, a_2, ..., a_p$,
- the numerator parameters $a_1, ..., a_p$ and the denominator parameters $b_1, ..., b_q$ take on complex

values, $b_j, j = 1, ..., q$, being non-negative integers,

- $[(a_r)]_k = \prod_{i=1}^{r} (a_i)_k$. By convention, a product over the empty set is 1.
- $(a)_k$ is the Pochhammer's symbol.

Thus, if a numerator parameter is a negative integer or zero, the ${}_pF_q$ series terminates and then we are led to a generalized hypergeometric polynomial.

The widely-used Pochhammer symbol $(\lambda)_\nu \;\; (\lambda, \nu \in \mathbb{C})$ is defined by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1, & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}), \\ \lambda(\lambda+1)\ldots(\lambda+n-1), & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}), \end{cases} \tag{2}$$

it being understood *conventionally* that $(0)_0 = 1$ and assumed *tacitly* that the Γ quotient exists.

In 1856, Karl Weierstrass gave a different definition of gamma function

$$\frac{1}{\Gamma(z)} = z \exp(\gamma z) \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n}\right) \exp\left(-\frac{z}{n}\right) \right], \tag{3}$$

where $\gamma = 0.577215664901532860606512090082402431042\ldots$, is called the Euler-Mascheroni constant and $\frac{1}{\Gamma(z)}$ is an entire function of z and

$$\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots.. + \frac{1}{n} - \ell n\,(n)\right).$$

The function

$$\psi(z) = \frac{d}{dz}\{\ell n\, \Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)}, \tag{4}$$

or, equivalently

$$\ell n\, \Gamma(z) = \int_1^z \psi(\zeta) d\zeta, \tag{5}$$

is the logarithmic derivative of the gamma function.

Gauss (1813) discovered an interesting formula for digamma (Psi) function as follows

$$\psi(p/q) = -\gamma - \ell n\,(q) - \frac{\pi}{2} \cot\left(\frac{\pi p}{q}\right) + \sum_{j=1}^{[\frac{q}{2}]}{}' \left\{ \cos\left(\frac{2\pi jp}{q}\right) \ell n \left(2 - 2\cos\frac{2\pi j}{q}\right) \right\}, \tag{6}$$

where $1 \leq p < q$ and p, q are positive integers and accent (prime) to right of the summation sign indicates the term corresponding to (last term) $j = \frac{q}{2}$ (when q is positive even integer) should be divided by 2.

A simplified treatment of the above formula was made by Murty and Saradha (2007) (see also, Lahmer (1975)) such that

$$\psi(p/q) = -\gamma - \ell n\,(2q) - \frac{\pi}{2} \cot\left(\frac{\pi p}{q}\right) + 2 \sum_{j=1}^{[\frac{q}{2}]} \left\{ \cos\left(\frac{2\pi pj}{q}\right) \ell n \sin\left(\frac{\pi j}{q}\right) \right\}, \tag{7}$$

where $p = 1, 2, 3, \ldots, (q-1),\; q = 2, 3, 4, \ldots; (p, q) = 1.$

In volume III of Prudnikov et al. (1990), summation theorems for Clausen's hypergeometric functions

$$
{}_3F_2\left[1,1,\frac{1}{4};2,\frac{5}{4};1\right], {}_3F_2\left[1,1,\frac{1}{3};2,\frac{4}{3};1\right], {}_3F_2\left[1,1,\frac{3}{8};2,\frac{11}{8};1\right], {}_3F_2\left[1,1,\frac{1}{2};2,\frac{3}{2};1\right],
$$
$$
{}_3F_2\left[1,1,\frac{5}{8};2,\frac{13}{8};1\right], {}_3F_2\left[1,1,\frac{2}{3};2,\frac{5}{3};1\right], {}_3F_2\left[1,1,\frac{3}{4};2,\frac{7}{4};1\right], {}_3F_2\left[1,1,\frac{9}{8};2,\frac{17}{8};1\right],
$$
$$
{}_3F_2\left[1,1,\frac{7}{8};2,\frac{15}{8};1\right], {}_3F_2\left[1,1,\frac{5}{2};2,\frac{7}{2};1\right], {}_3F_2\left[1,1,\frac{5}{4};2,\frac{9}{4};1\right], {}_3F_2\left[1,1,\frac{3}{2};2,\frac{5}{2};1\right],
$$
$$
{}_3F_2\left[1,1,\frac{11}{8};2,\frac{19}{8};1\right], {}_3F_2\left[1,1,\frac{7}{4};2,\frac{11}{4};1\right],
$$

are available.

In the paper of Qureshi et al. (2018), we have given summation theorems for Clausen's hypergeometric functions ${}_3F_2[1,1,n;2,n+1;1]$, where $n = 3,4,5,\ldots,52$.

Motivated by the work, recorded in the table of Prudnikov et al. (1990) and a paper of Qureshi et al. (2018), we have given some new and modified summation theorems for Clausen's hypergeometric functions using a celebrated formula of Gauss for digamma function in Sections 4 and 5 respectively.

3. Main Result

In this section, we shall establish an interesting formula for digamma function in the terms of Clausen's hypergeometric function and then connecting it with the Gauss formula for digamma function, we obtain our desired result.

Let us recall the Weierstrass definition of gamma function (3)

$$
\frac{1}{\Gamma(z)} = z\exp(\gamma z)\prod_{n=1}^{\infty}\left[\left(1+\frac{z}{n}\right)\exp\left(-\frac{z}{n}\right)\right].
$$

Taking $\log$ to the base e and making some manipulations, we get

$$
-\ell n\,\Gamma(z) = \ell n\, z + \gamma z + \sum_{n=1}^{\infty}\left\{\ell n\,\left(1+\frac{z}{n}\right) - \frac{z}{n}\right\}. \tag{8}
$$

Now, differentiating with respect to z and making some simplifications, we get

$$
\frac{\Gamma'(z)}{\Gamma(z)} = -\frac{1}{z} - \gamma - \sum_{n=1}^{\infty}\left\{\frac{1}{n+z} - \frac{1}{n}\right\},
$$
$$
\frac{\Gamma'(z)}{\Gamma(z)} = -\frac{1}{z} - \gamma + \sum_{n=1}^{\infty}\left\{\frac{z}{n(n+z)}\right\}. \tag{9}
$$

Applying the definition of generalized hypergeometric function (1) in the equation (9), we get new and general result for digamma function in terms of Clausen's hypergeometric function as follows

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\frac{1}{z} - \gamma + \left(\frac{z}{1+z}\right) {}_3F_2\left[\begin{matrix} 1, 1, 1+z; \\ \\ 2, 2+z; \end{matrix} \; 1\right], \tag{10}$$

where $z \neq 0, -1, -2, -3, \ldots$ and $\psi(z)$ denotes the Psi (or Digamma) function.

On setting $z = \frac{p}{q}$ in Equation (10), we get

$$\psi\left(\frac{p}{q}\right) = -\gamma - \frac{q}{p} + \left(\frac{p}{p+q}\right) \; {}_3F_2\left[\begin{matrix} 1, 1, \frac{p+q}{q}; \\ \\ 2, \frac{p+2q}{q}; \end{matrix} \; 1\right]. \tag{11}$$

Now, we will connect our above obtained result for digamma function (11) with the Gauss celebrated formula for digamma function (7), and get the desired result for Clausen's hypergeometric function in terms of π, and natural logarithm of rationals and irrationals as follows

$$\begin{aligned} {}_3F_2\left[\begin{matrix} 1, 1, \frac{p+q}{q}; \\ \\ 2, \frac{p+2q}{q}; \end{matrix} \; 1\right] = \left(\frac{q+p}{p}\right) \Bigg[\frac{q}{p} - \ell n \, 2q - \frac{\pi}{2} \cot\left(\frac{\pi p}{q}\right) \\ +2 \sum_{j=1}^{[\frac{q}{2}]} \left\{ \cos\left(\frac{2\pi p j}{q}\right) \ell n \sin\left(\frac{\pi j}{q}\right) \right\} \Bigg], \end{aligned} \tag{12}$$

$1 \leq p < q$.

4. Some new summation theorems for Clausen's function

For the application purpose of new result, we have derived some new summation theorems for Clausen's hypergeometric function using our formula (12).

$${}_3F_2\left[\begin{matrix} 1, 1, \frac{-1}{2}; \\ \\ 2, \frac{1}{2}; \end{matrix} \; 1\right] = \left\{ \frac{2}{3} - \frac{2}{3} \ell n \, 2 \right\}, \tag{13}$$

$${}_3F_2\left[\begin{matrix} 1, 1, \frac{7}{2}; \\ \\ 2, \frac{9}{2}; \end{matrix} \; 1\right] = \frac{7}{5}\left\{ \frac{46}{15} - 2\ell n \, 2 \right\}, \tag{14}$$

$${}_3F_2\left[\begin{matrix} 1, 1, -\frac{4}{3}; \\ \\ 2, -\frac{1}{3}; \end{matrix} \; 1\right] = \frac{4}{7}\left\{ \frac{15}{4} + \frac{\pi\sqrt{3}}{6} - \frac{3}{2} \ell n \, 3 \right\}, \tag{15}$$

$$ {}_3F_2\left[\begin{matrix}1,1,\frac{4}{3};\\ \\ 2,\frac{7}{3};\end{matrix}\ 1\right]=\left\{12-\frac{2\pi}{\sqrt{3}}-6\ell n\,3\right\}, \tag{16}$$

$$ {}_3F_2\left[\begin{matrix}1,1,\frac{10}{3};\\ \\ 2,\frac{13}{3};\end{matrix}\ 1\right]=\frac{10}{7}\left\{\frac{117}{28}-\frac{\sqrt{3}\pi}{6}-\frac{3}{2}\ell n\,3\right\}, \tag{17}$$

$$ {}_3F_2\left[\begin{matrix}1,1,\frac{6}{5};\\ \\ 2,\frac{11}{5};\end{matrix}\ 1\right]=6\left\{5-\ell n\,10-\left(\frac{1+\sqrt{5}}{\sqrt{(10-2\sqrt{5})}}\right)\frac{\pi}{2}+\frac{1}{2}\left(\sqrt{5}\ell n\left(\frac{\sqrt{5}-1}{2}\right)-\ell n\,\frac{\sqrt{5}}{4}\right)\right\}, \tag{18}$$

$$ {}_3F_2\left[\begin{matrix}1,1,\frac{7}{5};\\ \\ 2,\frac{12}{5};\end{matrix}\ 1\right]=\frac{7}{2}\left\{\frac{5}{2}-\ell n\,10-\left(\frac{\sqrt{5}-1}{\sqrt{(10+2\sqrt{5})}}\right)\frac{\pi}{2}+\frac{1}{2}\left(\sqrt{5}\ell n\left(\frac{\sqrt{5}+1}{2}\right)-\ell n\,\frac{\sqrt{5}}{4}\right)\right\}, \tag{19}$$

$$ {}_3F_2\left[\begin{matrix}1,1,\frac{8}{5};\\ \\ 2,\frac{13}{5};\end{matrix}\ 1\right]=\frac{8}{3}\left\{\frac{5}{3}-\ell n\,10+\left(\frac{\sqrt{5}-1}{\sqrt{(10+2\sqrt{5})}}\right)\frac{\pi}{2}+\frac{1}{2}\left(\sqrt{5}\ell n\left(\frac{\sqrt{5}+1}{2}\right)-\ell n\,\frac{\sqrt{5}}{4}\right)\right\}, \tag{20}$$

$$ {}_3F_2\left[\begin{matrix}1,1,\frac{9}{5};\\ \\ 2,\frac{14}{5};\end{matrix}\ 1\right]=\frac{9}{4}\left\{\frac{5}{4}-\ell n\,10+\left(\frac{\sqrt{5}+1}{\sqrt{(10-2\sqrt{5})}}\right)\frac{\pi}{2}+\frac{1}{2}\left(\sqrt{5}\ell n\left(\frac{\sqrt{5}-1}{2}\right)-\ell n\,\frac{\sqrt{5}}{4}\right)\right\}, \tag{21}$$

$$ {}_3F_2\left[\begin{matrix}1,1,\frac{1}{6},\\ \\ 2,\frac{7}{6};\end{matrix}\ 1\right]=\left\{\sqrt{3}\frac{\pi}{10}+\frac{3}{10}\ell n\,3+\frac{2}{5}\ell n\,2\right\}, \tag{22}$$

$$ {}_3F_2\left[\begin{matrix}1,1,\frac{7}{6},\\ \\ 2,\frac{13}{6};\end{matrix}\ 1\right]=7\left\{6-\ell n\,12-\sqrt{3}\frac{\pi}{2}-\ell n\,\sqrt{3}\right\}, \tag{23}$$

$$ {}_3F_2\left[\begin{matrix}1,1,\frac{11}{6},\\ \\ 2,\frac{17}{6};\end{matrix}\ 1\right]=\frac{11}{5}\left\{\frac{6}{5}-\ell n\,12+\sqrt{3}\frac{\pi}{2}-\ell n\,\sqrt{3}\right\}, \tag{24}$$

$$ {}_3F_2\left[\begin{matrix}1,1,\frac{11}{10};\\ \\ 2,\frac{21}{10};\end{matrix}\ 1\right]=11\left\{10-\ell n\,20-\left(\frac{\sqrt{(10+2\sqrt{5})}}{\sqrt{5}-1}\right)\frac{\pi}{2}+\frac{1}{2}\left(\sqrt{5}\ell n\,(\sqrt{5}-2)-\ell n\,\sqrt{5}\right)\right\}, \tag{25}$$

$$ {}_3F_2\left[\begin{matrix}1,1,\frac{13}{10};\\ \\ 2,\frac{23}{10};\end{matrix}\ 1\right]=\frac{13}{3}\left\{\frac{10}{3}-\ell n\,20-\left(\frac{\sqrt{(10-2\sqrt{5})}}{\sqrt{5}+1}\right)\frac{\pi}{2}+\frac{1}{2}\left(\sqrt{5}\ell n\,(\sqrt{5}+2)-\ell n\,\sqrt{5}\right)\right\}, \tag{26}$$

$$ {}_3F_2\left[\begin{matrix}1,1,\frac{17}{10};\\ \\ 2,\frac{27}{10};\end{matrix}\ 1\right]=\frac{17}{7}\left\{\frac{10}{7}-\ell n\,20+\left(\frac{\sqrt{(10-2\sqrt{5})}}{\sqrt{5}+1}\right)\frac{\pi}{2}+\frac{1}{2}\left(\sqrt{5}\ell n\,(\sqrt{5}+2)-\ell n\,\sqrt{5}\right)\right\}, \tag{27}$$

$$ {}_3F_2\left[\begin{matrix}1,1,\frac{19}{10};\\ \\ 2,\frac{29}{10};\end{matrix}\ 1\right]=\frac{19}{9}\left\{\frac{10}{9}-\ell n\,20+\left(\frac{\sqrt{(10+2\sqrt{5})}}{\sqrt{5}-1}\right)\frac{\pi}{2}+\frac{1}{2}\left(\sqrt{5}\ell n\,(\sqrt{5}-2)-\ell n\,\sqrt{5}\right)\right\}, \tag{28}$$

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$$ {}_3F_2\left[\begin{array}{c}1,1,\frac{13}{12};\\ \\ 2,\frac{25}{12};\end{array} 1\right] = 13\left\{12 - \ell n\, 24 - \left(2+\sqrt{3}\right)\frac{\pi}{2} + \sqrt{3}\ell n\,(2-\sqrt{3}) - \ell n\,\sqrt{3}\right\}, \tag{29} $$

$$ {}_3F_2\left[\begin{array}{c}1,1,\frac{17}{12};\\ \\ 2,\frac{29}{12};\end{array} 1\right] = \frac{17}{5}\left\{\frac{12}{5} - \ell n\, 24 - \left(2-\sqrt{3}\right)\frac{\pi}{2} + \sqrt{3}\ell n\,(2+\sqrt{3}) - \ell n\,\sqrt{3}\right\}, \tag{30} $$

$$ {}_3F_2\left[\begin{array}{c}1,1,\frac{19}{12};\\ \\ 2,\frac{31}{12};\end{array} 1\right] = \frac{19}{7}\left\{\frac{12}{7} - \ell n\, 24 + \left(2-\sqrt{3}\right)\frac{\pi}{2} + \sqrt{3}\ell n\,(2+\sqrt{3}) - \ell n\,\sqrt{3}\right\}, \tag{31} $$

$$ {}_3F_2\left[\begin{array}{c}1,1,\frac{23}{12};\\ \\ 2,\frac{35}{12};\end{array} 1\right] = \frac{23}{11}\left\{\frac{12}{11} - \ell n\, 24 + \left(2+\sqrt{3}\right)\frac{\pi}{2} + \sqrt{3}\ell n\,(2-\sqrt{3}) - \ell n\,\sqrt{3}\right\}. \tag{32} $$

Proof (Independent proof of summation theorem (25)):

On setting $z = \frac{1}{10}$ in our derived formula (10), we get

$$ \psi\left(\frac{1}{10}\right) = -\gamma - 10 + \frac{1}{11}{}_3F_2\left[\begin{array}{c}1,1,\frac{11}{10};\\ \\ 2,\frac{21}{10};\end{array} 1\right]. \tag{33} $$

Now, we calculate the value of $\psi\left(\frac{1}{10}\right)$ using the formula (7) as follows

$$ \psi\left(\frac{1}{10}\right) = -\gamma - \ell n\, 20 - \left(\frac{\sqrt{(10+2\sqrt{5})}}{\sqrt{5}-1}\right)\frac{\pi}{2} + \frac{1}{2}\{\sqrt{5}\ell n\,(\sqrt{5}-2) - \ell n\,\sqrt{5}\}. \tag{34} $$

On comparing the equations (33) and (34), we get the desired result (25). Similarly, we can derive other summation theorems. ■

5. Some modified summation theorems for Clausen's function

In this section, we have presented some corrected forms of summation theorems given in the table of Prudnikov et al. (1990). The errata are found in the following summation theorems

$$ {}_3F_2\left[\begin{array}{c}1,1,\frac{13}{8};\\ \\ 2,\frac{21}{8};\end{array} 1\right] \circeq \frac{13}{50}\left\{16 - 5(\sqrt{2}-1)\pi - 40\ell n\, 2 + 10\sqrt{2}\ell n\,(1+\sqrt{2})\right\}, \tag{35} $$

$$ {}_3F_2\left[\begin{array}{c}1,1,\frac{15}{8};\\ \\ 2,\frac{23}{8};\end{array} 1\right] \circeq \frac{15}{196}\left\{32 + 7(1+\sqrt{2})\pi - 112\ell n\, 2 - 28\sqrt{2}\ell n\,(1+\sqrt{2})\right\}, \tag{36} $$

$$ {}_3F_2\left[\begin{array}{c}1,1,\frac{5}{3};\\ \\ 2,\frac{8}{3};\end{array} 1\right] \circeq \frac{5\sqrt{3}}{12}\left\{3\sqrt{3}(1 - \ell n\, 3) - \pi\right\}, \tag{37} $$

where the symbol $\stackrel{\circ}{=}$ exhibits the fact that each of the equations (35), (36) and (37) does not hold true as stated.

The modified forms of the above summation theorems are as follows

$$ {}_3F_2\left[\begin{array}{c}1,1,\frac{13}{8};\\ \\ 2,\frac{21}{8};\end{array} 1\right] = \frac{13}{50}\left\{16+5(\sqrt{2}-1)\pi-40\ell n\, 2+10\sqrt{2}\ell n\,(1+\sqrt{2})\right\}, \tag{38} $$

$$ {}_3F_2\left[\begin{array}{c}1,1,\frac{5}{3};\\ \\ 2,\frac{8}{3};\end{array} 1\right] = \frac{5\sqrt{3}}{12}\left\{3\sqrt{3}(1-\ell n\, 3)+\pi\right\}, \tag{39} $$

$$ {}_3F_2\left[\begin{array}{c}1,1,\frac{15}{8};\\ \\ 2,\frac{23}{8};\end{array} 1\right] = \frac{15}{196}\left\{32+14(1+\sqrt{2})\pi-112\ell n\, 2-28\sqrt{2}\ell n\,(1+\sqrt{2})\right\}. \tag{40} $$

Remark 5.1.

We can not obtain the summation theorems of Sections 4 and 5 with the help of summation theorems of Whipple, Dixon, Watson and other associated contiguous relations of ${}_3F_2$.

6. Conclusion

We derived new summation formula for Clausen's hypergeometric function with unit argument using the digamma function. We conclude our present investigation by observing that several hypergeometric summation theorems for Clausen's hypergeometric function with unit argument have been deduced using our main result in an analogous manner. These new results may use in the further study of digamma function and Clausen's hypergeometric function with unit argument.

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REFERENCES

Appell, P. and Kampé de Fériet, J. (1926). *Fonctions hypergéométriques et hypersphériques. Polynomes d'Hermite*, Gauthier-Villars, Paris.

Bailey, W.N. (1935). *Generalised Hypergeometric Series,* Cambridge University Press, Cambridge, England.

Chu, W. (2012). Analytical formulae for extended $_3F_2$-series of Watson-Whipple-Dixon with two extra integer parameters, Math. Comp., Vol. 81, No. 277, pp. 467-479.

Gauss, C.F. (1813). Disquisitiones generales circa seriem infinitam etc..., Comm. Soc. Reg. Sci. Gott. Rec., Vol II, pp. 1-46.

Gradshteyn, I.S. and Ryzhik, I.M. (2014). *Table of Integrals, Series and Products*, 8th ed., Academic Press Inc., San Diego, CA.

Karlsson, P.W. (1974). Reduction of hypergeometric functions with integral parameter differences, Indagationes Mathematicae (Proceedings), Vol. 77, No. 3, pp. 195-198.

Khan, W.A., Khan, I.A., Ahmad, M. and Nisar, K.S. (2019). Sufficiency for general hypergeometric transform associated with conic region, New Trends in Mathematical Sciences, Vol. 2, No. 7, pp. 179-187.

Kim, Y.S., Rathie, A.K. and Paris, R.B. (2014). On two Thomae-type transformations for hypergeometric series with integral parameter differences, Math. Comm., Vol. 19, pp. 111-118.

Krattenhalter, C. and Rivoal, T. (2006). How can we escape Thomae′s relations, J. Math. Soc. Japan, Vol. 58, No. 1, pp. 183-210.

Lavoi, J.L., Grondin, F. and Rathie, A.K. (1996). Generalizations of Whipple's theorem on the sum of a $_3F_2$, J. Comp. Appl. Math., Vol. 72, No. 2, pp. 293-300.

Lehmer, D.H. (1975). Euler constants for arithmetical progressions, Acta Arith., Vol. 27, pp. 125-142.

Lewanowicz, S. (1997). Generalized Watson's summation formula for $_3F_2(1)$, J. Comp. Appl. Math., Vol. 86, No. 2, pp. 375-386.

Milgram, M. (2007). Comment on a paper of Rao et al., an entry of Ramanujan and a new $_3F_2(1)$, J. Comp. Appl. Math., Vol. 201, No. 1, pp. 1-2.

Miller, A.R. and Paris, R.B. (2011). Certain transformations and summations for generalized hypergeometric series with integral parameter differences, Int. Transf. Spec. Func., Vol. 22, No. 1, pp. 67-77.

Miller, A.R. and Paris, R.B. (2012). Clausen's series $_3F_2(1)$ with integral parameter differences and transformations of the hypergeometric function $_2F_2(x)$, Int. Transf. Spec. Func., Vol. 23, No. 1, pp. 21-33.

Miller, A.R. and Srivastava, H.M. (2010). Karlsson-Minton summation theorems for the generalized hypergeometric series of unit argument, Int. Transf. Spec. Func., Vol. 21, pp. 603-612.

Mubeen, S., Nisar, K.S., Rahman, G. and Arshad, M. (2007). Inequalities of Extended (p,q)-beta and confluent hypergeometric function. https://arxiv.org/abs/1703.08852

Murty, M.R. and Saradha, N. (2007). Transcendental values of the digamma function, J. Num. Theo., Vol. 125, pp. 298-318.

Nisar, K.S., Feng, Q., Ragman, G., Mubeen, S. and Arshad, M. (2018). Some inequalities involving the extended gamma function and the Kummer confluent hypergeometric k-function, J. Ineq. Appl., Vol. 135, pp. 1-12.

Nisar, K.S., Ragman, G., Mubeen, S. and Arshad, M. (2017). Generalized hypergeometric k-functions via (k, s)-fractional calculus, J. Nonl. Sci. Appl., Vol. 10, pp. 1791-1800.

Prudnikov, A.P., Brychkov, Y.A. and Marichev, O.I. (1990). *Integrals and Series, Vol. 3: More Special Functions*, Gordon and Breach Science Publishers.

Qureshi, M.I., Jabee, S. and Shadab, M. (2018). Truncated Gauss hypergeometric series and its application in digamma function (Communicated). https://arxiv.org/abs/1805.12548

Rainville, E.D. (1960). *Special Functions,* The Macmillan Co. Inc., New York.

Rao, K.S., Berghe, G.V. and Krattenhalter, C. (2005). An entry of Ramanujan on hypergeometric series in his notebooks, J. Comp. Appl. Math., Vol. 173, No. 2, pp. 239-246.

Rathie, A.K. and Paris, R.B. (2013). Extension of some classical summation theorems for the generalized hypergeometric series with integral parameter differences, J. Classical Anal., Vol. 3, No. 2, pp. 109-127.

Rathie, A.K. and Rakha, M.A. (2008). A study of new hypergeometric transformations, J. Phys. A: Math. Theor., Vol. 41, No. 44, Art. No. 445202, pp. 15.

Shpot, M.A. (2007). A massive Feynman integral and some reduction relations for Appell functions, J. Math. Phys., Vol. 48, No. 12, Art No. 123512-1-13.

Slater, L.J. (1966). *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge.

Srivastava, H.M. and Manocha, H.L. (1984). *A Treatise on Generating Functions,* Halsted Press (Ellis Horwood Ltd., Chichester, U.K.), John Wiley and Sons, New York, Chichester, Brisbane and Toronto.

Srivastava, H.M., Rahman, G. and Nisar, K.S. (1984). A treatise on generating functions, Iranian J. Sci. Technol. Trans. Sci., Vol. 43, pp. 2601-2606.