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Existence of resolvent for conformable fractional Volterra integral equations

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Abstract

In this paper, we consider the conformable fractional Volterra integral equation. We study the existence of a resolvent kernel corresponding to conformable fractional Volterra integral equation. The technique of proof involves Lebesgue dominated convergence theorem. Our results improve and extend the results obtained in literature.

Keywords: Kernel; Resolvent kernel; conformable; Volterra integral equation

MSC 2020 No.: 47H10, 47H09, 54H25, 47J25, 46B20

1. Introduction

Fractional calculus is a generalization of classical differentiation and integration into an arbitrary (non-integer) order and it is as old as calculus. The theory goes back to mathematicians like

Leibniz (1646-1716), Liouville (1809-1882), Riemann (1826-1866), Letnikov (1837-1888), Grünwald (1838-1920) and others. Since the last three centuries fractional calculus, like all fields of science, engineering and mathematics, is one of the most intensively developed fields of mathematical assessment. Due to its numerous applications in engineering, economics and finance, signal processing, earthquake dynamics, geology, probability and statistics, chemical engineering, physics, splines, thermodynamics, neural networks and so on (see Carvalho et al. (2018), Sweilam and Al-Mekhlafi (2016)), the fractional calculus has always drawn researcher's interest (see Anderson and Camrud (2019), Carvalho et al. (2018), Gao et al. (2020), Gao et al. (2019) and Khalil et al. (2014)). There are several definitions of fractional operators like Riemann-Liouville, Caputo and Grünwald-Letnikov, Weyl, Hadamard, Marchaud and Riesz (see Khalil et al. (2014), Miller (1971)). But it should be noted that these types of derivatives do not meet the fundamental formulas of the product derivative (the quotient) of two functions and the chain rule etc.

Recently, Khalil et al. (2014) has introduced a new well-behaved simple fractional derivative that is called the conformable fractional derivative based on the basic limit definition of the derivative. Under this definition, all the classical characteristics of the derivative retain and satisfy the chain rule. This new definition attracted many researchers and some results were obtained for the fundamental properties of the conformable fractional derivative in Abdeljawad (2015). For further features, also see (Abdeljawad et al. (2017), Al-Rifae et al. (2017), and Jarad et al. (2017)).

In the literature survey, there are some articles on the existence, uniqueness and boundedness of the solutions of integral equation on time scale Svetlin (2016). Kulik and Tisdell (2008) discussed the qualitative and quantitative properties of the solutions of Volterra integral equations on time scale. Whereas, in Adivar and Raffoul (2010) the existence of a resolvent Kernel corresponding to the Volterra integral equations on time scale has been discussed and its special cases are integral, summation and q-integral equations, which are also part of this reference. But there is no remarkable literature for existence of resolvent of Volterra integral equations on conformable fractional calculus. In this assertion, we extend the theory established in Adivar and Raffoul (2010) to the conformable fractional Volterra integral equations. The generalized Volterra integral equations arise in many scientific applications such as the population dynamics, spread of epidemics and semi-conductor devices. Resolvents are used to express the solutions of Volterra integral equations. For some recent paper on the qualitative behaviors of solutions of Volterra integro-differential equations (see Tunç (2016; 2017), Tunç and Tunç (2017; 2018; 2019) and the bibliography therein).

2. Basic notions

Given a function $f:[a,\infty)\to\mathbb{R},\ a\geq 0$, the conformable fractional integral of f is defined by

$$\left(I_\alpha^a f\right)(t)=\int_a^t f\left(x\right)d_\alpha x=\int_a^t\left(x-a\right)^{\alpha-1} f\left(x\right)dx\ ,$$

where the integral is the usual Riemann improper integral, and $\alpha\in(0,1)$ (see Abdeljawad

(2015)).

Assume that $f: [a,\infty) \to \mathbb{R}$ is continuous and $0 < \alpha \leq 1$. Then, for all $t > a$ we have

$$\left[I_\alpha^a f(t)\right]^{(\alpha)} = f(t),$$

(see Abdeljawad (2015)).

Assume that $f: (a,b) \to \mathbb{R}$ be differentiable and $0 < \alpha \leq 1$. Then, for all $t > a$ we have

$$I_\alpha^a f^{(\alpha)}(t) = f(t) - f(a),$$

(see Abdeljawad (2015)).

Assume $f, g: (a,\infty) \to \mathbb{R}$ be α-differentiable functions, where $0 < \alpha \leq 1$. Let $h(t) = f\big(g(t)\big)$. Then $h(t)$ is α-differentiable and for all t with $t > a$ we have

$$h^{(\alpha)}(t) = f'\big(g(t)\big).g'(t).(t-a)^{1-\alpha}.$$

If moreover, and $g(t) \neq a$ or g is one-to-one, then

$$h^{(\alpha)}(t) = f^{(\alpha)}\big(g(t)\big).g^{(\alpha)}(t).\big(g(t)-a\big)^{\alpha-1}$$

and

$$h^{(\alpha)}(a) = \lim_{t\to a^+} f^{(\alpha)}\big(g(t)\big).g^{(\alpha)}(t).\big(g(t)-a\big)^{\alpha-1},$$

(see Abdeljawad (2015)).

3. Construction of the resolvent equation

Firstly, chain rule for conformable fractional derivative of a function of two variables is discussed below.

Lemma 3.1.

Consider that $f: (0,\infty) \to \mathbb{R}$ and $g: (0,\infty) \to \mathbb{R}$ are α-differentiable functions, where $\alpha \in (0,1]$. Let

$$h(x) = G\big(f(x), g(x)\big). \tag{1}$$

Then, $h(x)$ is α-differentiable $\forall\ x$ with $x \neq 0$, and we have

$$h^{(\alpha)}(x) = \frac{\partial G(f(x))}{\partial f(x)} f^{(\alpha)}(x) + \frac{\partial G(g(x))}{\partial g(x)} g^{(\alpha)}(x), \tag{2}$$

where $h^{(\alpha)}(x)$ is conformable fractional derivative.

Proof:

For $t \neq 0$ and $\alpha \in (0,1]$, we have

$$h^{(\alpha)}(x) = x^{1-\alpha} h^{'(x)} = x^{1-\alpha} \frac{\partial G(f(x))}{\partial f(x)} f'(x) + \frac{\partial G(g(x))}{\partial g(x)} g'(x)$$

and, hence, the result follows. ■

To differentiate the iterated integrals, we will employ the following theorem.

Theorem 3.2.

Let $k(x,t)$ be continuous, $g(x)$ and $h(x)$ are α-differentiable with $g(x) \geq 0$, $h(x) \geq 0$, where $\alpha \in (0,1]$ and

$$K(x) = \int_{h(x)}^{g(x)} k(x\ t)\, d_\alpha t. \tag{3}$$

Then,

$$\begin{aligned} K^{(\alpha)}(x) &= (x - \mathrm{a})^{1-\alpha} k(x, g(x)) x^{1-\alpha} \frac{d}{dx} g(x) \\ &\quad - (x - \mathrm{a})^{1-\alpha} k(x, h(x)) (h(x) - \mathrm{a})^{1-\alpha} \frac{d}{dx} h(x) \\ &\quad + \int_{h(x)}^{g(x)} \frac{\partial^\alpha}{\partial_\alpha x} k(x,t)\, d_\alpha t \end{aligned} \tag{4}$$

Proof:

Let

$$G_1(u,x) = \int_a^u k(x,t)\, d_\alpha t, \tag{5}$$

where $u = h(x)$, $a \geq 0$, and

$$G_2(x,w) = \int_w^a k(x,t)\, d_\alpha t, \tag{6}$$

where $w = g(x),\ a \geq 0$. Evaluating partial derivatives of equations (5) and (6) with respect to u and w, respectively, and applying Lemma 3.1, we get our required result. ■

Corollary 3.3.

1. If

$$K(x) = \int_a^x k(x,t)\, d_\alpha t,$$

Then,

$$K^{(\alpha)}(x) = \int_a^x \frac{d^\alpha}{d_\alpha x} k(x,t)\, d_\alpha t + k(x,x).$$

2. If

$$K(x) = \int_x^a k(x,t)\, d_\alpha t,$$

Then,

$$K^{(\alpha)}(x) = \int_x^a \frac{d^\alpha}{d_\alpha x} k(x,t)\, d_\alpha t - k(x,x).$$

Here, we use some characteristics of multiple α-conformable fractional integrals to develop the resolvent equations related to the integral equations for linear and nonlinear systems.

Consider interval $[0, M]$ and let

$$E_1 = \{(x,y) \in M \times M : 0 \leq x < y, 0 \leq y < v\}.$$

Theorem 3.4.

Suppose $h: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous mapping. Then,

$$\int_0^v d_\alpha y \int_0^y h(x,y)\, d_\alpha x = \int_0^v d_\alpha x \int_0^y h(x,y)\, d_\alpha y.$$

That is,

$$\int_0^v \int_0^y h(x,y)\, d_\alpha x d_\alpha y = \int_0^v \int_x^v h(x,y)\, d_\alpha y d_\alpha x.$$

Proof:

Let

$$G(v)=\int_0^v\int_0^y h(x,y)\,d_\alpha x d_\alpha y-\int_0^v\int_x^v h(x,y)\,d_\alpha y d_\alpha x. \tag{7}$$

Now, we take the conformable fractional derivative of $G(v)$, i.e., $G^{(\alpha)}(v)$. We now consider the results of Corollary 3.3. If

$$f(t)=\int_0^t g(t,\tau)\,d_\alpha \tau\,.$$

Then,

$$f^{(\alpha)}(t)=\int_0^t \frac{\partial^\alpha}{\partial_\alpha t} g(t,\tau)\,d_\alpha \tau+g(t,t).$$

Let

$$\int_x^v h(x,y)\,d_\alpha y=g(x,v)$$

and

$$f(v)=\int_0^v g(x,v)\,d_\alpha x.$$

That is,

$$f(v)=\int_0^v\int_x^v h(x,y)\,d_\alpha y d_\alpha x.$$

Hence, it follows that

$$f^{(\alpha)}(v)=\int_0^v \frac{\partial^\alpha}{\partial_\alpha v} g(x,v)\,d_\alpha x+g(v,v).$$

That is,

$$\begin{aligned} f^{(\alpha)}(v)&=\int_0^v \frac{\partial^\alpha}{\partial_\alpha v}\left[\int_x^v h(x,y)\,d_\alpha y\right]d_\alpha x+\int_v^v h(v,y)\,d_\alpha y \\ &=\int_0^v \frac{\partial^\alpha}{\partial_\alpha v}\left[\int_0^v h(x,y)\,d_\alpha y-\int_0^x h(x,y)\,d_\alpha y\right]d_\alpha x. \end{aligned}$$

By fundmental theorem of conformable fractional calculus we have

$$f^{(\alpha)}(v)=\int_0^v h(x,v)\,d_\alpha x.$$

Now, we have to solve the following integral

$$\int_0^v\int_0^y h(x,y)\,d_\alpha x d_\alpha y=\int_0^v L(y)\,d_\alpha y, \tag{8}$$

where

$$L(y)=\int_0^y h(x,y)\,d_\alpha x.$$

When we calculate the α-conformable fractional derivative of the integral (8), it follows that

$$\frac{d^{\alpha}}{d_{\alpha}v}\int_{0}^{v}L(y)d_{\alpha}y=L(v)$$

and

$$\frac{d^{\alpha}}{d_{\alpha}v}\int_{0}^{v}\int_{0}^{y}h(x,y)d_{\alpha}xd_{\alpha}y=\int_{0}^{v}h(x,v)d_{\alpha}x.$$

Equation (7) shows that

$$G^{(\alpha)}(v)=\int_{0}^{v}h(x,v)d_{\alpha}x-\int_{0}^{v}h(x,v)d_{\alpha}x,$$

that is,

$$G^{(\alpha)}(v)=0.$$

The last equalitiy implies that $G(v)=C$. Consider the initial value problem

$$G^{(\alpha)}(v)=0 \text{ with } G(0)=0.$$

Thus $G(v)=0$, $\forall\ v\in\mathbb{R}$. That is,

$$\int_{0}^{v}\int_{0}^{y}h(x,y)d_{\alpha}xd_{\alpha}y=\int_{0}^{v}\int_{x}^{v}h(x,y)d_{\alpha}yd_{\alpha}x,$$

which gives the desired result. ■

Consider the linear conformable fractional Volterra integral equation of the following form:

$$\phi(v)=g(v)+\int_{0}^{v}b(v,x)\phi(x)d_{\alpha}x. \tag{9}$$

The corresponding resolvent equation related with kernel $b(v,x)$ is mentioned by

$$R(v,x)=-b(v,x)+\int_{x}^{v}R(v,y)b(y,x)d_{\alpha}y. \tag{10}$$

If the corresponding resolvent equation (10) has a solution $R(v,x)$, then the solution of the linear system (9) can be written in terms of g as below:

$$\phi(v)=g(v)-\int_{0}^{v}R(v,y)g(y)d_{\alpha}y. \tag{11}$$

To see this equality, we multiply both sides of equation (9) by $R(v,x)$ to obtain

$$\int_0^v R(v,y)\phi(y)\,d_\alpha y - \int_0^v R(v,y)\,g(y)\,d_\alpha y = \int_0^v R(v,y)\int_0^v b(y,x)\phi(x)\,d_\alpha x\,d_\alpha y,$$

which implies that

$$\int_0^v R(v,y)\phi(y)\,d_\alpha y - \int_0^v R(v,y)\,g(y)\,d_\alpha y = \int_0^v \left\{\int_x^v R(v,y)\,b(y,x)\,d_\alpha y\right\}\phi(x)\,d_\alpha x.$$

Equation (10) gives that

$$\int_x^v R(v,y)\,b(y,x)\,d_\alpha y = R(v,x) + b(v,x). \tag{12}$$

Therefore, by using equation (12), we have

$$\int_0^v R(v,y)\phi(y)\,d_\alpha y - \int_0^v R(v,y)\,g(y)\,d_\alpha y = \int_0^v \{R(v,x) + b(v,x)\}\phi(x)\,d_\alpha x,$$

which implies that

$$\int_0^v b(v,x)\phi(x)\,d_\alpha x = -\int_0^v R(v,y)\,g(y)\,d_\alpha y. \tag{13}$$

Hence equation (9) becomes

$$\phi(v) = g(v) - \int_0^v R(v,y)\,g(y)\,d_\alpha y.$$

Thus, we arrive at equation (11).

On the other hand, one may also show, by using equation (13), that equation (11) implies equation (9) as follows.

In fact, by equation (13), we obtain

$$\phi(v) = g(v) + \int_0^v b(v,x)\phi(x)\,d_\alpha x,$$

which is equation (9).

We now consider the following nonlinear conformable fractional integral equation:

$$\tilde{\phi}(v) = g(v) + \int_0^v b(v,x)\left\{\tilde{\phi}(x) + L\left(x,\tilde{\phi}(x)\right)\right\}d_\alpha x, \tag{14}$$

where $L(v,\tilde{\phi})$ refers to the higher-order terms of $\tilde{\phi}$.

If the solution $\tilde{\phi}$ of equation (14) is known, then this equation can be redefined as

$$\tilde{\phi}(v) = H(v) + \int_0^v b(v,x)\tilde{\phi}(x)\,d_\alpha x,$$

where

$$H(v) = g(v) + \int_0^v b(v,x)L\left(x,\tilde{\phi}(x)\right)d_\alpha x.$$

If the resolvent $R(v,x)$ is known, then we obtain

$$\tilde{\phi}(v) = H(v) - \int_0^v R(v,x)H(x)\,d_\alpha x.$$

Hence, it follows that

$$\begin{aligned}\tilde{\phi}(v) = g(v) &+ \int_0^v b(v,x)L\left(x,\tilde{\phi}(x)\right)d_\alpha x \\ &- \int_0^v R(v,x)\left\{g(x) + \int_0^v b(x,y)L\left(y,\tilde{\phi}(y)\right)d_\alpha y\right\}d_\alpha x,\end{aligned}$$

that is,

$$\begin{aligned}\tilde{\phi}(v) = g(v) &- \int_0^v R(v,x)g(x)\,d_\alpha x + \int_0^v b(v,x)L\left(x,\tilde{\phi}(x)\right)d_\alpha x \\ &- \int_0^v\left\{\int_y^v R(v,x)b(x,y)\,d_\alpha x\right\}L\left(y,\tilde{\phi}(y)\right)d_\alpha y.\end{aligned}$$

By equation (10), we find

$$\begin{aligned}\tilde{\phi}(v) = g(v) &- \int_0^v R(v,x)g(x)\,d_\alpha x + \int_0^v b(v,x)L\left(x,\tilde{\phi}(x)\right)d_\alpha x \\ &- \int_0^v\left\{R(v,y) + b(v,y)\right\}L\left(y,\tilde{\phi}(y)\right)d_\alpha y,\end{aligned}$$

which implies that

$$\tilde{\phi}(v) = g(v) - \int_0^v R(v,x)g(x)\,d_\alpha x - \int_0^v R(v,y)L\left(y,\tilde{\phi}(y)\right)d_\alpha y. \tag{15}$$

Equivalently, we have

$$\tilde{\phi}(v) = \phi(v) - \int_0^v R(v,y)L\left(y,\tilde{\phi}(y)\right)d_\alpha y. \tag{16}$$

By making use of equations (15), (9) and (11), one can easily check that equation (16) infers equation (14).

In the next portion, we examine the existence of resolvent $R(v,x)$ corresponding to the linear integral equation (9). We also use Theorems 4.6 and 4.8 to demonstrate that

$$\int_x^v R(v,y)b(y,x)d_\alpha y = \int_x^v b(v,y)R(y,x)d_\alpha y.$$

This allows us to rewrite equation (10) as

$$R(v,x) = -b(v,x) + \int_x^v b(v,y)R(y,x)d_\alpha y. \tag{17}$$

4. Existence of resolvent

Let $\Omega = \{(v,x) \in \mathbb{R} \times \mathbb{R}: 0 \leq x \leq v \leq M\}$, and $1 \leq p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. We indicate the matrix norm for any $n \times n$ matrix E by $|E|$ such that

$$|E| = \sup_{|x|\leq 1} \|Ex\|,$$

where $\|Ex\|$ refers to the vector norm of E.

We now define the following functions:

$$E(v) = \int_0^v |b(v,x)|^q d_\alpha x, v \in J, F(v) = \int_v^M |b(x,v)|^p d_\alpha x, v \in J, \tag{18}$$

and

$$a(v,x) = \int_x^v E(y)^{p/q} d_\alpha y, (v,x) \in \Omega. \tag{19}$$

Next, we define a class of $m \times m$ matrix-valued functions $g: \Omega \longrightarrow \mathbb{R}^{m\times m}$ such that the following conditions are fulfilled:

(C.1) $g(v,x)$ is measurable in $(v,x) \in \Omega$ with $g(v,x) = 0$, which holds almost everyehere when $x > v$.

(C.2) For almost all v in J, the integral $\int_0^M |g(v,x)|^q d_\alpha x$ exists, and for almost all x in J, the integral $\int_0^M |g(v,x)|^p d_\alpha v$ exists.

(C.3) The numbers $\int_0^M \left\{\int_0^M |g(v,x)|^q d_\alpha x\right\}^{p/q} d_\alpha v$ and $\int_0^M \left\{\int_0^M |g(v,x)|^p d_\alpha v\right\}^{q/p} d_\alpha x$ are both finite.

Definition 4.1.

An $m \times m$ matrix-valued function $g(v,x)$ is said to be of type (L^p, M) iff the conditions (C.1)-(C.3) hold.

Example 4.2.

Any function $g(v,x)$, which is continuous in (v,x) for $(v,x) \in \Omega$, is of type (L^p, M) for each $p > 1$ and $M > 0$.

Definition 4.3.

An $m \times m$ matrix-valued function $g(v,x)$ is called of type LL_p iff for all $M > 0$, $g(v,x)$ is of type (L^p, M).

Assume the kernel $b(v,x)$ be of type (L^p, M). Describe the sequence $\{R_m(v,x)_{m\in\mathbb{N}}\}$ via

$$R_1(v,x) = b(v,x), \tag{20}$$

$$R_{m+1}(v,x) = \int_x^v b(v,y) R_m(y,x) d_\alpha y, \tag{21}$$

for $(v,x) \in \Omega$ and $R_m(v,x) = 0$ for $0 \le v < x \le M$.

The following lemma plays a significant role in the proof of an inequality to be given.

Lemma 4.4.

Assume $1 < p < \infty$ and the kernel $b(v,x)$ is of type (L^p, M). Then the equality

$$\frac{\left\{a(v,x)^m\right\}^{(\alpha)_v}}{m!} = E(v)^{p/q} \frac{a(v,x)^{m-1}}{(m-1)!} \tag{22}$$

is valid for every positive integers $m > 1$ and $(v,x) \in \Omega$.

Proof:

We benefit from the following formula:

$$\left\{g^{m+1}(v)\right\}^{(\alpha)} = (m+1) g^{(\alpha)_v}(v) \left(g(v)\right)^m. \tag{23}$$

As $b(v,x)$ is of type (L^p, M), it is clear that

$$
\begin{aligned}
a(v,x) &= \int_x^v E(y)^{p/q}\, d_\alpha y \\
&= \int_x^v \left\{ \int_0^y |b(y,x)|^q\, d_\alpha x \right\}^{p/q} d_\alpha y, (v,x) \in \Omega,
\end{aligned}
$$

is α-conformable fractional differentiable in both of its variables and

$$
a^{(\alpha)_v}(v,x) = E(v)^{p/q}, a^{(\alpha)_x}(v,x) = -E(x)^{p/q}.
$$

It follows that a increases in v and decreases in x. Thus from equation (23) we obtain

$$
\begin{aligned}
\left\{a(v,x)^m\right\}^{(\alpha)_v} &= \frac{(m-1)!}{(m-1)!}\left\{m\left[a(v,x)\right]^{m-1} a^{(\alpha)_v}(v,x)\right\}, \\
\frac{\left\{a(v,x)^m\right\}^{(\alpha)_v}}{(m-1)!} &= \frac{m}{(m-1)!} a(v,x)^{m-1} E(v)^{\frac{p}{q}}, \\
\frac{\left\{a(v,x)^m\right\}^{(\alpha)_v}}{m(m-1)!} &= \frac{1}{(m-1)!} a(v,x)^{m-1} E(v)^{\frac{p}{q}}, \\
\frac{\left\{a(v,x)^m\right\}^{(\alpha)_v}}{m!} &= E(v)^{\frac{p}{q}} \frac{a(v,x)^{m-1}}{(m-1)!}.
\end{aligned}
$$

This completes the proof. ■

Lemma 4.5.

Assume $1 < p < \infty$ and the kernel $b(v,x) = R_1(v,x)$ is of type (L^p, M). Then, for all positive integer $m \geq 1$, the function $R_m(v,x)$ is of type (L^p, M). Furthermore, for all nonnegative integer $m \geq 0$ and for $(v,x) \in \Omega$, the following inequality

$$
|R_{m+2}(v,x)| \leq E(v)^{1/q} F(x)^{1/p} \left\{ \frac{a(v,x)^m}{m!} \right\}^{1/p}, \tag{24}
$$

is valid.

Proof:

If $0 \leq v < x \leq M$, then $R_{m+2}(v,x) = 0$ and equation (24) holds.

Assume that $x \leq v$ for each $(v,x) \in \Omega$. We continue through induction. For $m = 0$, we derive

$$\left|R_2(v,x)\right| = \left|\int_x^v b(v,y)\,R_1(y,x)\,dy\right|$$
$$\leq \int_x^v \left|b(v,y)\,R_1(y,x)\right| d_\alpha y.$$

Since $R_1(y,x) = b(y,x)$, then we have

$$\left|R_2(v,x)\right| \leq \int_x^v \left|b(v,y)\,b(y,x)\right| d_\alpha y$$
$$\leq \int_x^v \left|b(v,y)\right|\left|b(y,x)\right| d_\alpha y.$$

By the Hölder's inequality (see Sarikaya and Budak (2017), Lemma 1), we find that

$$\left|R_2(v,x)\right| \leq \left\{\int_x^v \left|b(v,y)\right|^q d_\alpha y\right\}^{1/q} \left\{\int_x^v \left|b(y,x)\right|^p d_\alpha y\right\}^{1/p}.$$

We also derive the following inequalities:

$$E(v) = \int_0^v \left|b(v,y)\right|^q d_\alpha y \geq \int_x^v \left|b(v,y)\right|^q d_\alpha y$$

and

$$F(x) = \int_x^M \left|b(y,x)\right|^p d_\alpha y \geq \int_x^v \left|b(y,x)\right|^p d_\alpha y.$$

Therefore, we obtain

$$R_2(v,x) \leq E(v)^{1/q}\,F(x)^{1/p}.$$

Since the kernel $b(v,x)$ is of type (L^p, M), therefore $E(v)$, $F(x)$ are of (L^p, M) type, the product of $E(v)F(x)$ is of (L^p, M) type. Hence $R_2(v,x)$ is of (L^p, M) type.

Suppose that $R_1, R_2, \cdots, R_{m+1}$ are all kernels of type (L^p, M) and the equation (24) holds for $m-1$. Then, it follows that

$$\left|R_{m+2}(v,x)\right| = \left|\int_x^v b(v,y)\,R_{m+1}(y,x)\,d_\alpha y\right|$$
$$\leq \int_x^v \left|b(v,y)\,R_{m+1}(y,x)\right| d_\alpha y.$$

By the Hölder's inequality (see Sarikaya and Budak (2017), Lemma 1), it is derived that

$$\left|R_{m+2}(v,x)\right| \leq \left\{\int_x^v \left|b(v,y)\right|^q d_\alpha y\right\}^{1/q} \left\{\int_x^v \left|R_{m+1}(y,x)\right|^p d_\alpha y\right\}^{1/p}.$$

This inequality implies that

$$\left|R_{m+2}(v,x)\right| \le E(v)^{1/q}\left\{\int_x^v \left|R_{m+1}(y,x)\right|^p d_\alpha y\right\}^{1/p}. \tag{25}$$

Therefore, equation (25) becomes:

$$\begin{aligned}\left|R_{m+2}(v,x)\right| &\le E(v)^{1/q}\left\{\int_x^v E(y)^{p/q} F(x)^{p/p}\left\{\frac{a(y,x)^{m-1}}{(m-1)!}\right\}d_\alpha y\right\}^{1/p}\\ &\le E(v)^{1/q}\left\{F(x)\int_x^v E(y)^{p/q}\frac{a(y,x)^{m-1}}{(m-1)!}d_\alpha y\right\}^{1/p}\\ &\le E(v)^{1/q}F(x)^{1/p}\left\{\int_x^v E(y)^{p/q}\frac{a(y,x)^{m-1}}{(m-1)!}d_\alpha y\right\}^{1/p}.\end{aligned}$$

By using the equality

$$\frac{\left\{a(v,x)^m\right\}^{(\alpha)_v}}{m!} = E(t)^{p/q}\frac{a(v,x)^{m-1}}{(m-1)!},$$

from equation (25), we obain

$$\begin{aligned}\left|R_{m+2}(v,x)\right| &\le E(v)^{1/q}F(x)^{1/p}\left\{\int_x^v \frac{\left\{a(y,x)^m\right\}^{(\alpha)_y}}{m!}d_\alpha y\right\}^{1/p}\\ &\le E(v)^{1/q}F(x)^{1/p}\left\{\frac{a(v,x)^m}{m!}\right\}^{1/p}.\end{aligned}$$

It implies that R_{m+2} is of type (L^p, M). Hence, Lemma 4.5. is proved. ■

Theorem 4.6.

If $1 < p < \infty$ and the kernel $b(v,x)$ be of type (L^p, M), then $\exists$ a kernel $R(v,x)$ is of type (L^p, M) which satisfies the resolvent equation *(17)* for almost all in $(v,x) \in \Omega$.

Proof:

Suppose

$$R(v,x) = -\sum_{m=1}^{\infty} R_m(v,x) \text{ for } (v,x) \in \Omega \tag{26}$$

and $R(v,x) = 0$ if $0 \le v < x \le M$. Taking modulus on both sides of equation (26), we get the following equalities:

$$\begin{aligned}|R(v,x)| &= \left|-\sum_{m=1}^{\infty} R_m(v,x)\right| = \left|\sum_{m=1}^{\infty} R_m(v,x)\right| \\ &= |R_1(v,x)| + \sum_{m=2}^{\infty} |R_m(v,x)|.\end{aligned}$$

From equation (24), we obtain

$$|R(v,x)| \le |b(v,x)| + E(v)^{1/q} F(x)^{1/p} \sum_{m=2}^{\infty} \left\{\frac{c^{m-2}}{(m-2)!}\right\}^{1/p},$$

where

$$c = \int_0^M E(y)^{p/q} \, d_\alpha y \, .$$

Hence, it follows that

$$|R(v,x)| \le |b(v,x)| + E(v)^{1/q} F(x)^{1/p} \sum_{m=2}^{\infty} \left\{\frac{c^{m}}{m!}\right\}^{1/p}. \tag{27}$$

For any $c > m$ it is clear that, $a_m = \left\{\frac{c^m}{m!}\right\}^{1/p}$, $a_{m+1} = \left\{\frac{c^{m+1}}{(m+1)!}\right\}^{1/p}$. Hence , we derive

$$\begin{aligned}\frac{a_{m+1}}{a_m} &= \left\{\frac{c^{m+1}}{(m+1)!} \times \frac{m!}{c^m}\right\}^{1/p} \\ &= \left\{\frac{c}{m+1}\right\}^{1/p}.\end{aligned}$$

Taking the limit $m \longrightarrow \infty$, we obtain

$$\lim_{m\to\infty} \frac{a_{m+1}}{a_m} = \lim_{m\to\infty} \left\{\frac{c}{m+1}\right\}^{1/p} = 0 < 1.$$

This suggests that the series in equation (27) converges through the ratio test. We know that

$$E(v) = \int_0^v |b(v,x)|^q \, d_\alpha x,$$

and

$$F(v)=\int_v^M |b(v,x)|^p\, d_\alpha x, v\in[0,M].$$

Since $E(v)$, $F(x)$ and $b(v,x)$ are finite, then $R(v,x)$ is well defined for almost all and measurable in (v,x) for $(v,x)\in\Omega$. From equation (27), we conclude that $R(v,x)$ is of type (L^p,M). Finally, we attempt the "Lebesgue dominated convergence theorem" to obtain

$$\begin{aligned}\int_x^v b(v,y)R(y,x)d_\alpha y &= \int_x^v b(v,y)\left\{-\sum_{m=1}^{\infty}R_m(y,x)\right\}d_\alpha y\\ &= -\sum_{m=1}^{\infty}\int_x^t b(v,y)R_m(y,x)d_\alpha y = -\sum_{m=1}^{\infty}R_{m+1}(v,x)\\ &= R(v,x)+b(v,x).\end{aligned}$$

Consequently,

$$R(v,x)=-b(v,x)+\int_x^v b(v,y)R(y,x)d_\alpha y.$$

This indicates that R defined in equation (26) solves resolvent equation (17). ∎

Lemma 4.7.

If $R_1(v,x)=b(v,x)$ be of type (L^p,M), then, for all positive integers c and d with $c+d=m+1$,

$$R_{m+1}(v,x)=\int_x^v R_c(v,y)R_d(y,x)d_\alpha y. \tag{28}$$

Proof:

The proof is trivial for $m=1$. That is,

$$R_2(v,x)=\int_x^v R(v,y)R(y,x)d_\alpha y,$$

where $c=d=1$ such that $c+d=1+1=m+1$, which implies that

$$c+d=2=m+1.$$

Let equation (28) be true for $c_0+d_0\le m,\ m\ge 1$. That is, $c_0+d_0\le 1$. Given $c,\ d\ge 1$ such that $c+d=m+1$, define

$$I(c,d):=\int_x^v R_c(v,y)R_d(y,x)d_\alpha y. \tag{29}$$

We know that

$$R_{m+1}(v,x)=\int_x^v b(v,y)R_m(y,x)d_\alpha y.$$

By using $R_{m+1}(v,x)$ in equation (29), we have

$$\begin{aligned}I(c,d)&=\int_x^v R_c(v,y)\left\{\int_x^y b(y,u)R_{d-1}(u,x)d_\alpha u\right\}d_\alpha y\\&=\int_x^v\int_x^y R_c(v,y)b(y,u)R_{d-1}(u,x)d_\alpha u d_\alpha y\\&=\int_x^v\int_u^v R_c(v,y)b(y,u)R_{d-1}(u,x)d_\alpha y d_\alpha u\\&=\int_x^v\left\{\int_u^v R_c(v,y)b(y,u)d_\alpha y\right\}R_{d-1}(u,x)d_\alpha u.\end{aligned}$$

Since

$$R_{c+1}(v,u)=\int_u^v R_c(v,y)b(y,u)d_\alpha y,$$

then we get

$$\begin{aligned}I(c,d)&=\int_x^v R_{c+1}(v,u)R_{d-1}(u,x)d_\alpha u\\&=I(c+1,d-1).\end{aligned}$$

Hence, we arrive at

$$I(1,m)=I(2,m-1)=I(3,m-3)=\ldots=I(m,1).$$

This relation proves the result for $m+1$. ■

Theorem 4.8.

If $b(v,x)$ be a kernel of type LL_p, then there exists a kernel $R(v,x)$ of type LL_p that satisfies both resolvent equations (10) and (17) for almost every $(v,x)\in\Omega$.

Proof:

By Theorem 4.6, we deduce that the kernel $R(v,x)$ is of type LL_p with the property that $R(v,x)$ satisfies equation (17) for almost all $(v,x)\in\Omega$. Now, we have to prove that $R(v,x)$ satisfies equation (10) for almost all $(v,x)\in\Omega$. By using equation (26), we obtain

$$\begin{aligned}\int_x^v R(v,y)b(y,x)d_\alpha y&=\int_x^v\left\{-\sum_{n=1}^{\infty}R_n(v,y)\right\}b(y,x)d_\alpha y\\&=-\sum_{n=1}^{\infty}\int_x^v R_n(t,y)b(y,x)d_\alpha y,\end{aligned}$$

where the "Lebesgue dominated convergence theorem" allows us to interchange summation and integration. Since $b(y,x) = R_1(y,x)$, then we derive that

$$\int_x^v R(v,y)b(y,x)d_\alpha y = -\sum_{n=1}^{\infty}\int_x^v R_n(v,y)R_1(y,x)d_\alpha y.$$

By Lemma 4.7, we have

$$\int_x^v R(v,y)b(y,x)d_\alpha y = -\sum_{n=1}^{\infty} R_{n+1}(v,x). \tag{30}$$

By equation (26), we know that

$$R(v,x) = -\sum_{n=1}^{\infty} R_n(v,x).$$

Replacing n by $n+1$, it follows that

$$\begin{aligned} R(v,x) &= -\sum_{n+1=1}^{\infty} R_{n+1}(v,x) = -\sum_{n=0}^{\infty} R_{n+1}(v,x) \\ &= -R_1(v,x) - \sum_{n=1}^{\infty} R_{n+1}(v,x) = -b(v,x) - \sum_{n=1}^{\infty} R_{n+1}(v,x). \end{aligned}$$

This equality implies that

$$-\sum_{n=1}^{\infty} R_{n+1}(v,x) = R(v,x) + b(v,x).$$

Therefore, equation (30) becomes

$$\int_x^v R(v,y)b(y,x)d_\alpha y = R(v,x) + b(t,x).$$

Then, we have

$$R(v,x) = -b(v,x) + \int_x^v R(v,y)b(y,x)d_\alpha y.$$

Hence, $R(v,x)$ satisfies equation (10). The proof is complete. ■

Example 4.9.

We now solve the comfortable integral equation $y(x) = sinx + \int_0^x e^{x-t} y(t) d_\alpha t$. Let

$$R_1(x,t) = b(x,t) = e^{x-t}.$$

It is known that

$$R_m(x,t)=\int_t^x b(x,z)R_{m-1}(z,t)\,d_\alpha z.$$

Hence, for $m=2$, we have

$$\begin{aligned}R_2(x,t)&=\int_t^x b(x,z)R_1(z,t)\,d_\alpha z\\&=\int_t^x e^{x-z}e^{z-t}\,d_\alpha z=\int_t^x e^{x-t}\,d_\alpha z\\&=e^{x-t}\int_t^x (z-t)^{\alpha-1}\,dz=e^{x-t}\left.\frac{(z-t)^\alpha}{\alpha}\right|_t^x\\&=e^{x-t}\frac{(x-t)^\alpha}{\alpha}.\end{aligned}$$

For $m=3$, it is obvious that

$$\begin{aligned}R_3(x,t)&=\int_t^x b(x,z)R_2(z,t)\,d_\alpha z\\&=\int_t^x e^{x-z}\frac{e^{z-t}}{\alpha}(z-t)^\alpha\,d_\alpha z\\&=\int_t^x \frac{e^{x-t}}{\alpha}(z-t)^\alpha\,d_\alpha z\\&=\int_t^x \frac{e^{x-t}}{\alpha}(z-t)^\alpha(z-t)^{\alpha-1}\,dz\\&=\int_t^x \frac{e^{x-t}}{\alpha}(z-t)^{2\alpha-1}\,dz\\&=\frac{e^{x-t}}{\alpha}\int_t^x (z-t)^{2\alpha-1}\,dz=\frac{e^{x-t}}{\alpha}\left|\frac{(z-t)^{2\alpha}}{2\alpha}\right|_t^x\\&=\frac{e^{x-t}}{2\alpha^2}(x-t)^{2\alpha}.\end{aligned}$$

For $n=4$, we get

$$\begin{aligned}R_4(x,t)&=\int_t^x b(x,z)R_3(z,t)\,d_\alpha z\\&=\int_t^x e^{x-z}\frac{e^{z-t}}{2\alpha^2}(z-t)^{2\alpha}\,d_\alpha z\\&=\frac{e^{x-t}}{2\alpha^2}\int_t^x (z-t)^{2\alpha}(z-t)^{\alpha-1}\,dz\\&=\frac{e^{x-t}}{2\alpha^2}\int_t^x (z-t)^{3\alpha-1}\,dz\end{aligned}$$

so that

$$R_4(x,t)=\frac{e^{x-t}}{2\alpha^2}\left|\frac{(z-t)^{3\alpha}}{3\alpha}\right|_t^x$$
$$=\frac{e^{x-t}}{6\alpha^3}(x-t)^{3\alpha}=\frac{e^{x-t}}{3!\alpha^3}(x-t)^{3\alpha}.$$

As a result, we obtain

$$R_m(x,t)=\frac{(x-t)^{(m-1)\alpha}}{(m-1)!\alpha^{m-1}}e^{x-t}.$$

Therefore, it is clear that

$$R(x,t)=-\sum_{m=1}^{\infty}R_m(x,t)$$
$$=-\left[R_1(x,t)+R_2(x,t)+\cdots\right]$$
$$=-\sum_{m=1}^{\infty}\frac{(x-t)^{(m-1)\alpha}}{(m-1)!\alpha^{m-1}}e^{x-t}.$$

Using the Definition of conformable exponential function [8, Definition 2.3.], we obtain

$$R(x,t)=-e^{\frac{(x-t)^\alpha}{\alpha}}e^{x-t}=-e^{\frac{(x-t)^\alpha}{\alpha}+(x-t)}.$$

We now obtain the solution of the given integral equation as

$$y(x)=\sin x-\int_0^x R(x,t)\,f(t)\,d_\alpha t.$$

Hence, we have

$$y(x)=\sin x+\int_0^x\left(e^{\frac{(x-t)^\alpha+\alpha(x-t)}{\alpha}}\right)\sin t\left(t^{\alpha-1}\right)dt.$$

For $\alpha=\frac{1}{2}$, equation (31) implies that

$$y(x)=\sin x+\int_0^x\left(e^{2(x-t)^{\frac{1}{2}}+(x-t)}\right)\sin t\left(t^{-\frac{1}{2}}\right)dt.$$

which is the solution of the given conformable fractional Volterra integral equation.

5. Conclusion

The solutions of Volterra integral equations have a significant role in the field of science and engineering. We have discussed the existence of a resolvent kernel corresponding to conformable fractional Volterra integral equation by using a strategy which is different from other authors approach. The notion of the resolvent equation to study boundedness and integrability of the solutions of the Conformable fractional Volterra integral equation. In particular, the existence of bounded solutions with various L^p properties has have been studied under suitable conditions on the functions involved in the above Volerra integral equation. Our results improved and extended the results obtained in the literature.

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